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ICA-based and second-order separability of nonlinear models involving reference signals: general properties and application to quantum bits

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Abstract

Relatively few results have been reported about the separability of given classes of nonlinear mixtures by means of statistical criteria such as ICA. We here first prove the ICA separability of a wide class of nonlinear global (i.e. mixing + separating) models involving "reference signals", i.e. unmixed signals. We also show the second-order separability of sub-classes of the above class of models. This work therefore concerns nonlinear extensions of (linear) adaptive noise cancellation. We illustrate the usefulness of our general results by applying them to a quantum information processing problem, which involves a model of Heisenberg-coupled quantum states (i.e. qubits). This paper opens the way to practical ICA-based and second-order blind source separation (BSS) methods for nonlinear mixtures encountered in various applications. These BSS methods are also outlined in this paper.

Keywords: blind source separation, independent component analysis, nonlinear adaptive noise cancellation, separability, nonlinear mixture, quantum bit (qubit)

1. Introduction

A generic signal processing problem consists in extracting one or several unknown source signals of interest from several observations, which are mixtures of these signals of interest and possibly of additional, undesired, source signals. A first generation of such problems was especially studied by Widrow et al. and e.g. reported in 1975 in [8]. It is known as adaptive noise cancellation (ANC). It typically corresponds to configurations where almost all observations are "reference signals", i.e. unmixed signals. An extended version of this problem, known as blind source separation (BSS), has then been widely studied since the 1980's [1], [6]. It mainly concerns cases when *all* observations are mixtures of all source signals.

The "mixing model" involved in these problems is most often the functional form which defines the expression of the vector of observed signals with respect to the vector of source signals and to the parameters of that functional form. The values of these parameters are unknown in the *blind* version of the source separation problem. The development of a complete BSS method for a given mixing model typically consists of the following steps. Step 1: analyze the invertibility of this mixing model if possible, and define a separating model, which essentially aims at implementing the inverse of the considered mixing model. Step 2: select a separation criterion for estimating the values of the parameters of the separating model. Step 3 (closely linked to Step 2): determine if this criterion ensures the separability of the considered models (at least for some classes of source signals). This consists in determining if this criterion is met only when the outputs of the separating system are equal to the sources up to "acceptable" indeterminacies. Step 4: develop practical estimation algorithms associated with the considered criterion.

The above procedure has been widely applied to simple mixing models, i.e. linear (and especially instantaneous) ones. It has been much less explored and is much tougher for nonlinear models [1]. Its Step 1, i.e. the definition of separating *structures*, has e.g. been addressed for a wide class of nonlinear models in [4]. A natural way to tackle its Step 2 consists in considering Independent Component Analysis (ICA) methods, which have been widely used for linear mixtures [1], [6]. The relevance of these methods should then be proved for the considered nonlinear models, by investigating *separability*, which corresponds to Step 3 of the above procedure. Although some general ICA (non-)separability properties have been reported, very few results are available for the specific models which have been considered in the literature.

The above separability issue is the main topic that we address in this paper: in Sections 2 and 3, we respectively analyze the separability of all considered general nonlinear models by means of ICA-based and second-order criteria. Then, in Section 4, we outline the practical BSS methods which directly result from the above separability properties. The details of these BSS methods are beyond the scope of this paper, which is focused on separability. In Section 5, we then illustrate our general separability results by applying them to a specific model encountered in an application dealing with quantum signals. In Section 6, we eventually draw conclusions from our overall investigation, which yields a nonlinear extension of linear ANC.

2. Analysis of ICA-based separability

2.1. ICA-based separability of single-additive-target-source (SATS) global model

2.1.1. Considered global model

In all this paper, we only consider memoryless mixing and separating models, i.e. models whose outputs at a given time t only depend on their inputs at the same time. Moreover, we do not require the signals to have any temporal structure (nor do we use it if it exists). Therefore, we omit the argument "(t)" in signal notations hereafter, and we in fact consider the associated random variables at time t. All configurations studied below involve N source signals s_i which form a vector $s = [s_1, \ldots, s_N]^T$, where ^{T} stands for transpose. These signals are transferred through a mixing operator M, which belongs to a given class and depends on a set of parameters which form a vector θ, whose value is unknown in the framework of BSS. This yields N observed signals x_i which form a vector $x = [x_1, \ldots, x_N]^T$, defined as

$$x = M(s; \theta). \tag{1}$$

These signals are then processed by a separating or unmixing system, which corresponds to an operator U. This operator belongs to a fixed class and depends on a set of parameters which form a vector ϕ. The N output signals y_i of the unmixing system form a vector $y = [y_1, \ldots, y_N]^T$, defined as

$$y = U(x; \phi). \tag{2}$$

The class of operator U is "matched" to the class of operator M in the sense that there exists at least one value ϕ_{opt} of ϕ which depends on the considered value of θ and which is such that, when $\phi = \phi_{opt}$, the output signals y_i of the unmixing system are equal to the source signals s_j, up to a set of acceptable transformations, called indeterminacies (such as permutation, scaling, additive constants).

Combining (1) and (2), the global model from the source signals s_j to the unmixing system outputs y_i reads

$$y = G(s; \theta, \phi) \tag{3}$$

where the global operator $G = U \circ M$ is explicitly defined by

$$G(s; \theta, \phi) = U(M(s; \theta); \phi). \tag{4}$$

In the separability analyses presented hereafter, we only have to consider the global model G, i.e. we need not define the mixing and unmixing models M and U from which it is derived. We set the following conditions on G. Whatever θ and ϕ, only one of the output signals of the global model (and unmixing system) may be a mixture of source signals, whereas each of all other output signals only depends on a single source. The possibly mixed output is always the same, and we assign it index no. 1, i.e. the corresponding signal is y_1. Moreover, we request that the components of the global model may first be expressed as follows, with adequate ordering of the other outputs with respect to the sources:

$$y_1 = T(s_1; \theta, \phi) + I(s_2, \ldots, s_N; \theta, \phi) \tag{5}$$

$$y_i = H_i(s_i; \theta, \phi) \qquad \forall\, i = 2, \ldots, N. \tag{6}$$

The output of interest, y_1, is thus the sum of: (i) a term $T(s_1; \theta, \phi)$, which only depends on the *target* source, i.e. on the source s_1 that we aim at extracting from y_1, (ii) an *interference* term $I(s_2, \ldots, s_N; \theta, \phi)$, which may depend on all sources except the target source, and that we aim at "cancelling", by properly adjusting ϕ. Each signal $H_i(s_i;\theta,\phi)$ in (6) is a transformed version of source signal s_i. We denote it as s'_i, for $i = 2, \ldots, N$. For $i = 1$, we define

$$s'_1 = s_1. \tag{7}$$

Moreover, we consider the case when the model G is such that the term $I(s_2, \ldots, s_N;\theta,\phi)$ may also be expressed with respect to the transformed sources s'_2 to s'_N, instead of the original source signals s_2 to s_N (this is especially true when all operators H_i are invertible). The global model (5)-(6) may then be reformulated as

$$y_1 = T(s_1;\theta,\phi) + I'(s'_2, \ldots, s'_N;\theta,\phi) \tag{8}$$

$$y_i = s'_i \qquad \forall\ i = 2, \ldots, N. \tag{9}$$

Considering this data model, BSS aims at adjusting ϕ so as to extract the additive term $T(s_1;\theta,\phi)$ associated with the target source s_1 from the output signal y_1 defined in (8)[1]. The global model (8)-(9) is therefore referred to as the Single-Additive-Target-Source (or SATS) global model hereafter.

