Positron annihilation in a strong magnetic field

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Strong magnetice field, it means close to the value equal to:

$$B_0 \equiv \frac{c^3}{e}\frac{m_0^2}{\hbar} = 4.414 \times 10^{13}\ \mathrm{G}$$

In that field several interesting phenomena occur:

- *the electron mas is affected by the strong magnetic field, as B increases first it is getting slightly lower than m_0, then slightly higher.*

-*magneic field distorts atoms into long, thin cylinders and molecules into strong, polymer-like chains.*

-*photons can rapidly split and merge with each other.*

- *vacuume itself is unstable, at extremely large $B \sim 10^{51}$-10^{53} G.*

Duncan, arXiv:astro-ph/0002442v1 (2000)

Outline

1. Introduction,
2. Cross section for single and two-photon annihilation proceses
3. Positronium atom in strong magnetic field.
4. Summary

Classical electron gyrating in a magnetic field, satifying the equation:

$$\frac{d\vec{p}}{dt} = \frac{e}{c}\vec{v}\times\vec{B}$$

where $\vec{p} = m_0\vec{v}$

Cyklotron frequency:

$$\varpi_C = \frac{e}{c}\frac{B}{m_0}$$

critical magnetic field straight

$$\hbar\varpi_C = m_0 c^2 \Rightarrow B_0 = \frac{c^3}{e}\frac{m_0^2}{\hbar} = 4.414\times 10^{13}\ \mathrm{G}$$

radius of gyration:

$$a_{gyr} = \frac{c}{e}\frac{p}{B}$$

$$\Rightarrow a_{gyr} = \frac{\hbar}{m_0 c}\sqrt{\frac{B_0}{B}} = 0.386[\mathrm{pm}]\sqrt{\frac{1}{b}}$$

$$a_{gyr}\,p \simeq \hbar$$

$$b = \frac{B}{B_0}$$

Pulsars and magnetars:

	Magnetic field
SGR 1806-20	8×10^{14} G
PSR J1846-0258	4.9×10^{13} G
SGR 0418+5729	$< 7.5 \times 10^{12}$ G
SGR 1900+14	7×10^{14} G
Swift J1834.9-0846	1.4×10^{14} G
SGR 0501+4516	1.9×10^{14} G
CXO J164710.2-455216 [Wd 1]	$<1.5 \times 10^{14}$ G

From McGill SGR/AXP Online Catalog & astro-ph.SR 0910.4859

Positron annihilation or Positronium in a strong magnetic field have been considered by many authors:

-Kroupa and Robl (1954)
-Ternov et aL, 1968
-Wunner (1979)
-Daugherty and Bussard (1980)
-Kaminker et al,1987, 1990
-Herold, Ruder and Wunner (1985)
-Leinson and A. Perez (2000)

The Dirac equation in a magnetic field

$$\left[-\gamma^{\mu}(i\partial_{\mu}+qA_{\mu})-m_0\right]\psi^{(\pm)}(x^{\nu})=0,$$

(We adopted $\hbar = c = 1$ in many equations that follow)

$$A^{\mu}(x)=(0,\vec{A})=(0,0,xB,0)$$

Solution for electrons

$$\psi^{(+)}(x)=\sqrt{\frac{E_n^{(+)}+m_0}{2E_n^{(+)}}}\exp(-iE_n^{(+)}t)\; u_n^{\uparrow\downarrow}(x)$$

$$u_n^{\uparrow}(x)=\begin{pmatrix} I_{n-1} \\ 0 \\ \dfrac{p}{E_n^{(+)}+m_0}I_{n-1}^{(+)} \\ -\dfrac{\sqrt{2nb}\,m_0}{E_n^{(+)}+m_0}I_n^{(+)} \end{pmatrix}, \qquad u_n^{\downarrow}(x)=\begin{pmatrix} 0 \\ I_n^{(+)} \\ -\dfrac{\sqrt{2nb}\,m_0}{E_n^{(+)}+m_0}I_{n-1}^{(+)} \\ -\dfrac{p}{E_n^{(+)}+m_0}I_n^{(+)} \end{pmatrix}$$

$$I_n=\frac{i^n}{L(\pi\lambda_C/b)^{1/4}\sqrt{2^n n!}}\exp\left[-\frac{b(x-a)^2}{2\lambda_C^2}\right]H_n\left(\frac{x-a}{\lambda}\right)\exp\left(-\frac{iayb}{\lambda_C^2}\right)\exp(ipz)$$

Johonsn and Lippmann, PR, **76** (1949) 828,
Bhattacharya, arXiv:0705.4275v2 [hep-th]

Solution for positrons

$$\psi^{(-)}(x) = \sqrt{\frac{E_n^{(-)} + m_0}{2E_n^{(-)}}} \exp(iE_n^{(-)}t)\ \mathrm{v}_n^{\uparrow\downarrow}(x)$$

$$v_n^{\uparrow}(x) = \begin{pmatrix} \dfrac{-p}{E_n^{(-)} + m_0} I_{n-1} \\ \dfrac{-\sqrt{2nb}\, m_0}{E_n^{(-)} + m_0} I_n \\ I_{n-1} \\ 0 \end{pmatrix}, \qquad v_n^{\downarrow}(x) = \begin{pmatrix} -\dfrac{\sqrt{2nb}\, m_0}{E_n^{(-)} + m_0} I_{n-1} \\ \dfrac{p}{E_n^{(-)} + m_0} I_n \\ 0 \\ I_n^{(-)} \end{pmatrix},$$

$$I_n = \frac{i^n}{L(\pi\lambda_C / b)^{1/4}\sqrt{2^n n!}} \exp\left[-\frac{b(x-a)^2}{2\lambda_C^2}\right] H_n\left(\frac{x-a}{\lambda}\right) \exp\left(-\frac{iayb}{\lambda_C^2}\right) \exp(ipz)$$

