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N-qubit system in a pure state: a necessary and sufficient condition for unentanglement

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Abstract If a pure state of a qubit pair is developed over the four basis states, an equality between the four coefficients of that development, verified if and only if that state is unentangled, is already known. This paper considers an arbitrary pure state of an N-qubit system, developed over the 2^N basis states. It is shown that the state is unentangled if and only if a well-chosen collection of $[2^N - (N+1)]$ equalities between the 2^N coefficients of that development is verified. The number of these equalities is large a soon as $N \gtrsim 10$, but it is shown that this set of equalities may be classified into $(N-1)$ subsets, which should facilitate their manipulation. This result should be useful e.g. in the contexts of Blind Quantum Source Separation (BQSS) and Blind Quantum Process Tomography (BQPT), with an aim which should not be confused with that found when using the concept of equivalence of pure states through local unitary transformations.

Keywords Unentanglement condition · Entanglement · N-qubit system · Pure state

1 Introduction

If both parts S_1 and S_2 of a bipartite quantum system S are initially prepared in a pure state, then, at a time scale allowing one to neglect any coupling

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between S and the rest of the universe, S may be described as being in a time-dependent pure state $\mid \Psi(t) >$ which obeys the Schrödinger equation. But, at that time scale, if an internal coupling transiently exists between S_1 and S_2, then, after it disappeared, $\mid \Psi(t) >$ often cannot be equated with a tensorial product $\mid \Psi_1(t) > \otimes \mid \Psi_2(t) >$ describing S_1 and S_2 respectively: $\mid \Psi(t) >$ is entangled (or not separable) [22], [3]. Entanglement plays a significant role in Quantum Information (QI) [15], e.g. in the context of Quantum Computing. The idea of a Quantum Computer (QC) dates back to the 1980s [14], [5], and the word *qubit* to name the basic cell of the Quantum Computer appeared in 1995 (cf. [23] and its acknowledgements). The basic components of the future QC should be the qubit, the quantum register - a quantum device consisting of several qubits- and the quantum gate - a device aimed at controlling qubits and registers. It is presently possible to claim that Quantum Computing research is coming of age [17], and qubits, registers and gates on one side, quantum algorithms using abstract qubits on the other are under development. Qubits are generally supposed to have been initially prepared in a pure state, which is not always easily achieved with physical qubits, as shown e.g. for nuclear spins in [19] (cf. its page 324). Coupling between qubits induces entanglement between initially unentangled qubit pure states. In the context of QI, entanglement may be either desired, e.g. when considering a QC, because it may allow some form of parallel computing (Deutsch speaks of quantum parallelism [5]), or avoided, because e.g. coupling with the environment may cause decoherence, then stopping a calculation. It should then be useful to be able to know whether a given pure state of a system of abstract qubits is entangled or not.

The aim of this paper is to establish a necessary and sufficient unentanglement condition (for breviety, an *iff* condition) for a system consisting of N distinguishable qubits and which is in a pure state. This iff condition should generalize a result which we recently established for $N = 2$ qubits [10], in the context of a subfield of QI, based on Blind Source Separation (BSS), which started around 1985 in a classical context and which is now a mature field. In BSS [4], [9], typically, a set of users (the Writer) presents a set of simultaneous signals (input signals, called sources) at the input of a multi-user communication system (the Mixer). The sources, compelled to possess some general properties (e.g. mutual statistical independence), are then combined (mixed, in the BSS sense) in the Mixer. Another set of users (the Reader) receives the signals arriving at the Mixer output. The Writer possibly knows the sources, but the Reader does not know them, and cannot access the inputs of the Mixer. That Mixer uses one or several parameter values, unknown to the Reader, who only knows some of its general properties. The Reader's final task is the restoration of the sources (possibly up to some so-called acceptable indeterminacies) from the signals at the Mixer output, during the "inversion phase". An intermediate task is the determination of the unknown parameters of the Mixer, or of its inverse, made during an "adaptation phase". Since 2007, we have been developing a quantum version of BSS, namely Blind Quantum Source Separation (BQSS). In our previous papers ([11] and references therein), we considered two distinguishable qubits numbered 1 and 2, first prepared in a pure unentangled state $\mid \Psi(t_w) >$, by the Writer, at time t_w. The qubit pair is isolated from the rest of the world between t_w and the time t_r when the Reader can operate. At time t_r, the qubit pair is therefore still in a pure state $\mid \Psi(t_r) >$ but, because of an undesired coupling between the qubits (viewed as the action of a Mixer), $\mid \Psi(t_r) >$ is generally entangled. The Reader's task is the restoration of $\mid \Psi(t_w) > = \mid \Psi_1(t_w) > \otimes \mid \Psi_2(t_w) >$ from $\mid \Psi(t_r) >$ (again possibly up to some acceptable indeterminacies). In 2012, in this journal, we presented a paper devoted to methods for separating quantum sources [7]. We recently introduced Blind Quantum Process Tomography (BQPT [8], [12]), the blind version of Quantum Process Tomography (QPT [19]), which is defined in Section 5. The iff condition hereafter established is first aimed at extending the solution of the BQSS and BQPT problems beyond the qubit pair, but it is hoped that it could be used in other contexts, e.g that of the QC.

In Section 2, we first recall the iff condition for $N = 2$ qubits, and then present already existing criteria related to pure states or statistical mixtures of quantum systems composed of several parts, in situations which are more or less different from the present one - qubits, in arbitrary number. In Section 3 the method to be followed for finding this iff condition is presented. This *iff* condition is established in Section 4 , through an iterative process. The results and some of their applications are discussed in Section 5. The possibility of using the von Neumann concept is briefly tackled in the Appendix.