2.1.2. *Source properties and goal of investigation*

We here aim at determining if the above SATS global model is separable in the sense of ICA, for given source statistics and operators T and I'. Using the standard ICA formulation for an N-input to N-output global model, this ICA-separability problem may be defined as follows. We consider the situation when the random variables defined at time t by all original source signals $s_1(t)$ to $s_N(t)$ have given marginal statistics and are mutually statistically independent. The random variables defined at time t by all transformed source signals $s'_1(t)$ to $s'_N(t)$ are then also mutually statistically independent. We consider the random variables defined at time t by the separating system outputs $y_1(t)$ to $y_N(t)$. We denote these random variables as Y_1 to Y_N, and we aim at determining all cases when they are mutually statistically independent. If this only includes cases when the output signals are equal to the source signals up to acceptable indeterminacies, then the considered global model is said to be ICA-separable (up to these indeterminacies), for the considered type of sources.

Note that we thus consider ICA from a BSS perspective, i.e. as one of the possible tools for performing source extraction in a situation when is it known that the available observations result from a given number of source signals through a mixing model which belongs to a known class (and with unknown parameter values). This should be distinguished from the case when one applies ICA to observations without any knowledge about whether/how they may relate to source signals, and one aims at transforming these observations into independent output signals, especially in order to derive a more suitable representation of these observations.

In all this investigation, the source and output random variables are assumed to be *continuous*. Their statistics may therefore be defined by their probability density *functions* (pdf), i.e. without having to resort to representations based on distributions. Each of these pdf is non-zero on at least one interval. For such random variables, statistical independence and the associated ICA-separability criterion may be analyzed by considering the joint and marginal pdf of Y_1 to Y_N. Therefore, we first derive the expressions of these pdf hereafter. Moreover, we especially consider the case when the joint pdf of Y_1 to Y_N is continuous at some points of our analysis.

2.1.3. *Joint pdf of output random variables*

When expressing the outputs y_i with respect to the *transformed* sources s'_j, the global model (3) may be reformulated as

$$y = G'(s';\theta,\phi) \tag{10}$$

and is explicitly defined by (8)-(9). We consider the case when the operator G', defined from s' to y (in their considered domains) and for fixed θ and ϕ, is invertible. The pdf of the random vector Y composed of the random variables Y_i may then be expressed with respect to the pdf of the random vector S' composed of the random variables S'_i associated with the transformed sources, as

$$f_Y(y) = \frac{f_{S'}(s')}{|J_{G'}(s')|} \tag{11}$$

[1]The source signal s_1 may then be derived from $T(s_1;\theta,\phi)$ if operator T is invertible (in the considered domain).

where y and s' are linked by (10) and $J_{G'}(s')$ is the Jacobian of G', i.e. the determinant of the matrix composed of the partial derivatives of all components of G' with respect to all its arguments s'_i (see e.g. [6] for details). The expression of $J_{G'}(s')$ is derived from the explicit form (8)-(9) of the considered operator G'. It may be shown that this yields

$$J_{G'}(s') = \frac{\partial T}{\partial s_1}(s_1; \theta, \phi). \tag{12}$$

Moreover, since all transformed sources S'_i are independent, we have

$$f_{S'}(s') = \prod_{i=1}^{N} f_{S'_i}(s'_i). \tag{13}$$

Taking into account (7), (12), (13) and denoting

$$O_1(s_1; \theta, \phi) = \frac{f_{S'_1}(s_1)}{|\frac{\partial T}{\partial s_1}(s_1; \theta, \phi)|}, \tag{14}$$

Eq. (11) becomes

$$f_Y(y) = O_1(s_1; \theta, \phi) \prod_{i=2}^{N} f_{S'_i}(s'_i). \tag{15}$$

The right-hand term of (15) should be expressed with respect to the output signals y_i. To this end, we first use (9), which yields

$$f_{S'_i}(s'_i) = f_{S'_i}(y_i) = f_{Y_i}(y_i) \qquad \forall\ i = 2, \ldots, N. \tag{16}$$

Moreover, we consider the case when operator T, which is a function of s_1 with parameters θ and ϕ, is invertible (in the considered domain). Denoting T^{-1} the inverse of this operator, (8) and (9) then yield

$$s_1 = T^{-1}(y_1 - I'(y_2, \ldots, y_N; \theta, \phi); \theta, \phi). \tag{17}$$

Eq. (15) thus becomes

$$f_Y(y) = O_2(y_1 - I'(y_2, \ldots, y_N; \theta, \phi); \theta, \phi) \prod_{i=2}^{N} f_{S'_i}(y_i) \tag{18}$$

$$= O_2(y_1 - I'(y_2, \ldots, y_N; \theta, \phi); \theta, \phi) \prod_{i=2}^{N} f_{Y_i}(y_i) \tag{19}$$

where we define operator O_2 by

$$O_2(v; \theta, \phi) = O_1(T^{-1}(v; \theta, \phi); \theta, \phi). \tag{20}$$

2.1.4. Interpretation of operator O_2

Before proceeding to the next natural step of our ICA-separability investigation, we here analyze the nature of operator O_2 in more detail. This shows a feature of this operator that we will then require to derive separability properties. The target term that we aim at extracting from output signal y_1 is equal to $T(s_1; \theta, \phi)$. Let us denote this component in short as c, and the associated random variable as C. This random variable C is then a function of the random variable $S_1 = S'_1$, defined as

$$C = T(S'_1; \theta, \phi) \tag{21}$$

where T is an invertible transform. The pdf of these random variables are therefore linked by

$$f_C(c) = \frac{f_{S'_1}(s_1)}{|\frac{\partial T}{\partial s_1}(s_1; \theta, \phi)|} \tag{22}$$

where we have

$$s_1 = T^{-1}(c;\theta,\phi). \tag{23}$$

Eq. (22) may therefore be rewritten as

$$f_C(c) = \frac{f_{S_1'}(T^{-1}(c;\theta,\phi))}{|\frac{\partial T}{\partial s_1}(T^{-1}(c;\theta,\phi);\theta,\phi)|}. \tag{24}$$

Comparing this expression to (14) and (20) shows that the operator O_2 that we were led to introduce in our above standard ICA analysis is nothing but the pdf f_C. It is by the way not surprising that this pdf f_C is eventually a parameter of importance in our investigation, as opposed to the pdf of S_1: as shown by (8), the component of output signal y_1 which depends on s_1 is not equal to s_1 itself, but to $c = T(s_1;\theta,\phi)$.