Landau levels

The transverse motion of the positron (e+) and electron (e-) is quantized, they energy obeys the relation :

$$E_n^{(\pm)} = \sqrt{m_0^2 + (p^{(\pm)})^2 + 2n_\pm\, b\, m_0^2}$$

$$n_\pm = 0, 1, 2, \ldots$$

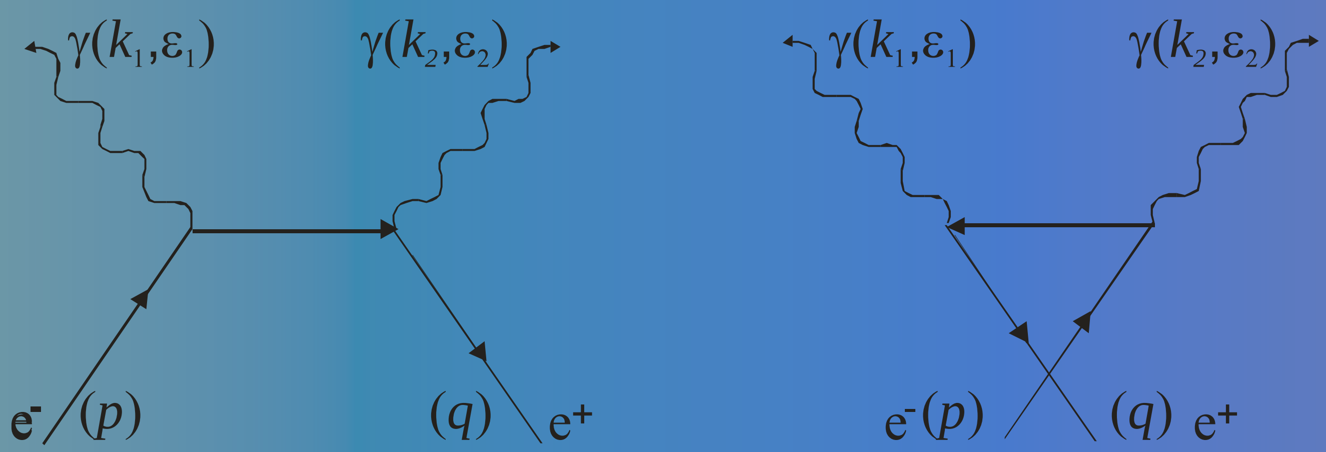
is the number of the Landau level,

p - is the longitudinal momentum, along the magnetic field line.

In calculations it is sufficient to assume: n=0

Two-photon annihilation

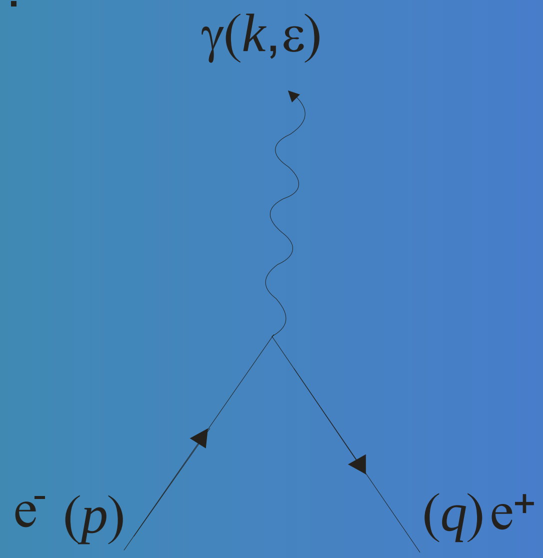
Positron annhilation is related to two photons emission



$$\text{cross section} \approx \alpha^2$$

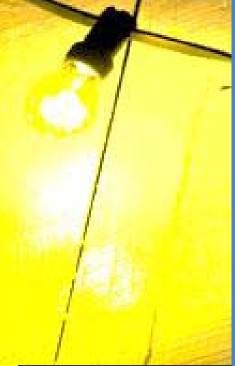
Single photon annihilation

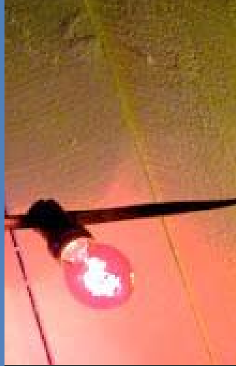
In a field *B* close to B_0 the single photon annihilation is possible:



(In free space this process is forbidden.)

$$\text{cross section} \approx \alpha$$





Differential cross section for single photon annihilation

(In the center of mass (CM) system)

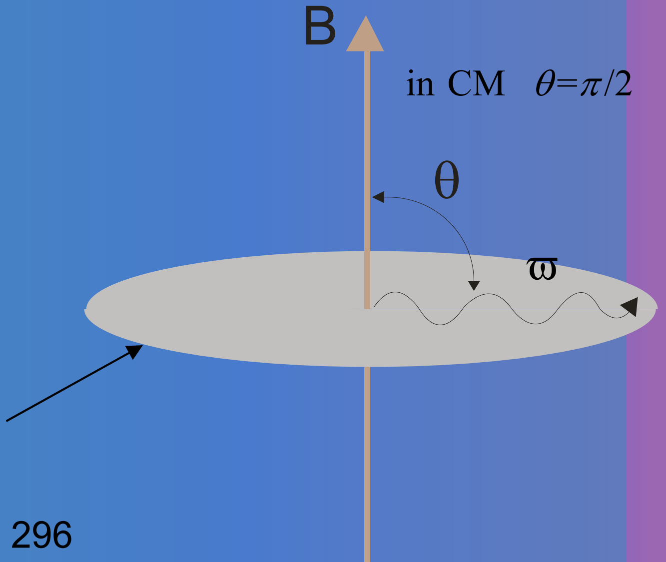
$$\frac{d^2\sigma_{1\gamma}}{d\Omega d\omega} = \frac{\alpha\lambda_C^2}{2}\frac{m_0^2}{\sqrt{E_0^2 - m_0^2}\,E_0}\frac{\varpi}{b}(1-\cos^2\theta)\exp\left[-\frac{(1-\cos^2\theta)}{2b}\left(\frac{\varpi}{m_0}\right)^2\right]\delta(\varpi\cos\theta)\,\delta(2E_0-\varpi)$$

