A CRITERION FOR DERIVING THE SUB/SUPER GAUSSIANITY OF SOURCE SIGNALS FROM THEIR MIXTURES

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ABSTRACT

In this paper, we consider the situation when two linear instantaneous mixtures of two independent source signals are available. We aim at determining whether the source signals are sub- or super-Gaussian, using only their observed mixtures. To this end, we propose a criterion based on the roots of a specific polynomial, whose coefficients depend on the estimated cross-cumulants of the observed signals. The effectiveness of this approach is demonstrated by means of experimental tests.

1 PROBLEM STATEMENT

Higher-order statistics have been a very active research field during the last decade. This domain involves non-Gaussian signals, which may be split in two classes i.e. sub-Gaussian and super-Gaussian signals. These classes resp. consist of the signals which have negative and positive kurtosis (see e.g. [1]-[2]; this parameter is redefined hereafter in Section 2). This distinction between sub- and super-Gaussian signals is of utmost importance in the blind source separation problem, which consists in estimating a set of sources signals only from a set of observed signals which are typically linear instantaneous mixtures of these source signals (see the surveys of this field in [3]-[4]). This is due to the fact that a large number of source separation methods only apply to one of these classes (e.g. all source signals should be sub-Gaussian), and others lead to several versions, depending on the class of each source signal (see e.g. [5]-[6]). As one seldom knows to which classes the considered source signals belong in practice, there is a need for a method that would allow one to determine the sign of kurtosis of each source only from the available mixed signals. This would make it possible to determine automatically if the considered source separation algorithm applies to the signals to be processed, or which version of this algorithm should be used for these signals. This paper provides a solution to this problem in the case when two mixtures of two sources are available.

2 PROPOSED METHOD

We assume that two observed signals $y_1(t)$ and $y_2(t)$ are available and that they are linear instantaneous mixtures of two unknown source signals $x_1(t)$ and $x_1(t)$, i.e:

$$y_1(t) = a_{11}x_1(t) + a_{12}x_2(t) \quad (1)$$
$$y_2(t) = a_{21}x_1(t) + a_{22}x_2(t), \quad (2)$$

where a_{ij} are unknown mixture coefficients. The source signals are supposedly stationary, statistically independent and centered for simplicity (non-centered signals are considered in Section 3). The mixture coefficients are real, constant and non-zero. They are also assumed to yield a non-degenerate mixture matrix, i.e. each observed signal is not a plain scaled version of the other one, or in other words:

$$a_{11}a_{22} - a_{12}a_{21} \neq 0. \quad (3)$$

Using only $y_1(t)$ and $y_2(t)$, we aim at determining the sign of the kurtosis of each source $x_i(t)$. We recall that this kurtosis is defined as the zero-lag 4^{th}-order cumulant of the considered signal [1]-[2], i.e:

$$CUM_4(x_i) = CUM[x_i(t), x_i(t), x_i(t), x_i(t)], \quad (4)$$

where the right-hand term of this equation is the cumulant of the considered four random variables, so that

$$CUM_4(x_i) = E\{x_i^4(t)\} - 3(E\{x_i^2(t)\})^2, \quad (5)$$

where $E\{.\}$ stands for mathematical expectation[1].

To reach the above-defined goal, we introduce the "conceptual" signal

$$s(t) = y_1(t) - cy_2(t), \quad (6)$$

where c is a real tunable coefficient. This signal is "conceptual" in the sense that it is only used as a tool in our approach, i.e. it will not be explicitly created in our

[1]Note that the approach proposed in this paper also applies to the normalized kurtosis [1], as the latter parameter has the same sign as the plain kurtosis considered hereafter.

eventual solution to the considered problem. This signal may also be expressed as:

$$s(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t), \qquad (7)$$

with:

$$\alpha_1 = a_{11} - c a_{21} \qquad (8)$$
$$\alpha_2 = a_{12} - c a_{22}. \qquad (9)$$

Using cumulant properties [1]-[2] and the independence of $x_1(t)$ and $x_2(t)$, (7) yields:

$$CUM_4(s) = \alpha_1^4 CUM_4(x_1) + \alpha_2^4 CUM_4(x_2). \qquad (10)$$

This leads to the following properties concerning the variations of the sign of $CUM_4(s)$ vs c, when $CUM_4(x_1)$ and $CUM_4(x_2)$ have given signs:

1. First consider the case when:

   $$CUM_4(x_1) > 0 \quad \text{and} \quad CUM_4(x_2) > 0. \qquad (11)$$

   If there existed a value c_z of c such that $\alpha_1 = 0$ and $\alpha_2 = 0$, this value would meet the conditions:

   $$c_z = \frac{a_{11}}{a_{21}} \qquad \text{and} \qquad c_z = \frac{a_{12}}{a_{22}}, \qquad (12)$$

   due to (8)-(9). But this requires:

   $$\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}}, \qquad (13)$$

   which is not possible, due to (3). Therefore

   $$\forall\, c, \quad \alpha_1^4 > 0 \quad \text{or} \quad \alpha_2^4 > 0. \qquad (14)$$