2.1.5. Marginal pdf of output random variables

Due to (9), the pdf of Y_2 to Y_N are equal to those of the corresponding transformed sources. The pdf of Y_1 is then obtained by integrating the joint pdf of all output random variables, i.e.

$$f_{Y_1}(y_1) = \int_{\mathbb{R}^{N-1}} f_Y([y_1, v_2, \ldots, v_N]^T) \prod_{i=2}^{N} dv_i. \tag{25}$$

Inserting (18) in (25), we obtain

$$f_{Y_1}(y_1) = \int_{\mathbb{R}^{N-1}} O_2(y_1 - I'(v_2,\ldots,v_N;\theta,\phi);\theta,\phi) \prod_{i=2}^{N} f_{S_i'}(v_i) \prod_{i=2}^{N} dv_i. \tag{26}$$

2.1.6. Condition for independent outputs: general properties

The random variables Y_i are statistically independent if and only if

$$f_Y(y) = \prod_{i=1}^{N} f_{Y_i}(y_i). \tag{27}$$

This condition is initially requested to be met for any y in $\mathbb{R}^N$. However, it may be studied only for a subset of $\mathbb{R}^N$, as will now be explained. The marginal pdf $f_{Y_i}(y_i)$ of any output random variable Y_i may be expressed with respect to the joint pdf $f_Y(y)$ of all output random variables as

$$f_{Y_i}(y_i) = \int_{\mathbb{R}^{N-1}} f_Y(v) \prod_{j=1,\ldots,N,\quad j\neq i} dv_j \tag{28}$$

with

$$v = [v_1,\ldots,v_{i-1},y_i,v_{i+1},\ldots,v_N]^T. \tag{29}$$

Let us assume that, for a given value of v, we have $f_Y(v) \neq 0$, and therefore $f_Y(v) > 0$. Then, if $f_Y(.)$ is assumed to be continuous, we also have $f_Y(y) > 0$ for any y in a neighbourhood of v. Moreover, for any y in $\mathbb{R}^N$, we have $f_Y(y) \geq 0$. The right-hand term of (28) is then strictly positive, and so is thus its left-hand term, i.e. $f_{Y_i}(y_i) > 0$. This proves, conversely, that if a marginal density is such that $f_{Y_i}(y_i) = 0$ for a given y_i, then we have

$$f_Y(v) = 0 \qquad \forall [v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_N]^T \in \mathbb{R}^{N-1}, \tag{30}$$

with v still defined by (29). So, let us denote as $\mathcal{D}_{1,N}$ the subset of $\mathbb{R}^N$ composed of all vectors $y = [y_1,\ldots,y_N]^T$ which are such that

$$f_{Y_i}(y_i) \neq 0 \qquad \forall\ i = 1,\ldots,N. \tag{31}$$

For any y which does *not* belong to $\mathcal{D}_{1,N}$, we have the following properties. On the one hand, (31) is not met, which implies

$$\prod_{i=1}^{N} f_{Y_i}(y_i) = 0. \tag{32}$$

On the other hand, (30) yields

$$f_Y(y) = 0. \tag{33}$$

Therefore, if y does not belong to $\mathcal{D}_{1,N}$, Condition (27) holds for that vector y, and this is true whatever the random variables Y_i (with a continuous joint pdf), i.e. not depending whether they are independent or not. Therefore, when using Condition (27) to determine in which cases the random variables Y_i are independent, one only has to consider it for $y \in \mathcal{D}_{1,N}$.

2.1.7. Condition for independent outputs: specific properties

The considerations about independence that we presented above apply to any random variables. We now proceed to more specific properties, i.e. properties which only apply to the outputs of the SATS global model that we consider in this paper. Taking into account (19), Eq. (27) becomes

$$O_2(y_1 - I'(y_2, \ldots, y_N; \theta, \phi); \theta, \phi) \prod_{i=2}^{N} f_{Y_i}(y_i) \quad = \quad \prod_{i=1}^{N} f_{Y_i}(y_i). \tag{34}$$

Condition (34) is initially considered for any y in $\mathbb{R}^N$ and may then be simplified as follows. Let us denote as $\mathcal{D}_{2,N}$ the subset of $\mathbb{R}^{N-1}$ composed of all vectors $[y_2, \ldots, y_N]^T$ which are such that

$$f_{Y_i}(y_i) \neq 0 \qquad \forall\ i = 2, \ldots, N. \tag{35}$$

Whatever the random variables Y_i, Eq. (34) is met for any y_1 and any vector $[y_2, \ldots, y_N]^T$ which does not belong to $\mathcal{D}_{2,N}$, because its left-hand and right-hand terms are then both equal to zero. The independence constraint actually set by (34) therefore reduces to

$$\begin{aligned} O_2(y_1 - I'(y_2, \ldots, y_N; \theta, \phi); \theta, \phi) \quad &= \quad f_{Y_1}(y_1) \\ \forall\ y_1 \in \mathbb{R}, \forall\ [y_2, \ldots, y_N]^T \in \mathcal{D}_{2,N} & \end{aligned} \tag{36}$$

where $f_{Y_1}(y_1)$ is defined by (26).

2.1.8. Consequences for ICA-based separability

Let us consider the case when the output random variables Y_i are independent. We aim at determining how these random variables are linked to the transformed source random variables S'_i when this independence condition is met. We showed above that we only have to consider this condition for the values of $[y_2, \ldots, y_N]^T$ which belong to $\mathcal{D}_{2,N}$, i.e. which are such that all corresponding pdf $f_{Y_i}(y_i)$, with $i = 2, \ldots, N$, are non-zero. We mentioned in Section 2.1.2 that this constraint $f_{Y_i}(y_i) \neq 0$ is met at least over one interval of values y_i, separately for each random variable Y_i. So, let us consider the situation when y_2 to y_N are *varied* within such intervals where $f_{Y_i}(y_i) \neq 0$, whereas y_1 takes an arbitrary fixed value. In that situation, the output independence condition may be expressed as in (36) for these output values y_i, i.e.

$$O_2(y_1 - I'(y_2, \ldots, y_N; \theta, \phi); \theta, \phi) \quad = \quad f_{Y_1}(y_1). \tag{37}$$