E_0 is the total energy of e^+

$$\lambda_C = 2.426\times 10^{-12}\,\mathrm{m}$$

is the Compton wavelenght

For annihilation at rest,
due to the delta function, it indicates the line of emission of the photon perpendicular to the magnetic field line.



a fan beam

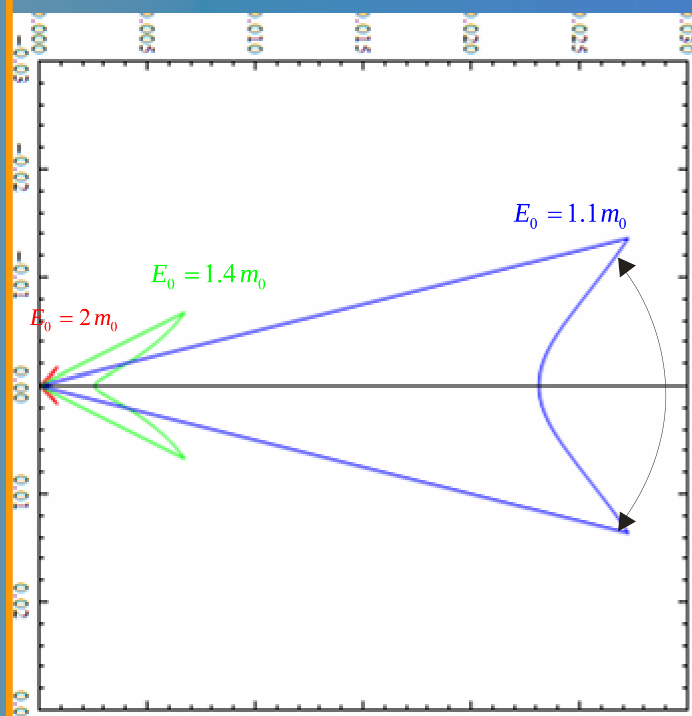
Daugherty and Bussard, The Astrophysical Journal, 238 (1980) 296

Differential cross section for single photon annihilation

$$\frac{d\sigma_{1\gamma}}{d\Omega}=\frac{\alpha\lambda_C^2}{2b}\frac{m_0^2}{\sqrt{E_0^2-m_0^2}\,E_0}\exp\left[-\frac{2\sin^2\theta}{b}\left(\frac{E_0}{m_0}\right)^2\right]\Theta\left[2E_0-\frac{2m_0}{\sin\theta}\right]\sin^2\theta$$

direction of magnetic line

b=1



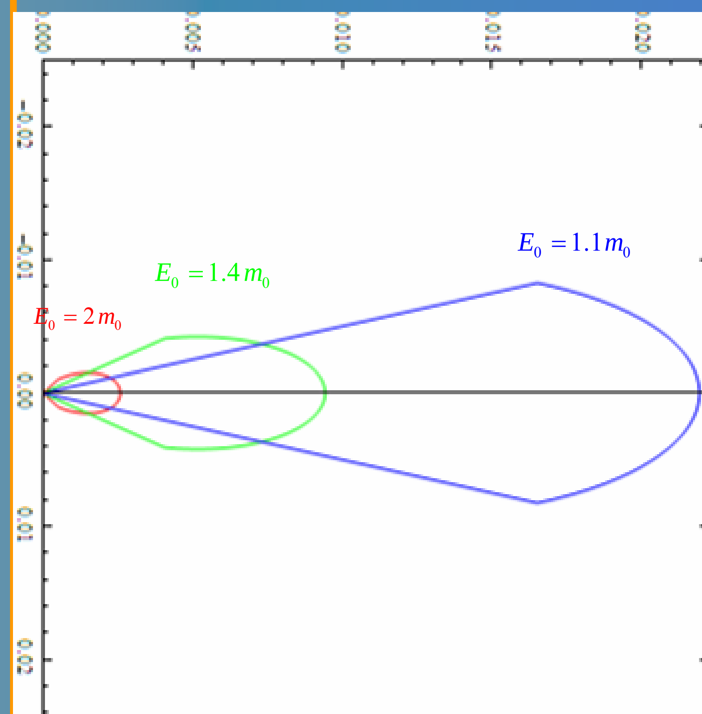
$$\Omega=\pi-2\arcsin\frac{m_0}{E_0}$$

Differential cross section for single photon annihilation

$$\frac{d\sigma_{1\gamma}}{d\Omega} = \frac{\alpha\lambda_C^2}{2b} \frac{m_0^2}{\sqrt{E_0^2 - m_0^2}\,E_0} \exp\left[-\frac{2\sin^2\theta}{b}\left(\frac{E_0}{m_0}\right)^2\right] \Theta\left[2E_0 - \frac{2m_0}{\sin\theta}\right]\sin^2\theta$$

direction of magnetic line

b=10



$$\Omega = \pi - 2\arcsin\frac{m_0}{E_0}$$

Total cross section for single photon annihilation.

