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# A unified stability analysis of the Hérault-Jutten source separation neural network

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**Abstract:** This paper presents an analysis of the stability of the equilibrium points of the Hérault-Jutten neural network. We show that a previously reported numerical analysis method only yields a sufficiency stability condition. By extending this method, we provide an analytical necessary and sufficiency condition, and we bridge the gap with another reported method.

**Résumé:** Ce papier présente une analyse de la stabilité des points d'équilibre du réseau de neurones de Hérault-Jutten. Nous montrons qu'une méthode d'analyse numérique précédemment publiée ne fournit qu'une condition suffisante de stabilité. En étendant cette méthode, nous fournissons une condition analytique nécessaire et suffisante, et nous faisons le lien avec une autre méthode connue.

# 1 Problem statement

Blind source separation is a generic signal processing problem which concerns e.g. antenna or microphone array processing [3]. In its simplest version, two sensors provide measured signals $E_1(t)$ and $E_2(t)$, which are unknown linear instantaneous mixtures of two unknown source signals $X_1(t)$ and $X_2(t)$, i.e:

$$E_1(t) = a_{11}X_1(t) + a_{12}X_2(t) \quad \text{and } E_2(t) = a_{21}X_1(t) + a_{22}X_2(t). \tag{1}$$

The problem is then to estimate the source signals $X_j(t)$ from the measured signals $E_i(t)$. In this paper, we consider the method proposed by Hérault and Jutten [3], which is based on the recursive neural network shown in Fig. 1. The adaptive weights $c_{12}$ and $c_{21}$ of this network are updated according to the following unsupervised adaptation rule:

$$dc_{ij}/dt = af[s_i(t)]g[s_j(t)], \tag{2}$$

where $a$ is a positive adaptation gain, $s_i(t)$ and $s_j(t)$ are the (estimated) centered signals corresponding to the network outputs $S_i(t)$ and $S_j(t)$, and $f$ and $g$ are odd functions [3]. In this paper, we use $f = (.)^3$ and $g = (.)$, since it may be shown that this avoids restrictions on the types of sources that this neural network can separate (this is partly explained in [1],[4]).

The convergence properties of this algorithm have been studied independently by three authors. While E. Sorouchyari [4] and J.C. Fort [2] used almost the same method (called the SF-method below), P. Comon et al. [1] presented another method (called the C-method below) which yields different results. This paper therefore aims at bridging the gap between these methods.

# 2 Stability analysis

## 2.1 Available results

The above algorithm has four equilibrium points [4]. The approach used in this paper applies to any of them, but for the sake of brevity it is only presented for the point which yields perfect source separation with no permutation, and which is denoted point $P$ below. This point corresponds to $c_{12} = a_{12}/a_{22}$ and $c_{21} = a_{21}/a_{11}$, and yields:

$$S_1(t) = a_{11}X_1(t) \quad \text{and } S_2(t) = a_{22}X_2(t). \tag{3}$$

As stated in Section 1, two approaches have been used to study the stability of this point. The SF-method is based on an analytical analysis, and yields the following stability condition[1,2] on the centered sources $x_1$ and $x_2$:

$$E\{x_1^4\}E\{x_2^4\} < 9(E\{x_1^2\})^2(E\{x_2^2\})^2. \tag{4}$$

The C-method uses an orthogonal mixture matrix $A = [\alpha\ \beta; -\beta\ \alpha]$. Point $P$ thus meets the condition $c_{21} = -c_{12}$. Therefore, only points in the weight space which meet this

[1] In addition, the condition $c_{12}c_{21} < 1$ should be met. This condition is skipped in the remainder of this paper, since it only corresponds to requiring the loop gain of the considered recursive network to be lower than one in order to ensure the stability of this network [3].

[2] For readability, the time argument $t$ of the signals is omitted in the remainder of this paper.

condition are considered. Each such point is defined by $\theta$, with $c_{12} = \theta$ and $c_{21} = -\theta$. This reduces the investigation to a one-dimensional problem. The authors then define:

$$\Phi_{ij}(\theta) = E\{s_i^3 s_j\} \quad \text{for } i = 1, 2;\ j = 1, 2;\ \text{such that } i \neq j. \tag{5}$$

They state that a set of conditions for point $P$ to be a stable equilibrium point is:

$$d\Phi_{12}(\theta)/d\theta < 0 \tag{6}$$

and

$$d\Phi_{21}(\theta)/d\theta > 0. \tag{7}$$

They analyze stability for given signals, by computing sample estimates of $\Phi_{12}(\theta)$ and $\Phi_{21}(\theta)$ for various values of $\theta$, and by checking graphically if conditions (6) and (7) are met at point $P$ for these signals.