2 About existing unentanglement criteria

In our previous papers, qubits were supposed to be physically implemented as spins 1/2. We hereafter first recall the notations used for writing an arbitrary pure state $\mid \Psi >$ of a qubit pair. $\mid \Psi >$ is developed as:

$$\mid \Psi >= \sum_{i=1}^{4} c_i \mid i > , \text{ where} \tag{1}$$

$$\mid 1 >=\mid ++ >, \quad \mid 2 >=\mid +- > , \quad \mid 3 >=\mid -+ >, \quad \mid 4 >\mid -- > \tag{2}$$

where $\mid ++ >$ is an abbreviation for $\mid 1, + > \otimes \mid 2, + >$, $\mid 1, + >$ being the eigenstate for the eigenvalue 1/2 in the standard basis of qubit 1. We now consider an arbitrary $N-$qubit system, and generalize this writing: for qubit number k in an N-qubit system:

$$s_{zk} \mid k, \pm >= \pm\frac{1}{2} \mid k, \pm > \tag{3}$$

and for a pure state $\mid \Psi >$ of this N-qubit system:

$$\mid \Psi >= \sum_{i=1}^{2^N} c_i \mid i > \tag{4}$$

where $\mid 1 > = \mid ++ .... ++ >$ (all qubits in state $\mid + >$), $\mid 2^N >= \mid --- .... -- >$ (all qubits in state $\mid - >$). Qubit 1 is in state $\mid + >$ in the first 2^{N-1} states, and in state $\mid - >$ in the 2^{N-1} remaining states. $\mid 2^{N-1}+1 > = \mid -+ .. ++ >$ (all qubits in state $\mid + >$, except qubit 1, in state $\mid - >$), $\mid 2^N - 2 > = \mid --- .... +- >$ (all qubits in state $\mid - >$, except qubit $N-1$, in state $\mid + >$).

We immediately prove (for $c_1 \neq 0$) the following property, which we established in more detail (including the $c_1 = 0$ case) in [10]: a pure state $\mid \Psi >$ of a qubit pair is unentangled if and only if the c_i coefficients obey the equality:

$$c_1 c_4 = c_2 c_3. \tag{5}$$

If $c_1 \neq 0$, $\mid \Psi >$ can always be written as:

$$\mid \Psi >= c_1 (\mid ++ > + \frac{c_2}{c_1} \mid +- > + \frac{c_3}{c_1} \mid -+ > + \frac{c_4}{c_1} \mid -- >). \tag{6}$$

When $c_1 c_4 = c_2 c_3$, $\mid \Psi >$ can then be written as:

$$\mid \Psi >= c_1 (\mid ++ > + \frac{c_2}{c_1} \mid +- > + \frac{c_3}{c_1} \mid -+ > + \frac{c_2 c_3}{c_1{}^2} \mid -- >) \tag{7}$$

$$= c_1 (\mid + > + \frac{c_3}{c_1} \mid - >) \otimes (\mid + > + \frac{c_2}{c_1} \mid - >) \tag{8}$$

which proves that when $N = 2$ and $c_1 c_4 = c_2 c_3$, $\mid \Psi >$ is unentangled.

Conversely, if $\mid \Psi >$ is unentangled, it may be written as:

$$\mid \Psi >= (a_1 \mid + > + b_1 \mid - >) \otimes (a_2 \mid + > b_2 \mid - >). \tag{9}$$

Eq. (9) must be consistent with Eq. (1), which imposes the following relations:

$$c_1 = a_1 a_2, \quad c_2 = a_1 b_2, \quad c_3 = b_1 a_2, \quad c_4 = b_1 b_2 \tag{10}$$

and therefore $c_1 c_4 = c_2 c_3$.

In this paper, the proposed *iff* condition will take the form of a set of equalities between the c_i coefficients. In Section 4 it will e.g. be shown that $\mid \Psi >$ is unentangled, when $N = 3$, if and only if the following two subsets (S1, S2) of equalities, corresponding to a total of four independent equalities, are simultaneously verified:

$$S_1 : \quad \frac{c_1}{c_2} = \frac{c_3}{c_4} = \frac{c_5}{c_6} = \frac{c_7}{c_8} \tag{11}$$

$$S_2 : \quad c_2 c_8 = c_4 c_6. \tag{12}$$

Coming now to existing criteria, one should first mention the Schmidt and the Peres-Horodecki ones. The so-called Schmidt decomposition allows one to express an *iff* condition for pure states of a bipartite system, in which the dimension of each part of the system is not restricted to 2. And an extension of the concept of entanglement to statistical mixtures is somewhat possible through the notions of separability and of entanglement witnesses [3]. The

Peres-Horodecki criterion [15] is an *iff* condition for the separability of the density matrix describing a state of a bipartite system, valid when the dimensions of the state spaces of $S_1$and S_2 are low. In [10], it was explained why these criteria are not appropriate for the context of BQSS. They are both restricted to bipartite systems, and therefore should presently be discarded, since this paper is devoted to the general case $N > 2$. With these presently strong restrictions, their aim is not very different from the one in this paper.