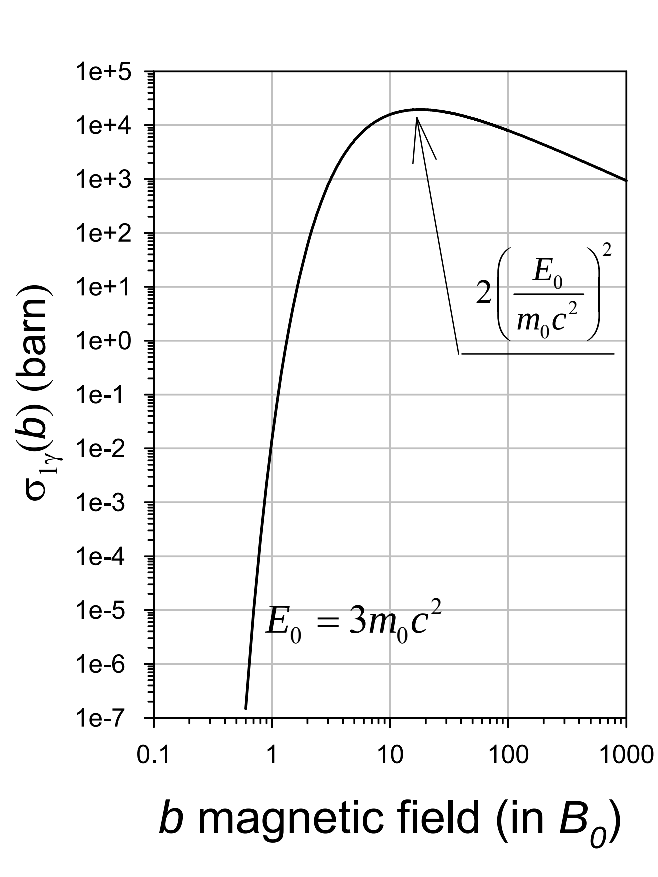
$$\sigma_{1\gamma} = \frac{\alpha\lambda_C^2}{2}\frac{m_0^2}{\sqrt{E_0^2 - m_0^2}\,E_0}\frac{1}{b}\exp\left[-\frac{2}{b}\left(\frac{E_0}{m_0}\right)^2\right]$$

(*n*=0, lowest Landau level)

E_0 is the total energy of e^+

$$\lambda_C = 2.426\times10^{-12}\,\mathrm{m}$$

is the Compton wavelenght

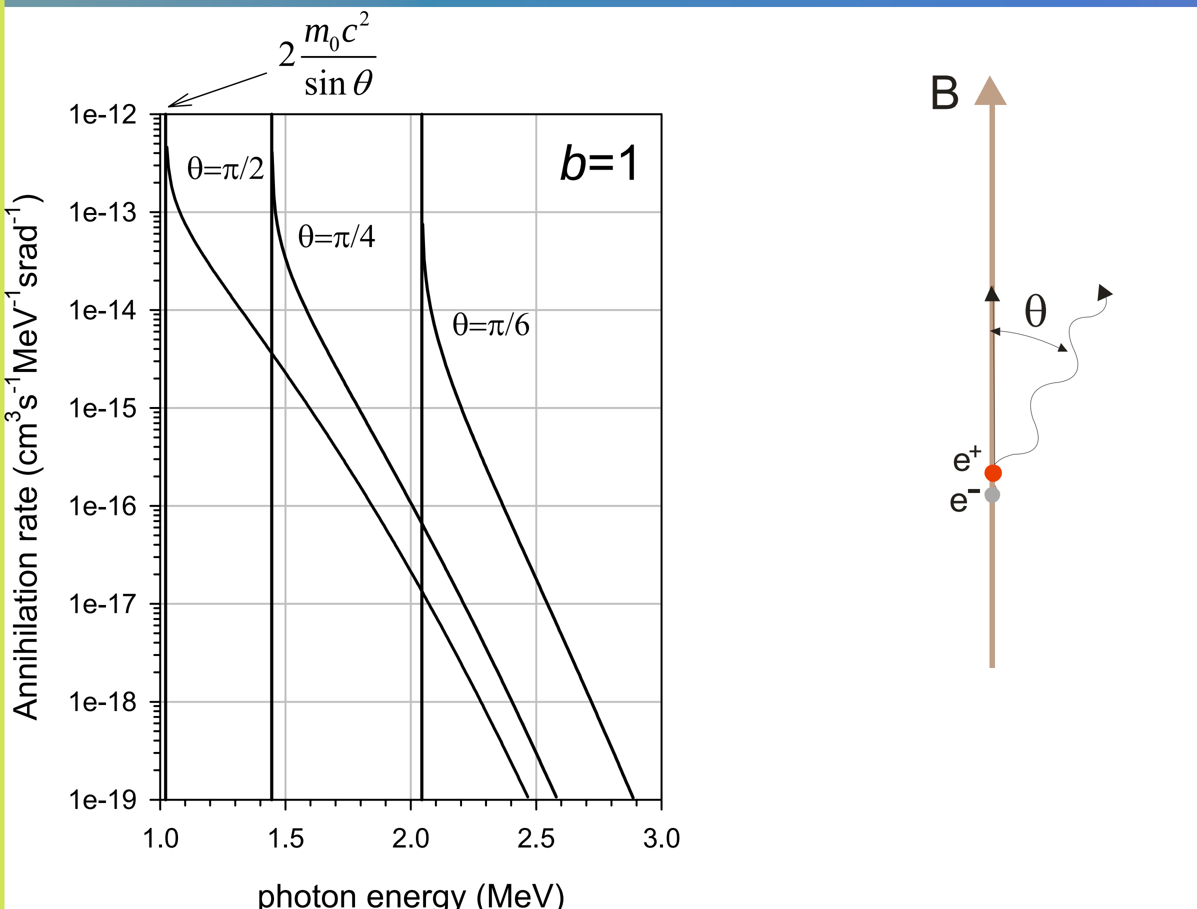


Wunner PRL 42 (1979) 82.