### 2.2 Discussion of available results; goal of proposed extensions

A clear difference appears between the SF-method, which is based on a theoretical analysis and which yields a single condition (4), and the C-method, which consists of an *a posteriori* numerical characterization of sample signals and which is based on two conditions (6) and (7). The remainder of this paper therefore aims at bridging the gap between these methods, by extending the C-method. For clarity, the analysis is again performed for an orthogonal mixture matrix $A = [\alpha\ \beta; -\beta\ \alpha]$.

### 2.3 First extension

It should first be pointed out that the set of conditions (6) and (7) is a necessary and sufficiency condition for the stable evolution towards point $P$ only along a trajectory such that $c_{21} = -c_{12}$. It is therefore only a sufficiency condition for the overall stability of point $P$ (because if these conditions are not met, a stable evolution towards $P$ may be possible through other trajectories anyway). The C-method therefore only provides partial insight into stability. We will remove this restriction in the second step of our approach.

Before this, as a first step of our approach, we still restrict ourselves to the trajectory such that $c_{21} = -c_{12}$ and we therefore only consider sufficiency conditions, but we move from the *a posteriori* numerical characterization of source signals used in the C-method, to a theoretical investigation of the type of sources for which $P$ is a stable point. To this end, we consider the exact functionals $\Phi_{ij}(\theta)$ defined in (5) instead of their sample estimates and we investigate the restrictions on the sources which correspond to the stability conditions (6) and (7). These conditions involve the first derivative of $\Phi_{ij}(\theta)$ with respect to $\theta$, which is derived from (5) and expressed as follows at any point in the weight space:

$$\frac{d\Phi_{ij}(\theta)}{d\theta} = E\left\{\frac{d(s_i^3 s_j)}{d\theta}\right\} = E\left\{3s_i^2 \frac{ds_i}{d\theta} s_j + s_i^3 \frac{ds_j}{d\theta}\right\}. \tag{8}$$

This expression depends on $s_1$ and $s_2$, which may be expressed as follows with respect to the centered mixed signals $e_1$ and $e_2$ when $c_{12} = \theta$ and $c_{21} = -\theta$:

$$s_1 = \frac{e_1 - \theta e_2}{1 + \theta^2}, \tag{9}$$

$$s_2 = \frac{\theta e_1 + e_2}{1 + \theta^2}. \tag{10}$$

By computing the first derivatives of these expressions with respect to $\theta$, one gets the following equations (the last expressions of (11) and (12) are obtained by deriving the expressions of $e_1$ and $e_2$ vs. $s_1$ and $s_2$ from (9) and (10) and by inserting them in the first expressions of (11) and (12)):

$$\frac{ds_1}{d\theta} = -\frac{2\theta e_1 + (1-\theta^2)e_2}{(1+\theta^2)^2} = -\frac{\theta s_1 + s_2}{1+\theta^2}, \tag{11}$$

$$\frac{ds_2}{d\theta} = \frac{(1-\theta^2)e_1 - 2\theta e_2}{(1+\theta^2)^2} = \frac{s_1 - \theta s_2}{1+\theta^2}. \tag{12}$$

Eventually, by inserting the last expressions of (11) and (12) in (8), one obtains:

$$\frac{d\Phi_{12}(\theta)}{d\theta} = \frac{1}{1+\theta^2} E\{s_1^4 - 3s_1^2 s_2^2\} - \frac{4\theta}{1+\theta^2}\Phi_{12}(\theta), \tag{13}$$

$$\frac{d\Phi_{21}(\theta)}{d\theta} = -\frac{1}{1+\theta^2} E\{s_2^4 - 3s_1^2 s_2^2\} - \frac{4\theta}{1+\theta^2}\Phi_{21}(\theta). \tag{14}$$