The major phenomenon in Eq. (37) is then that (i) the arguments y_2 to y_N of $I'(y_2, \ldots, y_N; \theta, \phi)$ in the left-hand term of (37) vary, (ii) meanwhile, the complete right-hand term of (37) remains constant: that term does not depend on y_2 to y_N, as shown not only by the notation $f_{Y_1}(y_1)$ used for that term, but also by its explicit expression (26). We can then derive consequences of this phenomenon by also taking into account the following property of operator O_2. We showed above that operator O_2 is equal to f_C, i.e. it is a pdf. We here consider the case when this pdf f_C is continuous, as explained in Appendix A. Therefore, there exists at least one interval of values of v where $O_2(v;\theta,\phi)$ is non-zero and varies with respect to v. We then consider values of y_1 to y_N such that, when y_2 to y_N are varied (in intervals such that $f_{Y_i}(y_i) \neq 0$), the resulting value $v = y_1 - I'(y_2,\ldots,y_N;\theta,\phi)$ is situated inside an interval where $O_2(v;\theta,\phi)$ is non-zero and varies with respect to v. As explained above, the right-hand term of (37) thus remains constant, and therefore its left-hand term $O_2(v;\theta,\phi)$ also remains constant. This implies that, although y_2 to y_N are varied, v remains constant, and therefore $I'(y_2,\ldots,y_N;\theta,\phi)$ also remains constant. Eq. (8) then shows that the output signal y_1 is equal to the target term $T(s_1;\theta,\phi)$ that we aim at extracting, up to a constant term (see comments in Appendix B), which is equal to $I'(s'_2,\ldots,s'_N;\theta,\phi)$. This proves that, under the above-defined mild conditions, the global model analyzed in this section is ICA-separable, i.e. output independence implies that the output signals are equal to the transformed source signals s'_i, up to an additive constant and an invertible function for the first output.

2.2. ICA-based separability of SATS global model with split interference term (SATS-SI)

We now consider a type of global models which is a sub-class of the SATS global model, that we defined by (8)-(9) and that we studied above. This sub-class corresponds to the situation when the interference term $I'(s'_2,\ldots,s'_N;\theta,\phi)$ in (8) is split as the product of two factors, where the first factor only depend on the vectors θ and ϕ of mixing and separating *parameters*, whereas the second factor only depends on the *source* signals, i.e.

$$I'(s'_2,\ldots,s'_N;\theta,\phi) = I'_p(\theta,\phi)I'_s(s'_2,\ldots,s'_N). \tag{38}$$

This sub-class of the SATS global model, with a Split Interference (SI) term, is therefore denoted as the SATS-SI global model hereafter.

Since we proved above the ICA separability of the overall SATS class, this obviously allows us to directly conclude that its SATS-SI sub-class is also ICA-separable. The reason why we introduce this sub-class however is that, when investigating the *second-order* separability of several classes of models further in this paper, we will show that focusing on the SATS-SI class yields additional attractive properties. For the same reason, we introduce a last sub-class of models hereafter, and we comment on its ICA-separability properties.

2.3. ICA-based separability of SATS-SI global model with fixed target source term (FSATS-SI)

We eventually consider a type of global models which is a sub-class of the SATS-SI global model, that we defined above by (8), (9) and (38). This sub-class corresponds to the situation when, in addition to the above properties, the term $T(s_1;\theta,\phi)$ of y_1, which appears is (8) and which corresponds to the target source s_1, is a "fixed" function of that source, i.e. it depends on the (unknown but) fixed vector θ of mixing parameters, but not on the tunable vector ϕ of separating parameters. This term $T(s_1;\theta,\phi)$ then reduces to

$$T(s_1;\theta,\phi) = T(s_1;\theta). \tag{39}$$

This sub-class of the SATS-SI global model, with a *Fixed* single-additive-target-source term, is therefore denoted as the FSATS-SI global model hereafter. This FSATS-SI sub-class is also ICA-separable, for the same reason as in the case of the SATS-SI model considered above.

3. Analysis of second-order separability

3.1. Motivation

The main historical approach for solving BSS problems is ICA. In the previous section, we derived new results concerning ICA-based separability, by considering a particular class of models, i.e. SATS global models, and by showing that they are ICA-separable. This completed our separability analysis from the ICA point of view. However, BSS may be achieved by other criteria than ICA in some configurations. In particular, ANC may be considered as a special type of linear BSS problem, and it is well-known that this specific problem may be solved by only resorting to second-order statistics, instead of the more general ICA criterion and its approximations, which involve higher-order statistics in various ways. One may therefore wonder whether the SATS global model, or its sub-classes, that we are considering in this paper are also separable by only using second-order statistics, or whether the ICA-based approach that we derived above is the only solution for these models, when the source signals are only assumed to be stochastic and statistically independent[2]. This question is especially relevant for our SATS global model, since it is essentially a nonlinear extension of ANC, as stated above in Section 1 and illustrated below in Section 5 by means of an example. So, the question is: does the NANC problem introduced in this paper result in second-order separability, in addition to its ICA-based separability that we showed above ? The answer to this question is not only of theoretical interest: showing the second-order separability of the considered model means that one can then develop associated second-order BSS algorithms, which are simpler than ICA methods (but possibly less accurate or robust, especially for highly nonlinear models).

The analysis of second-order separability that we present below consists of two main steps. We first describe the most natural approach to this problem, i.e. we consider: (i) the general SATS global model, as defined above, and (ii) the second-order cross-statistics of its outputs. We show by means of an example that this basic approach does not always guarantee separability. We then proceed further by considering sub-classes of the above model and, possibly, statistics associated with reformulated versions of these models. We thus show that separability is achieved by only resorting to second-order statistics for several sub-cases. Let us insist again that, in all this investigation, we only consider the signals at a single time and we set no restrictions on their temporal structure (as in Section 2). This should be constrasted with second-order BSS methods for general linear instantaneous mixtures, such as SOBI, which set constraints on source temporal structure.

3.2. Covariance-based non-separability of original global models: an example

For the sake of clarity, let us consider a specific mixing and separating configuration which yields a global model belonging to the SATS class, and also to its SATS-SI and FSATS-SI sub-classes. This configuration involves two observations defined as follows, again at a given time t:

$$x_1 = s_1 + a(s_2)^\beta \tag{40}$$
$$x_2 = s_2 \tag{41}$$

where s_1 and s_2 are the source values at the considered time, a is an unknown real-valued constant mixing coefficient, and the exponent β is a known non-zero real-valued constant. Therefore, the vector θ of unknown mixing parameters is here restricted to the scalar parameter a.

The most natural separating system for restoring s_1 and s_2 (s_2 is already available here) from the above observations has two outputs defined as

$$y_1 = x_1 - b(x_2)^\beta \tag{42}$$
$$y_2 = x_2 \tag{43}$$

where b is the only real-valued tunable coefficient of the separating system, which corresponds to ϕ defined in Section 2.