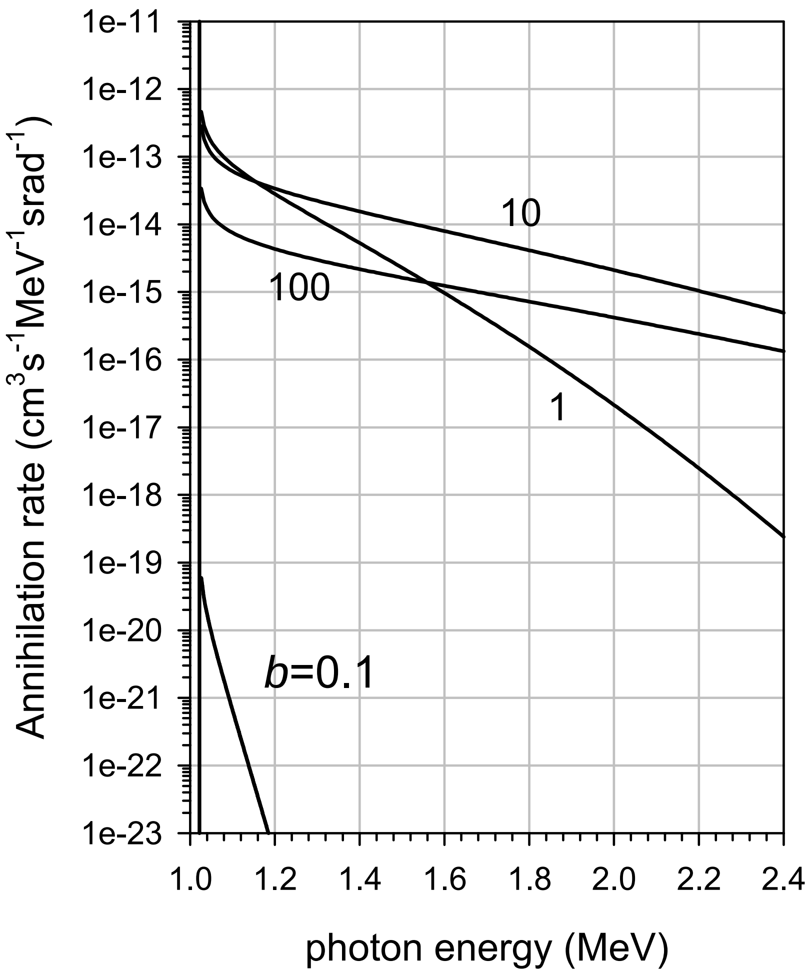
The single photon annihilation spectra for different angles related to the direction of a magnetic field.

Annihilation rate

$$\lambda \sim n_0^+ n_0^- \int dp^+ \eta(p^+) \int dp^- \eta(p^-) \left| \frac{p^+}{\sqrt{(p^+)^2 + m_0^2}} + \frac{p^-}{\sqrt{(p^-)^2 + m_0^2}} \right| d\sigma_{1\gamma}$$



The single photon annihilation spectra for different magnetic field strength.



Total cross section for two-photon annihilation.

$$\sigma_{2\gamma} = \frac{\sqrt{\pi}\,\alpha^2\lambda_C^2}{2}\frac{m_0^3E_0^2}{\sqrt{E_0^2-m_0^2}}\frac{1}{\sqrt{b}}\exp\left[-\frac{1}{b}\left(\frac{E_0}{m_0}\right)^2\right]\mathrm{Erfi}\left[\frac{1}{\sqrt{b}}\left(\frac{E_0}{m_0}\right)\right]$$

(n=0, lowest Landau level)

limit for small E_0 $\quad \sigma_{2\lambda} \cong \dfrac{\alpha^2\lambda_C^2 m_0}{\sqrt{E_0^2-m_0^2}}$