As stated above, these equations hold for any point in the weight space. Especially, at equilibrium point $P$, due to the mixture matrix $A$ considered here, $\theta = \beta/\alpha$ and (3) yields $s_1 = \alpha x_1$ and $s_2 = \alpha x_2$. Moreover, at this point, $\Phi_{12}(\theta) = 0$ and $\Phi_{21}(\theta) = 0$. In addition, the sources $x_1$ and $x_2$ are supposed to be independent. (13) and (14) then become:

$$\frac{d\Phi_{12}(\theta)}{d\theta} = \frac{\alpha^4}{1+\left(\frac{\beta}{\alpha}\right)^2}\left[E\{x_1^4\} - 3E\{x_1^2\}E\{x_2^2\}\right], \tag{15}$$

$$\frac{d\Phi_{21}(\theta)}{d\theta} = -\frac{\alpha^4}{1+\left(\frac{\beta}{\alpha}\right)^2}\left[E\{x_2^4\} - 3E\{x_1^2\}E\{x_2^2\}\right]. \tag{16}$$

By taking into account (15) and (16), the stability conditions (6) and (7) at point $P$ are eventually rewritten as follows:

$$E\{x_1^4\} - 3E\{x_1^2\}E\{x_2^2\} < 0 \tag{17}$$

and

$$E\{x_2^4\} - 3E\{x_1^2\}E\{x_2^2\} < 0. \tag{18}$$

In other words, the result obtained at this stage is that, if (17) and (18) are met, point $P$ is stable according to this first extension of the C-method. This result is coherent with those of the SF-method, because if (17) and (18) are met, (4) is met, so that the SF-method also concludes that $P$ is stable.

## 2.4 Second extension

The first step of our approach presented above provides an analytical condition on the source signals, but does not yet completely fill the gap with the SF-method, because it still only provides a sufficiency stability condition. To obtain a condition which is also necessary, all the trajectories in the weight space should be considered, instead of the single one used above. Unfortunately, this would require to go back from a one-dimensional analysis (vs. $\theta$) to a two-dimensional one (vs. $c_{12}$ and $c_{21}$). An intermediate solution consists in considering a class of trajectories containing point $P$, where each trajectory in the class is defined by the value of a parameter $\lambda$, and the position on each trajectory is defined by a single parameter $\theta$, thus still providing a one-dimensional analysis for each trajectory. The more general the class of trajectories, the more general the condition on source signals. One may even expect that for a general enough class, the stability condition of $P$ with respect to this complete class is a necessary and sufficiency condition for the stability of $P$ with respect to any trajectory. To illustrate this, let us consider the following class of trajectories: $c_{12} = \theta$, and $c_{21} = -\theta[1 + \lambda(\theta - \beta/\alpha)]$, designed so that all these trajectories go through point $P$, but along a direction which depends on $\lambda$. We first consider a single trajectory in this class, corresponding to a fixed value $\lambda$, and we use the same method as in Sub-section 2.3 to determine the restrictions on the sources which correspond to the stability conditions (6) and (7) for this new trajectory[3]. Here again, the investigation is based on the first derivative of $\Phi_{ij}(\theta)$ with respect to $\theta$ which is defined by (8). However, $s_1$ and $s_2$ are expressed as follows with respect to the centered sources $x_1$ and $x_2$ for the trajectory considered here:

$$s_1 = \alpha \frac{(1 + \theta\frac{\beta}{\alpha})x_1 + (\frac{\beta}{\alpha} - \theta)x_2}{1 + \theta^2[1 + \lambda(\theta - \frac{\beta}{\alpha})]}, \tag{19}$$

$$s_2 = \alpha \frac{\{-\frac{\beta}{\alpha} + \theta[1 + \lambda(\theta - \frac{\beta}{\alpha})]\}x_1 + \{1 + \frac{\beta}{\alpha}\theta[1 + \lambda(\theta - \frac{\beta}{\alpha})]\}x_2}{1 + \theta^2[1 + \lambda(\theta - \frac{\beta}{\alpha})]}. \tag{20}$$

One then computes the first derivatives of these expressions with respect to $\theta$. For the sake of brevity, only their values at point $P$ are provided hereafter:

$$\frac{ds_1}{d\theta} = -\alpha \frac{\frac{\beta}{\alpha}(1 + \lambda\frac{\beta}{\alpha})x_1 + x_2}{1 + \left(\frac{\beta}{\alpha}\right)^2}, \tag{21}$$

$$\frac{ds_2}{d\theta} = \alpha \frac{(1 + \lambda\frac{\beta}{\alpha})x_1 - \frac{\beta}{\alpha}x_2}{1 + \left(\frac{\beta}{\alpha}\right)^2}. \tag{22}$$