Combining the above mixing and separating equations, the resulting global model reads

$$y_1 = s_1 + (a-b)s_2^\beta \tag{44}$$
$$y_2 = s_2. \tag{45}$$

This model therefore indeed belongs to the SATS class defined in (8)-(9). It also belongs to its SATS-SI sub-class, since it meets condition (38). Besides, it belongs to its FSATS-SI sub-class, because it also meets condition (39), moreover with no actual dependency of $T(s_1;\theta)$ with respect to θ here. Note that this model is also a simple nonlinear extension of the basic version of the ANC model: for $\beta = 1$, the configuration considered here reduces to (linear) instantaneous ANC with two sources.

For the considered global model, BSS is achieved when

$$b = a. \tag{46}$$

[2]We do not consider the other main classes of BSS methods in this paper, especially sparse component analysis and non-negative matrix factorization, which require other source properties.

We here aim at determining if forcing the separating system outputs y_1 and y_2 to have zero covariance guarantees that b becomes equal to the value defined by (46), when the source signals s_1 and s_2 are assumed to be statistically independent. As an example, let us consider the case when s_2 is a zero-mean signal. The covariance of the separating system outputs is then easily shown to be

$$C_{y_1y_2} = (a-b)E\{s_2^{\beta+1}\} \tag{47}$$

where we use the standard definition of the covariance of two random variables X and Y, i.e.

$$C_{XY} = E\{(X - E\{X\})(Y - E\{Y\})\} = E\{XY\} - E\{X\}E\{Y\}. \tag{48}$$

This yields two possible cases:

1. the first case is when the source signal s_2 is such that $E\{s_2^{\beta+1}\} \neq 0$. Then, the covariance of the separating system outputs is equal to zero only if $b = a$. Therefore, in this case, the considered global model is separable by means of separating system output decorrelation.
2. The second case is when the source signal s_2 is such that

   $$E\{s_2^{\beta+1}\} = 0. \tag{49}$$

   Then, the covariance of the separating system outputs remains equal to zero whatever the value of b. Therefore, the sources cannot be separated by using this covariance parameter in that case. Note that condition (49) may actually be met in practice: this e.g. occurs when β is an even integer and s_2 has an even pdf.

The above example shows that directly using the covariance of the outputs of the original global model does not always guarantee separability, even for the most specific sub-class of global models that we considered above, i.e. the FSATS-SI model. A first solution to this problem may however be developed by modifying the way we handle the initial global model, i.e. by post-processing it (in practice, this means that this post-processing is applied to the outputs of the separating system involved in the original global model). This solution may appear in a rather natural way in the above simple example, but is much more general, as will now be shown.

3.3. Covariance-based separability of post-processed SATS-SI global model

We here investigate the situation when a global model has been derived from the considered mixing model and from the separating system which was originally designed for it. We consider the case when this global model belongs to the SATS-SI class, i.e. when it may be expressed according to (8)-(9) and it meets condition (38). We showed above that cancelling the covariance of the outputs of this model does not always guarantee separability, but we now claim that this separability is guaranteed when one cancels the covariance of an adequate *post-processed version* of the outputs of the above model. More precisely, we start from the original outputs (9), equal to $s'_2, \ldots, s'_N$, and we combine them according to the *known* operator I'_s involved in (38). Note that this procedure can be applied to the SATS-SI model because the interference term $I'(s'_2, \ldots, s'_N; \theta, \phi)$ meets (38), so that its factor $I'_s(s'_2, \ldots, s'_N)$ can be computed without knowing the value of the vector θ of mixing parameters. On the contrary, for a global model which only belongs to the overall SATS class, the value of $I'(s'_2, \ldots, s'_N; \theta, \phi)$ cannot be computed before estimating θ, so that the approach described in this section cannot be applied. This approach is therefore only proposed for the SATS-SI sub-class.

The above combination of the signals $s'_2, \ldots, s'_N$ yields one output, $I'_s(s'_2, \ldots, s'_N)$, of our modified separating and associated global models which include post-processing. The other output of these models is equal to y_1, defined by (8), to which we apply no post-processing. We denote as z_1 and z_2 the two outputs of our modified global model, which is obtained by combining (8)-(9) and (38) with the above-defined post-processing stage. This modified global model reads as follows (again at a given time t):

$$z_1 = T(s_1; \theta, \phi) + I'_p(\theta, \phi) I'_s(s'_2, \ldots, s'_N) \tag{50}$$

$$z_2 = I'_s(s'_2, \ldots, s'_N). \tag{51}$$

Using the modified model (50)-(51) that we derived above, the approach that we propose then uses the covariance of the outputs signals z_1 and z_2 as the separation criterion. When considering independent source signals, it is easily shown that this covariance reduces to

$$C_{z_1z_2} = I'_p(\theta, \phi)\sigma^2_{I'_s(s'_2,\ldots,s'_N)} \tag{52}$$

where $\sigma_X^2 = C_{XX}$ denotes the variance of a random variable X.

Let us first make it clear that the specific case when the variance of $I'_s(s'_2, \ldots, s'_N)$ is zero is not of interest, but yields no problem: in that case, $I'_s(s'_2, \ldots, s'_N)$ is essentially a constant, therefore (38) shows that $I'(s'_2, \ldots, s'_N; \theta, \phi)$ is a constant, and (8) proves that y_1 always (i.e. whatever the value of the vector ϕ of separating parameters) provides the target term $T(s_1; \theta, \phi)$, up to an additive constant.

Now consider the case of interest, i.e. when $\sigma^2_{I'_s(s'_2,\ldots,s'_N)} \neq 0$. Eq. (52) then shows that adapting the vector ϕ of separating parameters so as to cancel $C_{z_1 z_2}$ forces $I'_p(\theta, \phi)$ to become equal to zero. Eq. (38) and (8) then prove that y_1 becomes equal to the target term $T(s_1; \theta, \phi)$. This means that the *post-processed* SATS-SI global model is separable by means of output decorrelation (whatever the marginal source statistics).

As a consequence, the above result especially applies when considering the FSATS-SI global model and using the above post-processing. In addition, an alternative second-order approach may also be developed specifically for that FSATS-SI global model, as will now be shown.

3.4. Variance-based separability of original FSATS-SI global model

For two-source (linear) instantaneous ANC, it is well-known that separation may be achieved by two alternative second-order approaches:

- the first approach consists in cancelling the covariance of *both* output signals. This approach has a relationship with linear symmetric ICA, e.g. based on mutual information cancellation, whose separation criterion also involves *all* outputs.
- The second approach consists in minimizing the variance of *one* output signal, i.e. of the signal which is unmixed only when the separating parameters are tuned so as to cancel the interference term in this output. This approach is more similar to linear deflation-based statistical BSS, e.g. based on the maximization of the negentropy (or of the absolute value of the normalized kurtosis) of a *single* output.