   Combining this with (10) and (11) yields

   $$\forall\, c, \quad CUM_4(s) > 0. \qquad (15)$$

2. It may be shown in the same way that

   $$CUM_4(x_1) < 0 \quad \text{and} \quad CUM_4(x_2) < 0 \qquad (16)$$

   leads to

   $$\forall\, c, \quad CUM_4(s) < 0. \qquad (17)$$

3. Now consider the case when $CUM_4(x_1)$ and $CUM_4(x_2)$ have opposite signs. $c = a_{11}/a_{21}$ and $c = a_{12}/a_{22}$ resp. result in $CUM_4(s) = \alpha_2^4 CUM_4(x_2)$ and $CUM_4(s) = \alpha_1^4 CUM_4(x_1)$, which then have opposite signs. Moreover, combining (8)-(9) and (10) shows that $CUM_4(s)$ is a 4^{th}-order polynomial of c, and therefore a continuous function of c. Therefore, there exists at least one value of c such that $CUM_4(s) = 0$ in this case.

The above results define the properties of the kurtosis of the signal $s(t)$ with respect to those of the kurtosis of the source signals. But, conversely, one then easily derives from them the following properties of the kurtosis of the source signals vs those of the kurtosis of $s(t)$ [2]:

1. If $CUM_4(s)$, considered as a polynomial of c, has at least one real-valued root, then $CUM_4(x_1)$ and $CUM_4(x_2)$ have opposite signs, i.e. the considered couple of sources consists of one positive-kurtotic signal and one negative-kurtotic signal[3].

2. Otherwise, $CUM_4(x_1)$ and $CUM_4(x_2)$ have the same sign. Moreover, this sign may be determined as follows. $CUM_4(s)$ then has the same sign whatever c, and this sign is the same as that of both $CUM_4(x_i)$. This sign is e.g. the sign of $CUM_4(s)$ obtained when setting $c = 0$. But in this case

   $$s(t) = y_1(t), \qquad (18)$$

   so that

   $$CUM_4(s) = CUM_4(y_1). \qquad (19)$$

   Therefore, in this case the common sign of the kurtosis of the sources is obtained as the sign of the kurtosis of $y_1(t)$, which is an observable quantity.

The above discussion provides a root-based criterion for determining the signs of the kurtosis of the sources signals. This criterion is preliminary in the sense that, at this stage, we have provided no practical means to determine if the polynomial $CUM_4(s)$ has real roots (note that the expression (10) of this polynomial with respect to the source signals cannot be used to this end, as its coefficients are unknown in practice). The latter problem is solved by considering $s(t)$ and its kurtosis with respect to the mixed signals, instead of its relationship with respect to the sources which was considered above. We then derive $CUM_4(s)$ from (6), again using cumulant properties [1]-[2], but now with signals $y_1(t)$ and $y_2(t)$ which are not independent. This yields

$$\begin{aligned} CUM_4(s) \;=\; & c^4 CUM_4(y_2) - 4c^3 CUM_{31}(y_2, y_1) \\ & + 6c^2 CUM_{22}(y_2, y_1) - 4c\, CUM_{13}(y_2, y_1) \\ & + CUM_4(y_1), \end{aligned} \qquad (20)$$

where each term $CUM_{kl}(y_2, y_1)$ with $k + l = 4$ is the cumulant of the following random variables: $y_2(t)$ taken k times and $y_1(t)$ taken l times. $CUM_4(s)$ thus appears as another 4^{th}-order polynomial of c but, unlike in the previous case, its coefficients are known, or more precisely can be estimated, as they are cumulants of observed signals. This yields the final version of the criterion that we propose for determining the signs of kurtosis of the source signals, i.e:

- Consider the 4^{th}-order polynomial of c defined by (20) [4].

[2]For the sake of brevity, we omit the case when the kurtosis of either or both sources are zero.

[3]There is no sense wondering which of these sources has a positive kurtosis, as the order of the sources in the considered mixed signals is arbitrary.

[4]Several other equivalent expressions of a signal $s(t)$ created as a linear instantaneous combination of the two mixed signals may be considered instead of (6), leading to corresponding modified versions of our method and of the polynomial $CUM_4(s)$.