On the contrary, in the context of quantum communications, it is usual to speak of Local Unitary (LU) transformations of pure states, and when it is spoken of equivalent states the aim is different from the one in e.g. BQSS. If $\mid \Psi(1,2,...N) >$ is some pure state of a system composed of N distinguishable particles, and U a unitary operator acting on $\mid \Psi(1,2,...N) >$, then producing a transformed pure state $U \mid \Psi(1,2,...N) >$, the transformation is said to be local if $U = U(1) \otimes U(2) ... \otimes U(N)$, where $U(1)$, $U(2)$, ... $U(N)$ are themselves unitary operators, each acting upon a single particle. The reason for the choice of the word *Local* may be guessed if 1) one considers the transformation of an unentangled state $\mid \Psi(1,2,...N) >$, 2a) one considers two qubits, 1 in space-time zone 1, and 2 in space-time zone 2, separated by a spacelike interval, 2b) one imagines two experimenters, A(lice) and B(ob), A being able to access qubit 1 (only) and B qubit 2 (only), 2c) this situation is true for all distinct pairs of an N-qubit system. On the contrary, an instance of the effect of a non-LU transformation on an unentangled state is given in Section 5. Once LU transformations have been defined, two pure states are said to be equivalent if they differ by an LU transformation only (cf. e.g. the review article [27]). When trying to define a degree of entanglement in order to classify all the possible pure states of a multipartite system, one is led to make no distinction between equivalent states. On the contrary, in the context of BQSS, this concept of equivalence through an LU transformation plays no role. If a given pure state $\mid \Psi >$ is written at the input of the Mixer, and a state $\mid \Phi >$ is read at its output, the aim is not to get back some state $\mid \Xi >$ equivalent to $\mid \Psi >$, with the meaning that $\mid \Xi >$ and $\mid \Psi >$ differ by an LU transformation, but to get back state $\mid \Psi >$ itself, possibly up to some acceptable (in fact, far weaker than an arbitrary LU transformation) indeterminacies.

3 Towards a generalization of the relation between the c_i existing for a qubit pair

In order to extend entanglement-based BQSS methods beyond the simplest case, the qubit pair ($N = 2$), it is highly desirable to find a set of relations, if it does exist, which, when $N > 2$, could be substituted for the equality $c_1 c_4 = c_2 c_3$. We have to find a collection of equalities between the c_i which are obeyed if and only if $\mid \Psi >$ is unentangled. We will consider only normalized states: $< \Psi \mid \Psi >= 1$. Moreover only the projector $\mid \Psi > < \Psi \mid$ has a physical meaning: one should not distinguish between $\mid \Psi >$ and $e^{i\eta} \mid \Psi >$ (η is any real number). Therefore, if the complex numbers c_i are written $c_i = \rho_i e^{i\varphi_i}$ (ρ_i

and φ_i are real numbers and $i = 1, 2,...2^N$), then rather than the 2^N phases, only e.g. the $(2^N - 1)$ phase differences $(\varphi_i - \varphi_1)$ are meaningful when defining an arbitrary pure state $\mid \Psi >$. Consequently, an arbitrary pure state $\mid \Psi >$ of the N-qubit system is defined by the value of $(2^{N+1} - 2)$ independent real numbers: $(2^N - 1)$ moduli and $(2^N - 1)$ phases. However, an unentangled pure state may be written as:

$$\mid \Psi_{ue} >=\mid \psi_1 > \otimes \mid \psi_2 > \otimes ... \mid \psi_i > \otimes ... \otimes \mid \psi_N > \tag{13}$$

where each ordered factor describes the state of a given qubit, and therefore depends upon two real numbers (a modulus, a phase). An unentangled state $\mid \Psi_{ue} >$ therefore depends upon $2N$ real numbers only. Then, if $\mid \Psi_{ue} >$ is developed following (4), there should exist $2[2^N - (N+1)]$ relations between the 2^{N+1} real quantities $\{\rho_i, \varphi_i\}$ (besides those expressing that only $\mid \Psi >< \Psi \mid$ has a physical meaning and that $\mid \Psi >$ is normalized). This result strengthens the hope that there may exist, between the c_i coefficients, a set of $[2^N - (N+1)]$ equalities which are verified if and only if $\mid \Psi >$ is unentangled. This is presently only a hope, since normalization of $\mid \Psi >$ and the physical meaning of $\mid \Psi > < \Psi \mid$ lead to two constraints between real numbers, not to a constraint between the c_i themselves. The present paper will *not* try to directly prove the existence of such relations between the c_i for unentangled and only unentangled pure states, but will rather use an iterative approach in order to try and establish such general relations.

4 Finding a necessary and sufficient condition: an iterative approach

We will first examine the $N = 3$ and $N = 4$ cases in some detail, starting with $N = 3$. It is supposed that $c_i \neq 0$ for any i value. The case when $c_i = 0$ for at least one i value will be discussed in Section 5.

When $N = 3$, any pure state may be written as:

$$\begin{aligned} \mid \Psi > = (\mid ++ > + \frac{c_3}{c_1} \mid +- > + \frac{c_5}{c_1} \mid -+ > + \frac{c_7}{c_1} \mid -- >) \otimes c_1 \mid + > \\ + (\mid ++ > + \frac{c_4}{c_2} \mid +- > + \frac{c_6}{c_2} \mid -+ > + \frac{c_8}{c_2} \mid -- >) \otimes c_2 \mid - > \end{aligned} \tag{14}$$

where $\mid + >$ in the writing $c_1 \mid + >$ and $\mid - >$ in the writing $c_2 \mid - >$ refer to a state of qubit no. 3. When

$$\frac{c_3}{c_1} = \frac{c_4}{c_2}, \frac{c_5}{c_1} = \frac{c_6}{c_2}, \frac{c_7}{c_1} = \frac{c_8}{c_2} \tag{15}$$

it is possible to write $\mid \Psi >$ as:

$$\mid \Psi >= \underbrace{(\mid ++ > + \frac{c_4}{c_2} \mid +- > + \frac{c_6}{c_2} \mid -+ > + \frac{c_8}{c_2} \mid -- >)}_{\mid \Psi(1,2) >} \otimes \underbrace{(c_1 \mid + > + c_2 \mid - >)}_{\mid \Psi(3) >}. \tag{16}$$

When moreover the coefficients of $\mid \Psi(1,2)>$ obey Eq. (5), which is presently written as:

$$\frac{c_8}{c_2} = \frac{c_4}{c_2}.\frac{c_6}{c_2}, \ i.e. \ \ c_2 c_8 = c_4 c_6 \tag{17}$$

then $\mid \Psi(1,2)>$ is an unentangled state, and $\mid \Psi >$ is unentangled. Therefore,

when $N = 3$, if subsets (11) and (12) are obeyed, $\mid \Psi >$ is unentangled (subset (11) is equivalent to the three equations (15)).