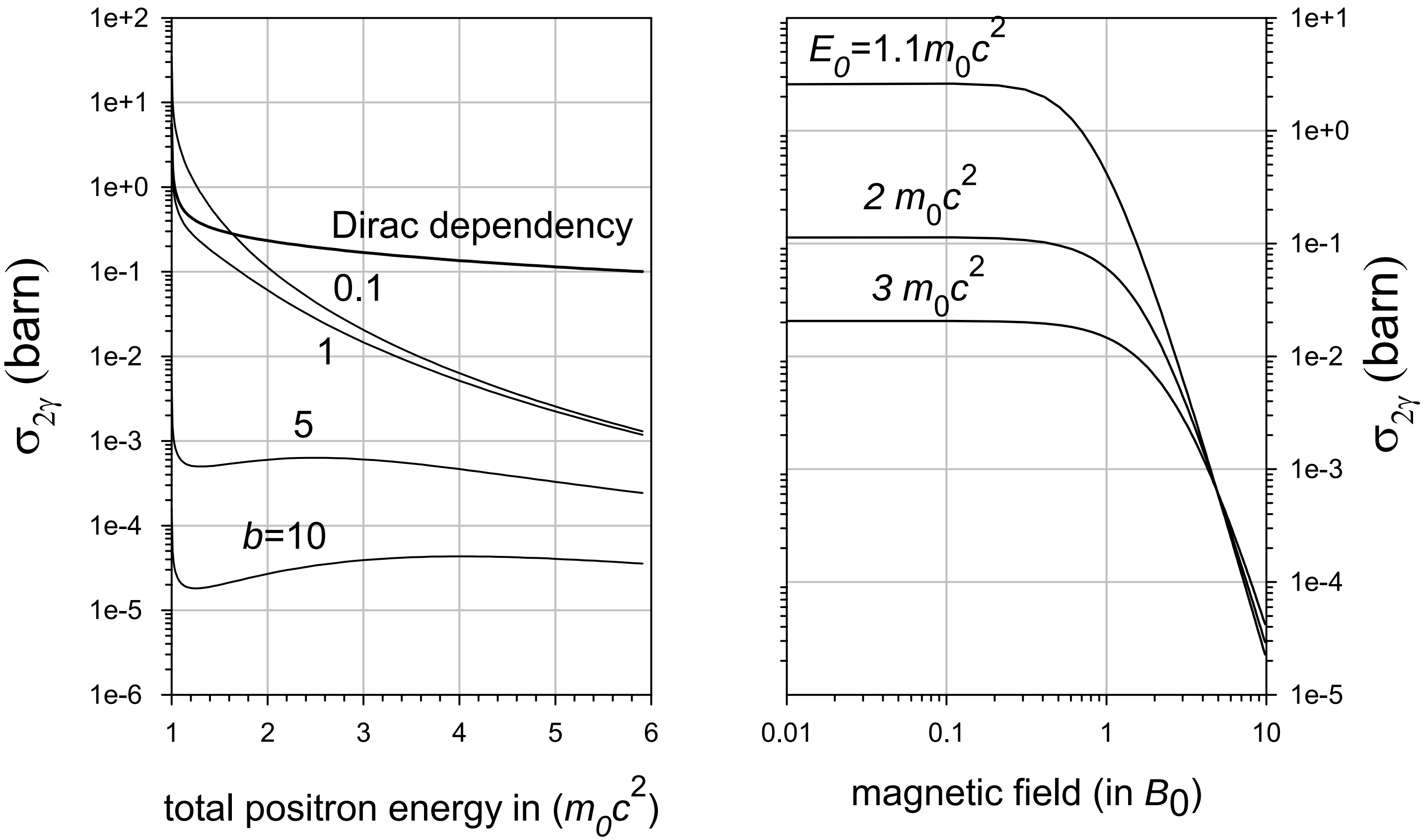
problem for small *b* or *b=0*

Total cross section for two-photon annihilation for free particles(Dirac formula) :

$$\sigma_{2\gamma} = \frac{1}{4\pi}\frac{\alpha^2\lambda_C^2 m_0}{(E_0+m_0)\left(E_0^2-m_0^2\right)}\left[(E_0-m_0)(E_0+5m_0)\ln\left(\frac{E_0+\sqrt{E_0^2-m_0^2}}{m_0}\right)-(E_0+3m_0)\sqrt{E_0^2-m_0^2}\right]$$

limit for small E_0: $\quad \sigma_{2\lambda} \cong \dfrac{1}{4\pi}\dfrac{\alpha^2\lambda_C^2 m_0}{\sqrt{E_0^2-m_0^2}}$

Total cross section for two-photon annihilation.

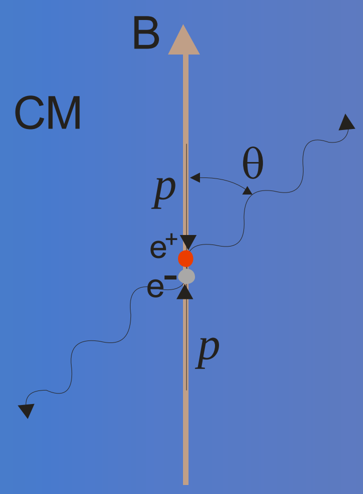


Differential cross section for two-photon annihilation in CM

(n=0, lowest Landau level)

$$E_0 = \sqrt{p^2 + m_0^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \lambda_C^2}{\pi} \frac{E_0^3 m_0^2}{p\left(E_0^2 + 2bm_0^2\right)^2} \frac{1}{b} \exp\left(-\frac{E_0^2}{bm_0^2} \sin^2\theta \right)$$