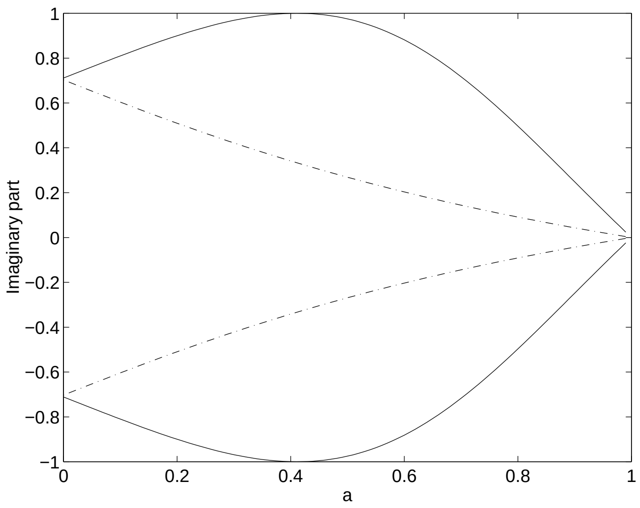

Figure 1: Variations of the imaginary parts of the four roots of the polynomial (20) vs mixture coefficient a, when both sources are sub-Gaussian.

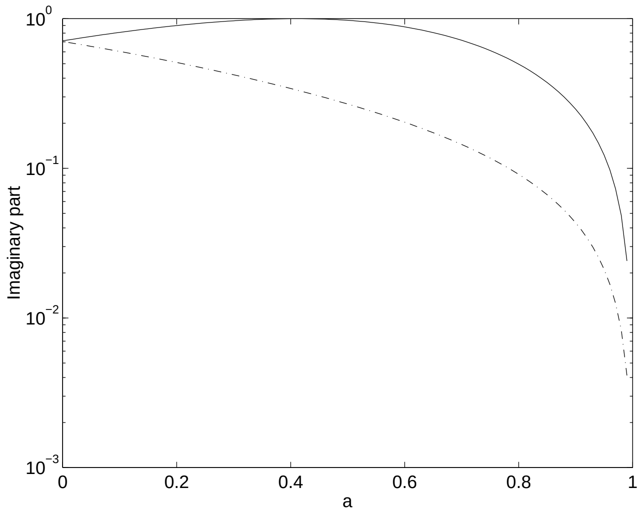

Figure 2: Variations of the two positive imaginary parts of the roots of the polynomial (20) vs mixture coefficient a, when both sources are sub-Gaussian.

- Estimate its coefficients from the available data, i.e. estimate the cross- and auto-cumulants of the (centered) mixed signals (auto-cumulants are estimated by the time averages associated to (5); cross-cumulants may be derived similarly from [1]-[2]).
- Check if this polynomial has at least one real-valued root. Note that this may be achieved e.g. i) by using classical numerical methods for determining the roots of any polynomial or the zeros of arbitrary functions, or more specifically ii) by means of Ferrari's analytical method for determining the roots of 4^{th}-order polynomials [7].
- Use the above-defined root-based criterion to eventually derive the signs of the kurtosis of the (centered) sources.

3 EXPERIMENTAL RESULTS

We have validated the above approach by means of experimental tests performed in the following conditions. Each source is a possibly-non-zero-mean binary-valued signal $X_i(t)$, which takes the values +1 et -1 resp. with the probabilities p_i et $1 - p_i$, where p_i is a parameter of the considered source. Two linear instantaneous mixtures of these sources are computed by using specific mixture coefficients a_{ij}. The method proposed in this paper is then applied to the centered versions of these mixed signals.

The first series of tests was performed with two sub-Gaussian sources. More precisely, the parameters p_1 and p_2 which resp. define the statistics of the centered versions $x_1(t)$ and $x_2(t)$ of the sources were both set to $\frac{1}{2}$. Our calculations showed that this leads to the following theoretical source kurtosis values: $CUM_4(x_1) = -2$ and $CUM_4(x_2) = -2$. We tested the proposed approach in detail by considering various mixtures of these sources. More precisely, the mixture coefficients a_{12} and a_{21} were set to the same value a, which was varied in the tests, whereas both mixture coefficients a_{11} and a_{22} were kept constant to 1. For each value of a, we computed the empirical cumulants of the centered mixed signals and we derived the four roots of the polynomial (20). Fig. 1 shows the evolution of the imaginary parts of these roots when a is varied from 0 to 0.99. These results may be used as follows for any single value of a. Fig. 1 shows that all these imaginary parts are non zero, i.e. that the experimental polynomial has no real roots. Based on this result, the above-defined criterion leads to the decision that $CUM_4(x_1)$ and $CUM_4(x_2)$ have the same sign, which may then be determined to be negative as explained above. For any value of a, this experimental decision is in full agreement with the theoretical kurtosis values that we provided above for this type of sources, thus showing the effectiveness of the proposed approach. It should be noted that this method is successful even for very highly mixed source signals, as these tests were performed with a varied up to 0.99 and no decision errors occured. This robustness with respect to high mixture levels becomes more apparent in Fig. 2, where the two positive imaginary parts are represented with semilogarithm scales: this shows that, even for high values of a, these imaginary parts remain significantly far from 0 (as opposed to the imaginary parts equal to 0 obtained in the other series of tests reported below). It should also be noted that the imaginary-part plots in Fig. 1 form two couples having symmetrical positions with respect to the X axis. This results from the fact that the four roots of the polynomial (20) here consists of two sets of conjugate roots. This is confirmed by Fig. 3: the real