Conversely, if $\mid \Psi >$ is unentangled, it can be written as:

$$\mid \Psi(1,2,3) >= \underbrace{(\sum_{i=1}^{4} C_i \mid i >)}_{\text{unentangled}} \otimes (a_3 \mid + > + b_3 \mid - >) \tag{18}$$

where the C_i coefficients obey the relation $C_1 C_4 = C_2 C_3$. Identifying Eq. (18) with Eq. (4) (with $N = 3$), one gets:

$$c_1 = C_1 a_3, \ c_2 = C_1 b_3, \ c_3 = C_2 a_3, \ ... \ c_8 = C_4 b_3 \tag{19}$$

and therefore Eqs. (11) and (12) (i.e. Conditions S_1 and S_2) are obeyed.

When $N = 3$, $|\Psi >$ is therefore unentangled if and only if the two subsets of equalities (11) and (12), expressing a total of 4 (independent) equalities, are obeyed. Therefore, the above-expressed hope of finding an *iff* condition for unentanglement using $[2^N - (N+1)]$ relations between the c_i is satisfied when $N = 3$.

Similarly, when $N = 4$, any pure state may be written as:

$$\begin{aligned} \mid \Psi > = (\mid +++> + \frac{c_3}{c_1} \mid ++-> + ... + \frac{c_{15}}{c_1} \mid --->) \otimes c_1 \mid + > \\ + (\mid +++> + \frac{c_4}{c_2} \mid ++-> + ... + \frac{c_{16}}{c_2} \mid --->) \otimes c_2 \mid - > \end{aligned} \tag{20}$$

where now $\mid + >$ in the writing $c_1 \mid + >$ and $\mid - >$ in the writing $c_2 \mid - >$ refer to a state of qubit no. 4. When the following set of equalities is satisfied:

$$\frac{c_3}{c_1} = \frac{c_4}{c_2}, \frac{c_5}{c_1} = \frac{c_6}{c_2}, ..., \frac{c_{15}}{c_1} = \frac{c_{16}}{c_2}, \tag{21}$$

which can also be written as:

$$\frac{c_1}{c_2} = \frac{c_3}{c_4} = \frac{c_5}{c_6} = ... = \frac{c_{15}}{c_{16}}, \tag{22}$$

$\mid \Psi >$ may be written as:

$$\mid \Psi >= \underbrace{(\mid +++> + \frac{c_4}{c_2} \mid ++-> + ... \frac{c_{16}}{c_2} \mid --->)}_{|\Psi(1,2,3)>} \otimes (c_1 \mid + > + c_2 \mid - >). \tag{23}$$

When moreover the coefficients of $\mid \Psi(1,2,3) >$ obey subsets (11) and (12), presently written as:

$$\frac{c_2}{c_4} = \frac{c_6}{c_8} = \frac{c_{10}}{c_{12}} = \frac{c_{14}}{c_{16}} \tag{24}$$

$$c_4 c_{16} = c_8 c_{12}, \tag{25}$$

$\mid \Psi >$ is unentangled. Therefore, when $N = 4$, if the three subsets of Eqs. (22), (24) and (25) are verified, $\mid \Psi >$ is unentangled.

Conversely, if $\mid \Psi >$ is unentangled, it can be written as:

$$\mid \Psi(1,2,3,4) >= \underbrace{(\sum_{i=1}^{8} C_i \mid i >)}_{\text{unentangled}} \otimes (a_4 \mid + > + b_4 \mid - >) \tag{26}$$

where the C_i coefficients now obey the following sets of relations:

$$\frac{C_1}{C_2} = \frac{C_3}{C_4} = \frac{C_5}{C_6} = \frac{C_7}{C_8} \tag{27}$$

$$C_2 C_8 = C_4 C_6. \tag{28}$$

Now identifying Eq. (26) with Eq. (4) (with $N = 4$), one gets:

$$c_1 = C_1 a_4,\ c_2 = C_1 b_4,\ c_3 = C_2 a_4 .....c_{16} = C_8 b_4. \tag{29}$$

These equalities were obtained through the same process as in Eq. (19), and therefore lead to the same type of equalities as in Eq. (11), but with now 16 and not 8 c_i coefficients, namely:

$$\frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{15}}{c_{16}}. \tag{30}$$

Moreover, from Eqs.(27), (28) and (29), one gets:

$$\frac{c_2}{c_4} = \frac{c_6}{c_8} = \frac{c_{10}}{c_{12}} = \frac{c_{14}}{c_{16}} \tag{31}$$

$$c_4 c_{16} = c_8 c_{12}. \tag{32}$$

Therefore, when $N = 4$, $\mid \Psi >$ is unentangled if and only if the following three subsets of equalities are verified:

$$S_1: \quad \frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{15}}{c_{16}} \tag{33}$$

$$S_2: \quad \frac{c_2}{c_4} = \frac{c_6}{c_8} = \frac{c_{10}}{c_{12}} = \frac{c_{14}}{c_{16}} \tag{34}$$

$$S_3: \quad c_4 c_{16} = c_8 c_{12}. \tag{35}$$

When $N = 4$, this *iff* condition is therefore expressed through 3 subsets of equations, expressing a total of 11 equalities, which is also again the value of $[2^N - (N+1)]$, now for $N = 4$.