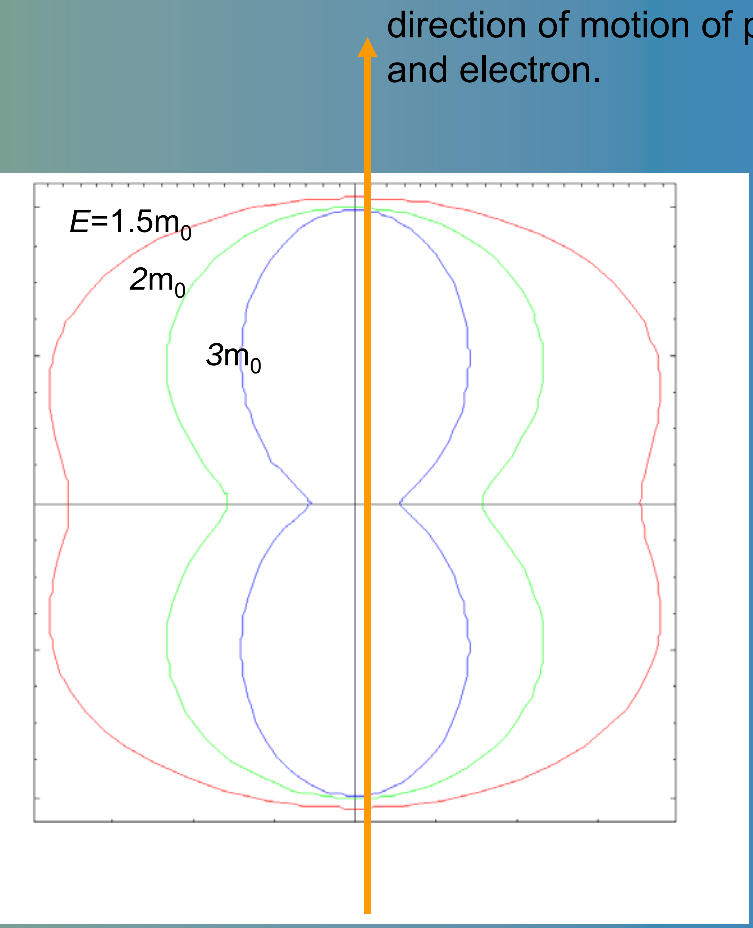


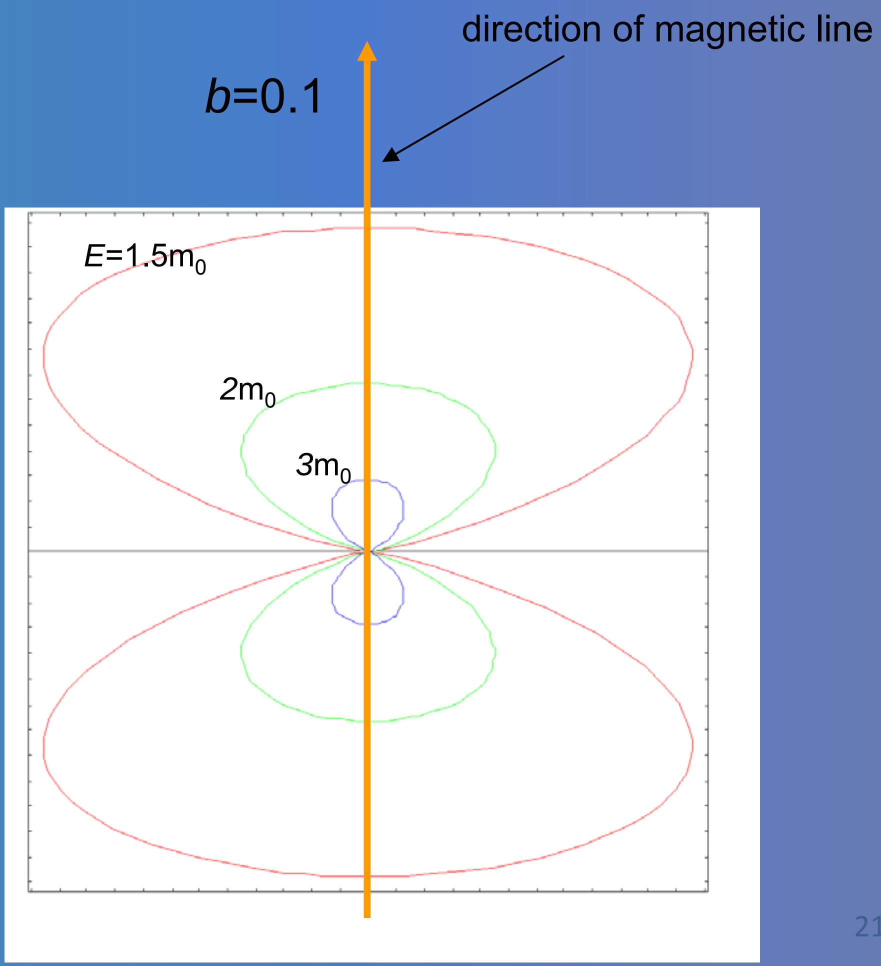
Dirac (Heitler) formula for free annihilation:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\mathrm{D,CM}} = \frac{\alpha^2 \lambda_C^2}{16\pi} \frac{m_0^2}{E_0\, p} \left[\frac{E_0^2 + p^2 + p^2 \sin^2\theta}{E_0^2 - p^2 \cos^2\theta} - \frac{2p^4 \sin^4\theta}{(E_0^2 - p^2\cos^2\theta)^2} \right]$$

Differential cross section for two-photon annihilation in CM system.

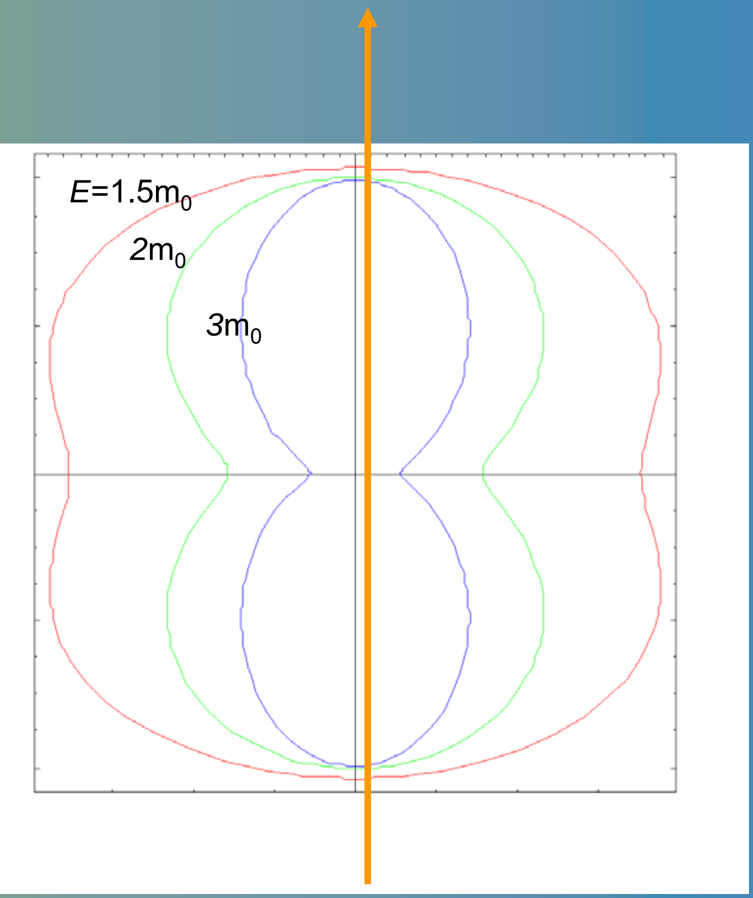
Free particles in **CM (Dirac formula)**