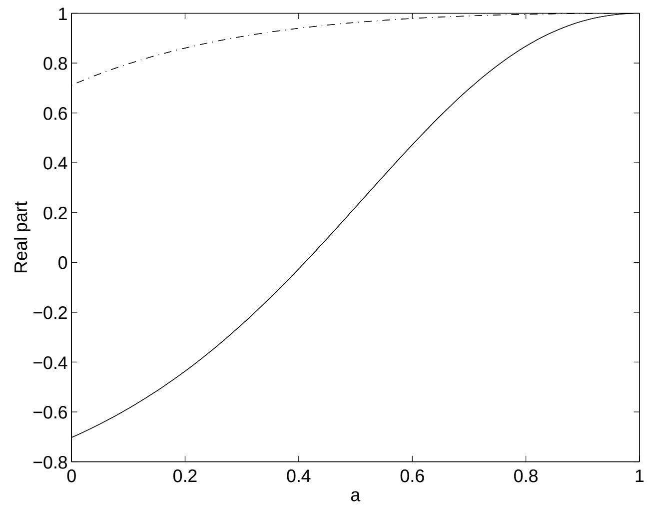

Figure 3: Variations of the real parts of the four roots of the polynomial (20) vs mixture coefficient a, when both sources are sub-Gaussian.

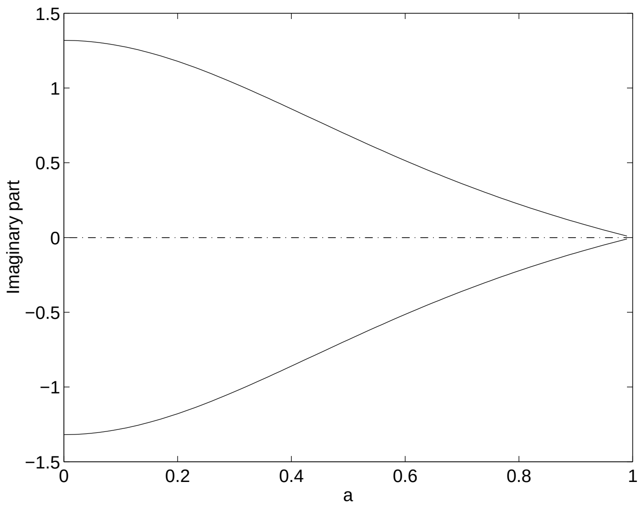

Figure 4: Variations of the imaginary parts of the four roots of the polynomial (20) vs mixture coefficient a, for 1 sub-Gaussian source and 1 super-Gaussian source.

parts of all four roots are plotted in this figure, but only two charts are visible because in each set of conjugate roots both roots have the same real part.

The second series of tests was performed in the same conditions as above, except that the second source was super-Gaussian. More precisely, p_2 was set to $\frac{1}{10}$, resulting in the following theoretical kurtosis value: $CUM_4(x_2) = 0.6624$. Fig. 4 shows the variations of the imaginary parts of the roots of the polynomial (20) vs a. For any value of a, two of these imaginary parts are zero, i.e. the experimental polynomial has two real roots. The above-defined criterion then leads to the decision that $CUM_4(x_1)$ and $CUM_4(x_2)$ have opposite signs, which again agrees with the above-defined theoretical source kurtosis values. The semilogarithmic representation of the only positive imaginary part leads to a plot[5] situated between those of Fig. 2 and therefore to the same conclusion as above.

4 CONCLUSIONS AND FUTURE WORK

In this paper, we have proposed a method for determining if two source signals are sub- or super-Gaussian in the situation when only (two) linear instantaneous mixtures of these signals are available. We have shown the effectiveness of the proposed approach by means of various types of experimental tests. We plan to investigate extensions of this approach to two cases, which are also of interest from the point of view of their application to blind source separation problems, i.e: i) a larger number of observed and source signals and ii) observed signals which are convolutive mixtures of the considered sources.

[5]Due to space limitations, this chart and the charts associated to the real parts of the roots are not provided in this paper. It should be mentioned that, unlike in the previous series of tests, one of the real roots here takes large values for some values of a, because the highest-order coefficient $CUM_4(y_2)$ of the polynomial (20) may here be very small for some values of a.

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