Now taking an arbitrary N value, one may write any pure state $\mid \Psi >$ as:

$$\mid \Psi > = (\mid + + ... + > + \frac{c_3}{c_1} \mid + + ... - > + ... + \frac{c_{2^N-1}}{c_1} \mid - - ... - >) \otimes c_1 \mid + >$$
$$+ (\mid + + ... + > + \frac{c_4}{c_2} \mid + + .. - > + ... + \frac{c_{2^N}}{c_2} \mid - - ... - >) \otimes c_2 \mid - > \tag{36}$$

which generalizes Eqs. (14) and (20). The reasoning which led to Eq. (11) and (33) now leads to:

$$\frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{2^N-3}}{c_{2^N-2}} = \frac{c_{2^N-1}}{c_{2^N}}. \tag{37}$$

This subset contains $(2^{N-1}-1)$ independent equalities. When they are obeyed, $\mid \Psi >$ may be written as:

$$\mid \Psi > = \mid \Psi(1, 2, 3, ...N-1) > \otimes \mid \Psi(N) > . \tag{38}$$

This state $\mid \Psi >$ is unentangled if and only if, moreover, $\mid \Psi(1, 2, 3, ...N - 1) >$ itself is unentangled. If the results obtained for $N = 3$ and $N = 4$ may be generalized, this is obtained when $(N - 2)$ other subsets of inequalities, corresponding to a total of $[2^N - (N + 1)] - (2^{N-1} - 1)$ equalities between the c_i, i.e. $(2^{N-1} - N)$ independent equalities, are satisfied. This quantity is equal to 1 if $N = 3$ (subset S_2, Eq. (12)), and to 4 if $N = 4$ (subsets S_2, Eq. (34), and S_3, Eq. (35)).

We now momentarily suppose that the property established for $N = 3$ and $N = 4$ is true for $N - 1$ (with $N - 1 \geq 4$), i.e. that $\mid \Psi >$ is unentangled if and only if the following $(N - 2)$ subsets of equalities are simultaneously verified:

$$S_1: \quad \frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{2^{N-1}-3}}{c_{2^{N-1}-2}} = \frac{c_{2^{N-1}-1}}{c_{2^{N-1}}} \tag{39}$$
$$S_2: \quad \frac{c_2}{c_4} = \frac{c_6}{c_8} = ... = \frac{c_{2^{N-1}-2}}{c_{2^{N-1}}} \tag{40}$$
$$S_3: \quad \frac{c_4}{c_8} = \frac{c_{12}}{c_{16}} = ... = \frac{c_{2^{N-1}-4}}{c_{2^{N-1}}} \tag{41}$$
$$............ \tag{42}$$
$$S_{N-3}: \quad \frac{c_{2^{N-4}}}{c_{2*2^{N-4}}} = \frac{c_{3*2^{N-4}}}{c_{4*2^{N-4}}} = \frac{c_{5*2^{N-4}}}{c_{6*2^{N-4}}} = \frac{c_{7*2^{N-4}}}{c_{8*2^{N-4}}} \tag{43}$$
$$S_{N-2}: \quad c_{2^{N-3}} c_{4*2^{N-3}} = c_{2*2^{N-3}} . c_{3*2^{N-3}} . \tag{44}$$

In order to help the reader anxious to see more explicitly the meaning of the dots in Eq. (42), and in Eq. (48) hereafter, the 6 subsets for $N = 7$ are all written at the end of this section (four of them again with dots, but then with a rather obvious meaning). We now establish that any ket $\mid \Psi > = \sum_{i=1}^{2^N} c_i \mid i >$

describing a pure state of an N-qubit system is unentangled if and only if the $N-1$ following subsets are all obeyed:

$$S_1: \quad \frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{2^N-3}}{c_{2^N-2}} = \frac{c_{2^N-1}}{c_{2^N}} \tag{45}$$

$$S_2: \quad \frac{c_2}{c_4} = \frac{c_6}{c_8} = ... = \frac{c_{2^N-2}}{c_{2^N}} \tag{46}$$

$$S_3: \quad \frac{c_4}{c_8} = \frac{c_{12}}{c_{16}} = ... = \frac{c_{2^N-4}}{c_{2^N}} \tag{47}$$

$$............ \tag{48}$$

$$S_{N-2}: \quad \frac{c_{2^{N-3}}}{c_{2*2^{N-3}}} = \frac{c_{3*2^{N-3}}}{c_{4*2^{N-3}}} = \frac{c_{5*2^{N-3}}}{c_{6*2^{N-3}}} = \frac{c_{7*2^{N-3}}}{c_{8*2^{N-3}}} \tag{49}$$

$$S_{N-1}: \quad c_{2^{N-2}}.c_{4*2^{N-2}} = c_{2*2^{N-2}}c_{3*2^{N-2}}. \tag{50}$$

The approach already used for $N = 3$ and 4 is applied to an $N-$qubit system. Any pure state may be written as:

$$\begin{aligned} | \Psi > = (| ++ ...+ > + \frac{c_3}{c_1} | ++ ...- > +... + \frac{c_{2^N-1}}{c_1} | -- ...- >) \otimes c_1 | + > \\ +(| ++ ...+ > + \frac{c_4}{c_2} | ++ ...- > +... + \frac{c_{2^N}}{c_2} | -- ...- >) \otimes c_2 | - > \end{aligned} \tag{51}$$

where now $| + >$ in the writing $c_1 \mid + >$ and $| - >$ in the writing $c_2 \mid - >$ refer to a state of qubit no. N. When each c_{2^k-1}/c_1 quantity (for $k = 2, 3...N$) is equal to c_{2^k}/c_2, this collection of relations can collectively be written as S_1 subset (45), and $| \Psi >$ may be expressed as:

$$| \Psi >= \underbrace{(| +++ > + \frac{c_4}{c_2} | ++- > + \frac{c_{2^N}}{c_2} | --- >)}_{|\Psi(1,2,...N-1)>} \otimes (c_1 | + > +c_2 | - >). \tag{52}$$