In Section 3.3, we extended the multi-output, i.e. covariance-based, ANC approach to a class of nonlinear mixtures. We will now show that the single-output, i.e. variance-based, approach can be extended to the FSATS-SI global model (without post-processing). To this end, we consider the output signal of interest of this model, defined by (8), (38) and (39), which yield

$$y_1 = T(s_1; \theta) + I'_p(\theta, \phi) I'_s(s'_2, \ldots, s'_N). \tag{53}$$

When considering independent source signals, it is easily shown that the variance of this signal reduces to

$$\sigma^2_{y_1} = \sigma^2_{T(s_1;\theta)} + [I'_p(\theta, \phi)]^2 \sigma^2_{I'_s(s'_2,\ldots,s'_N)}. \tag{54}$$

Therefore, this variance reaches its minimum value when the vector ϕ of separating parameters is adapted to as to achieve

$$I'_p(\theta, \phi) = 0. \tag{55}$$

Eq. (53) then shows that the output signal y_1 becomes equal to the target term $T(s_1; \theta)$. This means that the FSATS-SI global model is separable by means of output variance minimization (whatever the marginal source statistics).

We thus obtained a variance-based separability criterion only for the simplest of the global models considered in this paper. Note that the above derivation does not extend to the other two classes, as will now be shown. For the SATS-SI model, (39) does not hold any more and (53) is replaced by

$$y_1 = T(s_1; \theta, \phi) + I'_p(\theta, \phi) I'_s(s'_2, \ldots, s'_N) \tag{56}$$

which leads to

$$\sigma^2_{y_1} = \sigma^2_{T(s_1;\theta,\phi)} + [I'_p(\theta, \phi)]^2 \sigma^2_{I'_s(s'_2,\ldots,s'_N)}. \tag{57}$$

Now, when adapting ϕ, not only $I'_p(\theta, \phi)$ is varied, but also $\sigma^2_{T(s_1;\theta,\phi)}$. Therefore, the minimum of $\sigma^2_{y_1}$ in general does not correspond to condition (55), i.e. to the cancellation of the interference term. The overall SATS class is analyzed in the same way, starting from (8) and deriving the associated expression of $\sigma^2_{y_1}$. This shows that this model, too, cannot be separated by minimizing $\sigma^2_{y_1}$.

4. Resulting separation methods

As stated above, this paper is primarily focused on the *separability* of various nonlinear models, not on practical separation: we mainly aim at unveiling statistical properties of output signals which ensure separation, not at detailing associated BSS algorithms. However, we here want to make it clear that, once these statistical properties have been derived, straightforward ways to develop corresponding BSS methods result from them. These approaches may be defined as follows, for the two types of separation properties that we respectively showed in Sections 2 and 3.

We first proved in Section 2 that adapting ϕ so as to make the outputs of the SATS model independent guarantees that separation is achieved (under mild conditions). To derive a corresponding BSS criterion, we must then define a parameter which measures the degree of mutual dependence of these output signals. A very natural candidate to this end is their mutual information, which is zero if the signals are independent and positive otherwise. The resulting BSS method for the SATS model then consists in adapting ϕ so as to minimize the mutual information of the outputs of the separating system. This shows that moving from the property derived above (i.e. independence of output signals) to a BSS criterion (minimization of output information) is straightforward for our ICA-based solution to the problem.

Things are even simpler for the two second-order approaches that we developed in Section 3. We defined the first proposed solution not only in terms of a statistical property (i.e. uncorrelateness) of the output signals, but also with respect to a corresponding parameter (i.e. output covariance). Moreover, we presented the corresponding practical second-order criterion that may first be used to achieve BSS: it just consists in cancelling the output covariance parameter. Our other second-order approach is directly defined in terms of a constraint imposed to a parameter: it consists in minimizing the variance of the first output.

The next step of the development of such practical BSS methods then consists in focusing on specific models belonging to the considered classes, deriving the corresponding expressions of the output information or (co)variance parameters, and addressing the optimization of these parameters. Depending on the complexity of the expressions of these parameters, closed-form solutions for their optimization may be obtained, or numerical algorithms must be considered. While the description of separation *algorithms* (or closed-form solutions) for specific models is beyond the scope of this paper, it is here worth it considering a *particular model* however, in order to more explicitly illustrate the above general *separability* results. This will show that there do exist practical applications where the global model belongs to several above-defined classes. The separability properties obtained in the previous sections are then of high importance, because they prove the relevance of the associated practical separation methods that we defined in the current section. We therefore now proceed to the description of such an application.

5. Application to coupled quantum bits

5.1. Mixing model

We now consider an application which concerns Quantum Information Processing (QIP) [7]. QIP is an emerging field, which widely uses quantum bits (qubits) instead of classical bits for performing computations [7]. A qubit, with index i, has a quantum state expressed as follows (for a pure state):

$$|\psi_i >= \alpha_i|+> +\beta_i|-> \tag{58}$$

where $|+>$ and $|->$ are basis vectors, whereas α_i and β_i are two complex-valued coefficients such that

$$|\alpha_i|^2 + |\beta_i|^2 = 1. \tag{59}$$

In [2], [3] we introduced the Blind Quantum Source Separation (BQSS) field, by considering the situation when two qubits of a physical system are separately initialized with states defined by (58) and then get “mixed” (in the BSS sense) due to the undesired coupling effect which exists in the considered system (Heisenberg coupling in our case). We proposed an approach which consists in repeatedly initializing the two qubits according to (58) and later measuring spin components associated with the system composed of these two coupled qubits. We showed that this yields four possible measured values, with respective probabilities p_1, p_2, p_3 and p_4. We derived the expressions of these probabilities with respect to the polar representation of the qubit parameters α_i and β_i, which reads

$$\alpha_i = r_i e^{i\theta_i}, \quad \beta_i = q_i e^{i\phi_i} \qquad \forall i \in \{1, 2\} \tag{60}$$

with $0 \leq r_i \leq 1$ and $q_i = \sqrt{1-r_i^2}$, due to (59). The above probabilities may then be expressed as follows (with different numbering as compared to [3]):

$$\begin{aligned} p_1 &= r_1^2 r_2^2 &(61)\\ p_2 &= r_1^2(1-r_2^2)(1-v^2) + (1-r_1^2)r_2^2 v^2 \\ &\quad -2r_1 r_2 \sqrt{1-r_1^2}\sqrt{1-r_2^2}\sqrt{1-v^2}\, v \sin\Delta_I &(62)\\ p_4 &= (1-r_1^2)(1-r_2^2) &(63)\end{aligned}$$

where

$$\Delta_I = (\phi_2 - \phi_1) - (\theta_2 - \theta_1) \tag{64}$$

and v is a parameter, defined in [3], which is such that $0 \leq v^2 \leq 1$, and whose value is unknown in most configurations. Note that probability p_3 is not considered in this investigation, since it is redundant with the above three ones: we always have