By inserting (21) and (22) in (8), one obtains at point $P$:

$$\frac{d\Phi_{12}(\theta)}{d\theta} = \frac{\alpha^4}{1 + \left(\frac{\beta}{\alpha}\right)^2}[(1 + \lambda\frac{\beta}{\alpha})E\{x_1^4\} - 3E\{x_1^2\}E\{x_2^2\}], \tag{23}$$

$$\frac{d\Phi_{21}(\theta)}{d\theta} = -\frac{\alpha^4}{1 + \left(\frac{\beta}{\alpha}\right)^2}[E\{x_2^4\} - 3(1 + \lambda\frac{\beta}{\alpha})E\{x_1^2\}E\{x_2^2\}]. \tag{24}$$

By taking into account (23) and (24), the stability conditions (6) and (7) at point $P$ are eventually rewritten as follows for the trajectory corresponding to the considered fixed value $\lambda$:

$$Q_1(\lambda) = (1 + \lambda\frac{\beta}{\alpha})E\{x_1^4\} - 3E\{x_1^2\}E\{x_2^2\} < 0 \tag{25}$$

and

$$Q_2(\lambda) = E\{x_2^4\} - 3(1 + \lambda\frac{\beta}{\alpha})E\{x_1^2\}E\{x_2^2\} < 0. \tag{26}$$

[3]The computations are presented in a slightly different way than in Sub-section 2.3 hereafter, in order to avoid complex equations.

Then, point $P$ is stable with respect to the whole class of trajectories if and only if (25) and (26) are met for at least one trajectory, i.e. one real value $\lambda$. Since $Q_1(\lambda)$ increases with $\lambda\beta/\alpha$, while $Q_2(\lambda)$ decreases, this necessary and sufficiency condition is equivalent to the following one:

$$Q_1(\lambda_0) < 0, \tag{27}$$

where $\lambda_0$ is the value of $\lambda$ such that:

$$Q_1(\lambda) = Q_2(\lambda). \tag{28}$$

By solving (28), one gets:

$$1 + \lambda_0 \frac{\beta}{\alpha} = \frac{E\{x_2^4\} + 3E\{x_1^2\}E\{x_2^2\}}{E\{x_1^4\} + 3E\{x_1^2\}E\{x_2^2\}}. \tag{29}$$

Inserting (29) in the expression of $Q_1(\lambda)$ provided in (25) yields:

$$Q_1(\lambda_0) = \frac{E\{x_1^4\}E\{x_2^4\} - 9(E\{x_1^2\})^2(E\{x_2^2\})^2}{E\{x_1^4\} + 3E\{x_1^2\}E\{x_2^2\}}. \tag{30}$$

(30) allows to rewrite (27), which is the overall necessary and sufficiency stability condition with respect to the complete class of trajectories. The final condition thus obtained is:

$$E\{x_1^4\}E\{x_2^4\} - 9(E\{x_1^2\})^2(E\{x_2^2\})^2 < 0. \tag{31}$$

The latter condition turns out to be identical to (4). This proves that the extended approach that we have defined at this stage is fully coherent with the SF-method, and that the class of trajectories considered here is general enough to obtain a necessary and sufficiency stability condition for point $P$ with respect to any trajectory.

## 3 Conclusions

In this paper, we have shown that the C-method only provides a sufficiency stability condition. We have extended this method: i) by using an analytical approach instead of a numerical one, and ii) by deriving a necessary and sufficiency stability condition. We have thus shown how the C-method can be transformed in order to become fully coherent with the SF-method. We have thus provided a unified view of the stability of the Hérault-Jutten network.

## References


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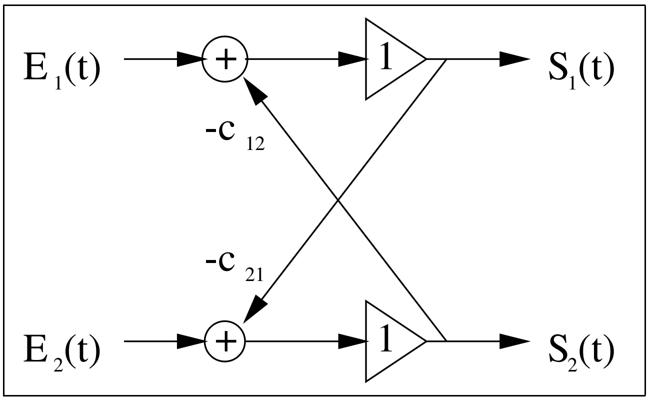


Figure 1: Basic Hérault-Jutten source separation neural network.