If moreover the c_{2^k}/c_2 coefficients in $| \Psi(1, 2, ...N-1) >$ obey all the equalities

expressed in Eq. (39) to (44), $| \Psi(1, 2, ...N-1) >$ is unentangled, and $| \Psi >$ itself is therefore unentangled. For instance, Eq. (44) presently takes the form:

$$c_{2^{N-2}}c_{4*2^{N-2}} = c_{2*2^{N-2}} * c_{3*2^{N-2}} \tag{53}$$

which is subset (50). More generally, Eq. (39) to (44) presently take the form of Eq. (46) to (50) respectively. Any reader aiming at establishing these equalities should appreciate that the c_i coefficients in Eq. (4) are generic quantities. For instance, c_1 is the coefficient for $| +++ >$ if $N = 3$, whereas it is the coefficient for $| +++++ >$ if $N = 5$.

Conversely, if $| \Psi(1, 2, ..., N >$ is unentangled, it can be written as:

$$\underbrace{(\sum_{i=1}^{2^{N-1}} C_i | i >)}_{\text{unentangled}} \otimes (a_N | + > +b_N | - >) \tag{54}$$

where the C_i coefficients obey equalities expressed through subsets (39) to (44), with the C_i instead of the c_i coefficients. Then, the method used for $N = 3$ and $N = 4$ is again used for expressing each c_i coefficient in expression $| \Psi >= \sum_{i=1}^{2^N} c_i \mid i >$ as a function of both a C_i coefficient and a_N or b_N. This allows us to show that subsets (45) to (50) are obeyed. Therefore, if it is true that a ket $| \Psi(1, 2, ...N - 1) >$ is unentangled if and only if subsets (39) to (44) are all verified, then it is true that a ket $| \Psi(1, 2, ...N >$ is unentangled if and only if subsets (45) to (50) are all verified.

Considering successively $N = 2$, 3 and 4, it was shown above that $| \Psi >$ is unentangled if and only if a collection of equalities, structured into $(N - 1)$ subsets, is obeyed. These results and this iterative discussion from $(N - 1)$ to N finally allow us to claim that subsets (45) to (50) do collectively express an iff condition for a ket $| \Psi >$ of an N-qubit system to be unentangled.

If e.g. $N = 7$ (the dimension of the state space is 128 then), it is tedious but quite possible, through successive iterations, to get the explicit expressions of the six subsets of equalities expressing unentanglement. They are respectively

$$S_1: \ \frac{c_1}{c_2} = \frac{c_3}{c_4} = ... = \frac{c_{125}}{c_{126}} = \frac{c_{127}}{c_{128}} \tag{55}$$

$$S_2: \ \frac{c_2}{c_4} = \frac{c_6}{c_8} = ... = \frac{c_{126}}{c_{128}} \tag{56}$$

$$S_3: \ \frac{c_4}{c_8} = \frac{c_{12}}{c_{16}} = ... = \frac{c_{124}}{c_{128}} \tag{57}$$

$$S_4: \ \frac{c_8}{c_{16}} = \frac{c_{24}}{c_{32}} = ... = \frac{c_{120}}{c_{128}} \tag{58}$$

$$S_5: \ \frac{c_{16}}{c_{32}} = \frac{c_{48}}{c_{64}} = \frac{c_{80}}{c_{96}} = \frac{c_{112}}{c_{128}} \tag{59}$$

$$S_6: \ c_{32}.c_{128} = c_{64}c_{96}. \tag{60}$$

They have been written here in order to help the reader interpret the dots in Eq. (42) and (48).

Our results for $N = 2$, 3 and 4 suggest that there are $[2^N - (N + 1)]$ independent equalities in Eqs. (45) to (50). We now establish this result, again using mathematical induction. We first suppose that it is true that Eqs. (39) to (44), for an $(N - 1)$-qubit system, contain $(2^{N-1} - N)$ such equalities. Now considering an N -qubit system, we may claim that the corresponding number of such equalities is the sum of two quantities: $(2^{N-1} - 1)$, the number of equalities associated with S_1 (cf. Eq. (45)) and $(2^{N-1} - N)$ new equalities expressing that $| \Psi(1, 2, 3, ...N - 1) >$ in Eq. (38) is itself unentangled, which does lead to a total of $[2^N - (N + 1)]$ such equalities. This expression, which was already known to be valid for $N = 2$, 3 and 4, is therefore valid for any $N \geq 2$.

5 Discussion

It is possible to build other sets of equalities which are obeyed if and only if an arbitrary pure state $| \Psi >$ is unentangled. When $N = 3$, for instance, keeping the same approach, it is easy to replace (12) with condition $c_1c_7 = c_3c_5$. Then, with the same approach for $N > 3$, all even c_i coefficients in the subsets S_k with $k > 1$ are suppressed. For $N = 6$, e.g. the S_5 subset becomes $c_1c_{49} = c_{17}c_{33}$. Use of even indices leads to simpler expressions, which explains the choice made in this paper.