$$p_1 + p_2 + p_3 + p_4 = 1. \tag{65}$$

Eq. (61)-(63) form the nonlinear "mixing model" (in BSS terms) of this investigation. The observations involved in this model are the probabilities p_1, p_2 and p_4 measured (in fact, estimated, using repeated qubit initializations) for each value of the couple of qubits. Using standard BSS notations, the observation vector is therefore $x = [x_1, x_2, x_3]^T$ with $x_1 = p_1, x_2 = p_2$ and $x_3 = p_4$. The source vector to be retrieved from these observations is $s = [s_1, s_2, s_3]^T$ with $s_1 = r_1, s_2 = r_2$ and $s_3 = \Delta_I$. The parameters q_i are then obtained as $q_i = \sqrt{1-r_i^2}$. The four phase parameters in (60) cannot be individually extracted from their combination Δ_I. Moreover, as explained in [3], Δ_I in fact only depends on one of the angular parameters $\theta_1, \theta_2, \phi_1, \phi_2$, which is used to store information, since all other three angular parameters are fixed. We anticipate that this data stream Δ_I and the other two, i.e. $s_1 = r_1$ and $s_2 = r_2$, will be handled separately when applying our approach to store information in qubit parameters of future quantum computers. These three data streams, i.e. the source signals of our BSS problem, may therefore be quite reasonably assumed to be (or created so as to be) statistically independent in the considered application. In blind configurations, retrieving the sources first requires one to estimate the only unknown mixing parameter of this model, i.e. v.

5.2. *Separating system*

In [3], we showed that the above mixing model is invertible (with respect to the considered domain of source values), for any fixed v such that $0 < v^2 < 1$, provided the source values meet the following conditions:

$$0 < r_1 < \tfrac{1}{2} < r_2 < 1 \tag{66}$$

$$-\tfrac{\pi}{2} \leq \Delta_I \leq \tfrac{\pi}{2}. \tag{67}$$

The separating structure that we proposed for retrieving the sources then yields an output vector $y = [y_1, y_2, y_3]^T$ which reads

$$y_1 = \sqrt{\frac{1}{2}\left[(1+p_1-p_4) - \sqrt{(1+p_1-p_4)^2 - 4p_1}\right]} \tag{68}$$

$$y_2 = \sqrt{\frac{1}{2}\left[(1+p_1-p_4) + \sqrt{(1+p_1-p_4)^2 - 4p_1}\right]} \tag{69}$$

$$y_3 = \arcsin\left[\frac{y_1^2(1-y_2^2)(1-\hat{v}^2) + (1-y_1^2)y_2^2\hat{v}^2 - p_2}{2y_1 y_2\sqrt{1-y_1^2}\sqrt{1-y_2^2}\sqrt{1-\hat{v}^2}\,\hat{v}}\right] \tag{70}$$

where $\hat{v}$ is the estimate of v used in the separating system. The outputs y_1, y_2 and y_3 respectively restore the sources $s_1 = r_1, s_2 = r_2$ and $s_3 = \Delta_I$.

5.3. Original global model

Starting from the above results obtained in [3], we here aim at analyzing the ICA-based and second-order separability of the resulting global model, which was not addressed in [2], [3], since other types of BQSS methods were considered in these papers. We first derive this global model, by combining the mixing model (61)-(63) and the separating model (68)-(70).

More precisely, one should preferably first consider the sub-model obtained by only combining (61) and (63) with (68)-(69). Under condition (66), the resulting global sub-model reads

$$y_1 = s_1 \tag{71}$$

$$y_2 = s_2 \tag{72}$$

where this model may either be obtained by direct calculation, or justified as follows. The first stage of this global sub-model, i.e. (61) and (63), transforms the sources $s_1 = r_1, s_2 = r_2$ into the observations $x_1 = p_1, x_3 = p_4$, without any dependence on the mixing parameter v. It is known from [3] that, under condition (66), this sub-model is invertible. The second stage of the considered global sub-model, i.e. (68)-(69), implements the exact inverse of the mixing sub-model composed of (61) and (63), without any dependence on the estimate $\hat{v}$ of v. The outputs y_1 and y_2 are therefore exactly equal to the original sources s_1 and s_2 (again ignoring the deviations which are due to the fact that p_1 and p_4 are estimated in practice).

The expression of the last output of the global system, i.e. y_3, may eventually be obtained by inserting (71), (72) and (62) into (70). This yields

$$y_3 = \arcsin\left[\frac{(\hat{v}^2 - v^2)(s_2^2 - s_1^2)}{2s_1 s_2 \sqrt{1-s_1^2}\sqrt{1-s_2^2}\sqrt{1-\hat{v}^2}\hat{v}} + \frac{\sqrt{1-v^2}v}{\sqrt{1-\hat{v}^2}\hat{v}} \sin s_3\right]. \tag{73}$$

Let us note that if $\hat{v} = v$, Eq. (73) reduces to

$$y_3 = \arcsin\left[\sin s_3\right] \tag{74}$$

and therefore, using condition (67)

$$y_3 = s_3, \tag{75}$$

without any ambiguity, which again shows the invertibility of the mixing model (61)-(63) in the considered conditions. Now, for any values of the mixing and separating parameters v and $\hat{v}$, the considered global model consists of (71), (72) and (73). Note that the property which consists for all signals except one in always being unmixed here applies to the outputs of the global model, but not to those of the mixing model (61)-(63) involved in it.

5.4. ICA-based separability

We now aim at investigating whether the results that we obtained in Section 2 may be applied to the quantum problem that we are considering in this section. Comparing the SATS class of global models defined by (8)-(9) to the specific quantum model defined by (71), (72) and (73) shows that the SATS class does not include this quantum model in its original form, but contains a slightly reformulated version of it, which is obtained as follows.

We first split the separating system defined in Section 5.2 as the cascade of two sub-systems. The first, and main, sub-system derives (i) the signals y_1 and y_2 respectively defined in (68) and (69), and (ii) the argument of the arcsin() function in the right-hand term of (70). The second sub-system is a fixed "post-distortion" stage which only consists in deriving y_3 by computing the arcsin() of the third output of the first sub-system, while leaving its first two outputs, i.e. y_1 and y_2, unchanged. The mutual independence of the outputs of the overall separating system is equivalent to that of the outputs of its first stage, and the same equivalence therefore holds for the ICA separability of the global models involving these two (i.e. partial or complete) separating systems. Hence, we only investigate the ICA separability of the global model involving the first stage of the separating system hereafter. Moreover, to map its notations with those of Section 2, we hereafter modify the meaning of notation y_3: we denote as y_3 the third output of this first stage, i.e. *before* applying the arcsin() post-distortion which yields the eventual expression in the right-hand side of (70).