In [10], with $N = 2$, if at least one of the c_i coefficients is equal to 0, it was shown that condition $c_1c_4 = c_2c_3$ is still valid. In Section 4 of the present paper, it was assumed that $c_i \neq 0$ for any i. When $N = 3$, if e.g. $c_5 = 0$, then in Eq. (14) the $(c_5/c_1) \mid -+>$ term is absent and$\mid \Psi >$ is then unentangled only if $c_6 = 0$ (cf. the presence of the $(c_6/c_2) \mid -+>$ term in Eq. (14)). When $N > 3$, if $c_5 = 0$, then in all the subsets expressing unentanglement, the c_6 terms will be absent. The reason is that in Eq. (36) the (unexplicitly written) c_5/c_1 term of qubits 1 to $(N-1)$ is associated with the $c_1 \mid + >$ state of qubit N, and the corresponding state of qubits 1 to $(N-1)$ associated with the $c_2 \mid - >$ state of qubit N has a c_6/c_2 coefficient. This reasoning may also be used if more than one c_i coefficient is equal to 0.

When $N = 20$, the dimension of the state space $\mathcal{E}_{20}$ is 2^{20}, which is roughly 10^6. An unentangled normalized state then depends upon 40 real numbers only. In the context of Quantum Information Processing, it is generally considered that the wealth of the quantum behaviour originates in the existence of entanglement, but it may be important to be able to decide whether a given pure state is entangled or not, e.g. in order to achieve BQSS or BQPT, and finding an *iff* condition is therefore significant. The present paper has shown that, when $\mid \Psi >$ is unentangled, there exist $[2^N - (N+1)]$ independent equalities between the c_i coefficients, the value of which is itself roughly 10^6 when $N = 20$. But it has also been found that these equalities may be classified into only $(N-1)$ subsets, e.g. 19 subsets when $N = 20$. It is hoped that this classification should allow tractable operations in numerical simulations or calculations.

We now come to recent papers making use of a Local Unitary (LU) transformation. Ninety years after the building of modern Quantum Mechanics (QM), there is a vast literature devoted to its foundations (see e.g. [1], [16], [26]), while most physicists use QM without discussing its deep content. The following lines just aim at drawing a link between these recent papers and this literature. Paty [20] has stressed that, historically, well before the 1935 EPR paper, Einstein, at the 1927 Solvay Congress [13] (p. 256), exposed his concern about what he would later on call the incompleteness of QM. In his 1995 paper, Paty clearly and convincingly asserts that "*it is only recently, indeed, that the concept of non-locality as a fundamental feature of quantum mechanics has been fully appreciated, and commentators have seldom realized that this was one of Einstein's main points"*, and that *"it is in the Einstein-Podolsky-Rosen's paper itself that non-locality is described and that emphasis is put on it"*. At the same 1927 meeting, Einstein stated that the interpretation of $\mid \Psi \mid^2$ as a probability density for a single particle (rather than for an ensemble of particles) implied, for him, "*a contradiction with the principle of relativity"*. After 1945, a deepening of the foundations of QM partly overlapped with the development of Classical and Quantum Information. Bell's contributions to these foundations, from his 1966/1964 papers down to his death in 1990 [1], especially stimulated the development of both experimental tests and so-called quantum communications. In the context of quantum communications, speaking of LU transformations (cf. Section 2) is usual. An LU transformation $U = U(1) \otimes U(2)$ transforms a pure unentangled state $\mid \Psi(1 > \otimes \mid \Phi(2) >$ into the unentangled state $(U(1) \mid \Psi(1 >) \otimes (U(2) \mid \Phi(2) >)$. On the contrary, a simple calculation shows that e.g. the non-LU transformation $U = e^{ias_{1x}s_{2x}}$ acting on the unentangled state $\mid +- >$,(a is some dimensional real constant) transforms it into the *entangled* state (cf. Eq. (5))

$$\frac{e^{ia/4}}{4}(\mid +x, +x > - \mid -x, -x >) - \frac{e^{-ia/4}}{4}(\mid +x, -x > - \mid -x, +x >) \quad (61)$$

where e.g. $\mid +x, -x >$ means $\mid 1, +x > \otimes \mid 2, -x >$, and $\mid i, +x >$ (resp. $\mid i, -x >$) is the eigenket for s_{ix} for the eigenvalue $1/2$ (resp. $-1/2$), with $i = 1$, 2. LU transformations should therefore be distinguished from entanglement-inducing transformations. The latter transformations are faced e.g. in BPQT and BQSS, which are important quantum information processing problems due to their applications, including those presented hereafter.

Blind or non-blind QPT may be defined as the identification (i.e. estimation) of a given quantum process or gate, called the direct process or gate hereafter, which receives a "source state". As discussed e.g. in [2], [18], [19], [24], [25], [28], (B)QPT is a major quantum information processing tool, since it especially allows one to characterize the actual behavior of quantum gates, which are the building blocks of the quantum computers considered in Section 1. The usual, i.e. non-blind, version of QPT requires one to know, hence to precisely control (i.e. prepare), the specific quantum source states used as inputs of the quantum gate to be characterized. The blind version of this tool, i.e. BQPT, then provides an attractive extension of QPT, since it allows one to use quantum source states whose values are unknown and arbitrary, except that they are requested to meet some general properties. These properties e.g. consist of unentanglement [12], which is one of the motivations for analyzing unentanglement in the present paper (more details about the operation of BQPT are available e.g. in [12], [11]).

BQSS may be seen a quantum information processing problem where one aims at handling the altered quantum state available at the output of a direct quantum process / gate which typically involves undesired coupling between its qubits, this process and its input being initially unknown. This BQSS problem is e.g.[1] handled by (i) first identifying that direct gate with BQPT, thus using only the output of that gate and unentanglement or other properties, (ii) then deriving a quantum gate that performs the inverse transform of that of the direct gate, and (iii) then feeding that "inverse gate" with the altered states available at the output of the direct gate during final operation, so as to restore the corresponding source, i.e. non-altered, states. One may anticipate that this

[1] Other approaches perform BQSS directly, i.e. without first resorting to BQPT.

approach will be useful e.g. in situations where data are stored in a register of qubits of a quantum computer, for subsequent use. Due to non-idealities of the physical implementation of that register, the qubits which form it may be coupled (e.g. when qubits are implemented as the spins of electrons which are close to one another). As time goes on, the register state will therefore evolve in a complicated way due to qubit coupling, thus making the final value of that register state not directly usable in the target application of the quantum computer. BQSS may then be used to restore the initially stored register state, before providing it to the part of the quantum computer which uses these nonaltered data to perform the target task of that computer.

6 Conclusion

In the 2009 review article devoted to entanglement [15], the Horodecki team noticed that *"it appears that this new resource is complex and difficult to detect"*. Experimental and theoretical aspects were both involved. If one focuses this comment on the idea that establishing whether a pure state is entangled or not is a cumbersome task, the following remarks may be made. In the present paper, devoted to an arbitrary number, N, of distinguishable qubits, it has been shown that if a pure state of that N-qubit system is developed over the 2^N basis states of the generalized standard basis (or of some arbitrary well-defined basis) as $\mid \Psi >= \sum_i c_i \mid i >$, one is then led to introduce $(N-1)$ subsets of equalities, which are verified if and only if $\mid \Psi >$ is unentangled. It should however be realized that if $N = 20$, these 19 subsets together collect $[2^N - (N+1)]$ equalities, which is here roughly equal to 2^{20}, i.e. approximalely 10^6. While the complexity of the problem is reflected in the fact that the number of equalities roughly grows as 2^N, it is hoped that the necessary and sufficient condition established in this paper, which introduced a systematic ordering within these equalities, through a classification into $(N-1)$ subsets, may in practice help in the manipulation of the entanglement concept.

A The von Neumann entropy and the establishment of the iff condition

The entropy concept, which did not appear in this paper yet, is briefly considered here. The von Neumann entropy of a quantum system in a pure or mixed state described by a density operator ρ is the trace $S = -Tr(\rho Ln\rho)$. This concept cannot directly be used in an attempt to find an *iff* condition for the unentanglement of a pure state $\mid \Psi >$ of an $N-$qubit system, since its von Neumann entropy is zero for both unentangled and entangled pure states. But this $N-$qubit system can be viewed as a bipartite system Σ, composed of parts Σ_A and Σ_B, and if Σ is described by ρ, one may first introduce reduced density operators $\rho_A = Tr_B\rho$ and $\rho_B = Tr_A\rho$ (see e.g. [21]). From now on, we focus on the situation when $\rho =\mid \Psi >< \Psi \mid$. Both Σ_A and Σ_B possess orthonormal basis states $\mid \varphi_i^A >$ and $\mid \chi_i^B >$ allowing to write any pure state $\mid \Psi >$ of Σ as $\mid \Psi >= \sum_i \lambda_i \mid \varphi_i^A > \otimes \mid \chi_i^B >$ (Schmidt decomposition), where the sum of the squares of the real non-negative so-called Schmidt coefficients λ_i is equal to 1 (see e.g. [19]). Moreover, ρ_A and ρ_B have the same eigenvalues, equal to λ_i^2 [19]. One introduces the entropies for Σ_A and Σ_B, respectively $S_A = -Tr_A(\rho_A Ln\rho_A)$ and

$S_B = -Tr_B(\rho_B Ln\rho_B)$, and, as a result of both the Schmidt decomposition and the just mentioned property of the eigenvalues of ρ_A and ρ_B, $S_A = S_B = -\sum_i \lambda_i^2 Ln\lambda_i^2$. Then $S = 0$, while $S_A = S_B \geq 0$, and $\mid \Psi >$ is unentangled if and only if $S_A = S_B$ is equal to zero. A means of establishing an iff condition through the reduced entropy concept therefore does in principle exist. But the fact that the reduced entropy of a bipartite system is related to the Schmidt decomposition immediately suggests that, if this concept is used as a tool for establishing an iff condition for the c_i introduced in this paper, the difficulty will be at least as great as the one found with the Schmidt criterion, already discussed in Section 2.

Let us first examine the two-qubit case: A is qubit 1 and B qubit 2. Then, keeping our previous notations, $\mid \Psi >= \sum_{i=1}^{4} c_i \mid i >$, one has first to express the condition $S_A = 0$ as a function of the c_i coefficients, but this means: 1) calculating the expression of ρ_A , 2) calculating its eigenvalues, 3) calculating S_A and solving the equation $S_A = 0$. The reader may verify that a tedious calculation leads to our well-known result:

$$(c_1 c_4 - c_2 c_3) = 0. \tag{62}$$

The next simplest situation is $N = 3$, and one may first introduce ρ_3, the reduced entropy for qubit no. 3, and focus on the corresponding reduced entropy $S_3 = -Tr_3(\rho_3 Ln\rho_3)$, which is zero *iff* $\mid \Psi >$ is unentangled. This necessitates first to calculate all the elements of the reduced density matrix ρ_3, each one a complicated sum involving our c_i coefficients, and secondly to find an analytical expression for the eigenvalues of ρ_3. But, once this is done, one knows that if and only if one and only one eigenvalue is non-zero, and therefore equal to one, then the state is unentangled. Considering the reduced entropy S_3, i.e manipulating sums of quantities involving logarithms, is therefore unnecessary.

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