Besides, we change the indices of the signals involved in this application, again to map its notations with those of Section 2. We thus reorder the source and output signals as follows:

$$\begin{array}{rcl} \text{notations in (71), (72), (73)} & \rightarrow & \text{new notations} \\ s_1 & \rightarrow & s_2 \\ s_2 & \rightarrow & s_3 \\ s_3 & \rightarrow & s_1 \\ y_1 & \rightarrow & y_2 \\ y_2 & \rightarrow & y_3 \\ y_3 & \rightarrow & y_1. \end{array}$$

The considered global model (without post-distortion) thus becomes

$$y_1 = \frac{\sqrt{1-v^2}v}{\sqrt{1-\hat{v}^2}\hat{v}} \sin s_1 + \frac{(\hat{v}^2-v^2)(s_3^2-s_2^2)}{2s_2s_3\sqrt{1-s_2^2}\sqrt{1-s_3^2}\sqrt{1-\hat{v}^2}\hat{v}} \tag{76}$$

$$y_2 = s_2 \tag{77}$$

$$y_3 = s_3. \tag{78}$$

This reformulated model belongs to the SATS class defined by (8)-(9), with

$$N = 3 \tag{79}$$

$$s'_i = s_i \qquad \forall\ i = 1, \ldots, N \tag{80}$$

$$\theta = v \tag{81}$$

$$\phi = \hat{v} \tag{82}$$

$$T(s_1;\theta,\phi) = \frac{\sqrt{1-v^2}v}{\sqrt{1-\hat{v}^2}\hat{v}} \sin s_1 \tag{83}$$

$$I'(s'_2,\ldots,s'_N;\theta,\phi) = \frac{(\hat{v}^2-v^2)(s_3^2-s_2^2)}{2s_2s_3\sqrt{1-s_2^2}\sqrt{1-s_3^2}\sqrt{1-\hat{v}^2}\hat{v}}. \tag{84}$$

Moreover, the properties requested in Section 2.1 are met here. Therefore, the general results that we derived in Section 2 directly show that this global model is ICA-separable (in the conditions defined in Section 2).

5.5. Covariance-based separability of post-processed model

The considered quantum global model (again without post-distortion) also belongs to the SATS-SI sub-class, since its term $I'(s'_2,\ldots,s'_N;\theta,\phi)$ defined in (84) meets (38), with

$$I'_p(\theta,\phi) = \frac{\hat{v}^2-v^2}{\sqrt{1-\hat{v}^2}\hat{v}} \tag{85}$$

$$I'_s(s'_2,\ldots,s'_N) = \frac{s_3^2-s_2^2}{2s_2s_3\sqrt{1-s_2^2}\sqrt{1-s_3^2}}. \tag{86}$$

Therefore, the general results that we derived in Section 3.3 guarantee that this quantum model is separable by means of the covariance-based method that we proposed in Section 3.3.

5.6. Unapplicability of variance-based method

The considered quantum global model does not belong to the FSAT-SI sub-class, because its term $T(s_1;\theta,\phi)$ defined in (83) indeed depends on $\phi = \hat{v}$. Therefore, the general results derived in Section 3.4 cannot be applied here to guarantee the separability of our quantum model by means of the variance-based method that we proposed in Section 3.4. On the contrary, that method is of interest for other applications, not detailed here due to space limitations.

6. Conclusions

In this paper, we investigated the separability of broad classes of nonlinear global models involving reference signals. We thus derived general properties concerning ICA-based and second-order (covariance-based or variance-based) separability. In addition, these investigations allowed us to outline practical BSS methods. These methods are e.g. applicable to the specific model involved in the quantum application that we presented in Section 5. The implementation of these BSS methods and the evaluation of their numerical performance is a topic beyond the scope of this paper, which will be reported in future articles. We will thus further develop the field that may be briefly called "nonlinear adaptive noise cancellation" (NANC), as a reference to (linear) ANC.

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Appendix A. Motivations for considering a continuous pdf

As stated in Section 2.1.8, in this paper we consider the case when the pdf f_C is continuous. We focus on this case for two reasons. On the one hand, this pdf continuity is not constraining because is it likely to be met in most cases of interest, due to (24) and to the fact that, in many practical cases, the source random variable S_1 has a continuous pdf and the transform T which yields the random variable $C = T(S_1; \theta, \phi)$ has a continuous derivative. On the other hand, this pdf continuity condition simplifies our approach, because it is sufficient to guarantee that the following property, used in Section 2.1.8, is met: $O_2(v; \theta, \phi)$ varies at least over one interval of values of v for which $O_2(v; \theta, \phi) \neq 0$.

The latter property may however also be obtained by considering broader classes of pdf f_C, i.e. by just requesting f_C to vary at least over one interval where it is non-zero. This includes many types of discontinuous pdf, and it only excludes from our subsequent investigation the case of uniform (over one or several intervals) pdf. Uniform pdf might be further analyzed. However, this can be avoided by considering that a uniform pdf (over a single interval) is the limit of a continuous pdf with a non-zero almost constant part, and continuous transitions towards zero or almost zero parts. The latter continuous pdf may e.g. be defined as a combination of sigmoidal functions with tunable smoothness parameters (see e.g. [5]). One may therefore avoid potential issues due to discontinuities in ideal uniform pdf, by considering instead their above continuous approximation with parameters tuned so as to achieve sharp transitions between zero and non-zero pdf values.

The continuity constraint for f_C may also be linked to the other pdf continuity assumptions that we made above.

Appendix B. Comments about constant output term

In Section 2.1.8, we stated that the output signal y_1 is equal to the target term $T(s_1; \theta, \phi)$ that we aim at extracting, up to a constant term, which is equal to $I'(s'_2, \ldots, s'_N; \theta, \phi)$. More precisely, if any of the random variables Y_2 to Y_N has a non-zero pdf on *several* disjoint intervals, the analysis of Section 2.1.8 proves that the term $I'(s'_2, \ldots, s'_N; \theta, \phi)$ is constant over each of these intervals, but it does not state whether these constant values may be different one from the other. However, one should also take into account that this term $I'(s'_2, \ldots, s'_N; \theta, \phi)$ is obtained by transferring the source values through mixing and separating models which do not depend on the signal values. The models encountered in practice are expected to be such that $I'(s'_2, \ldots, s'_N; \theta, \phi)$ takes the *same* constant value on all above-mentioned intervals. One may check that this is the case for the application considered in Section 5, as shown by (84): for this model, the fact that $I'(s'_2, \ldots, s'_N; \theta, \phi)$ is constant, and therefore zero, for some source value intervals implies that θ and ϕ are such that $I'(s'_2, \ldots, s'_N; \theta, \phi)$ is zero everywhere. Similarly, by first considering a single value of y_1, one derives properties of θ and ϕ which then imply that $I'(s'_2, \ldots, s'_N; \theta, \phi)$ remains constant for any value of y_1.

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Erratum: replace two terms $E\{r_i\}E\{q_i\}$ in (33) of [2] by $E\{r_i q_i\}$, since q_i depends on r_i: see (23) in [3].

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