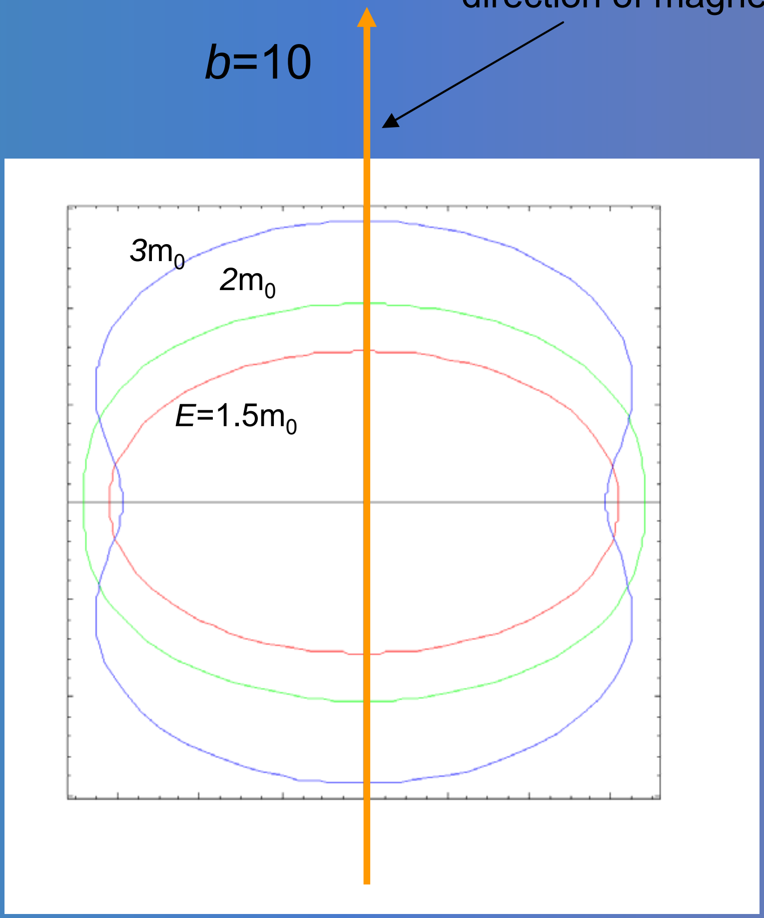
Differential cross section for two-photon annihilation in CM system.

Dirac formula in CM



direction of magnetic line

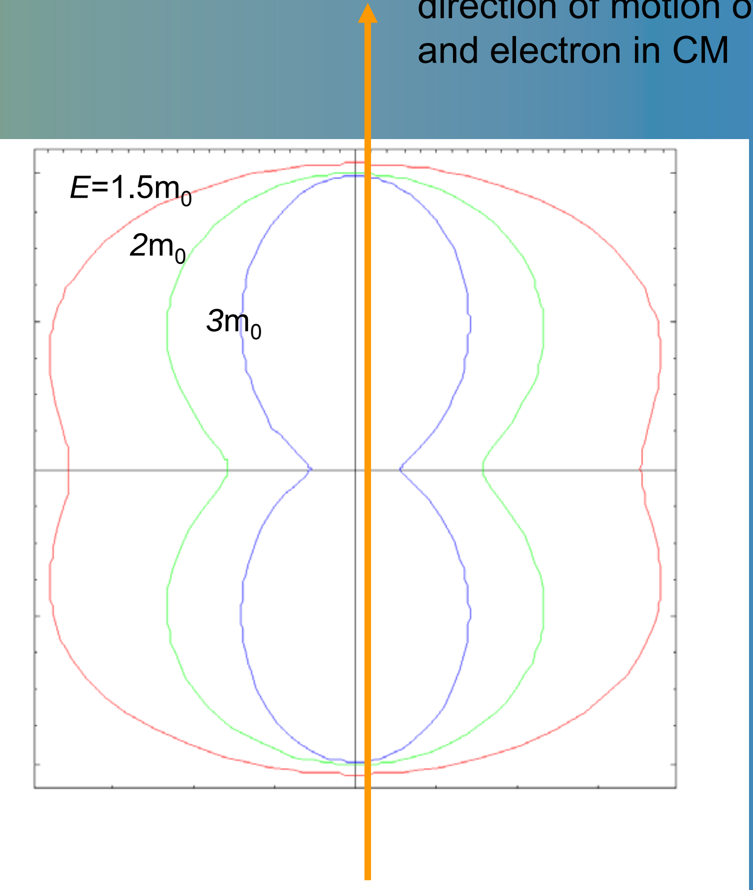
b=10



Differential cross section for two-photon annihilation in CM system.

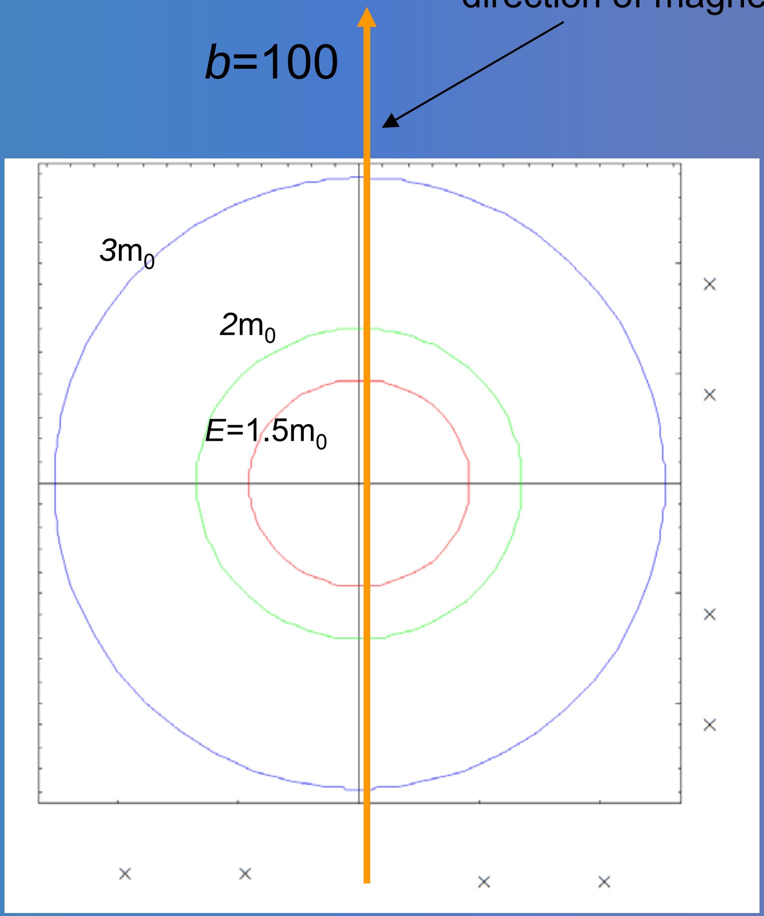
Dirac formula in CM

direction of motion of positron and electron in CM



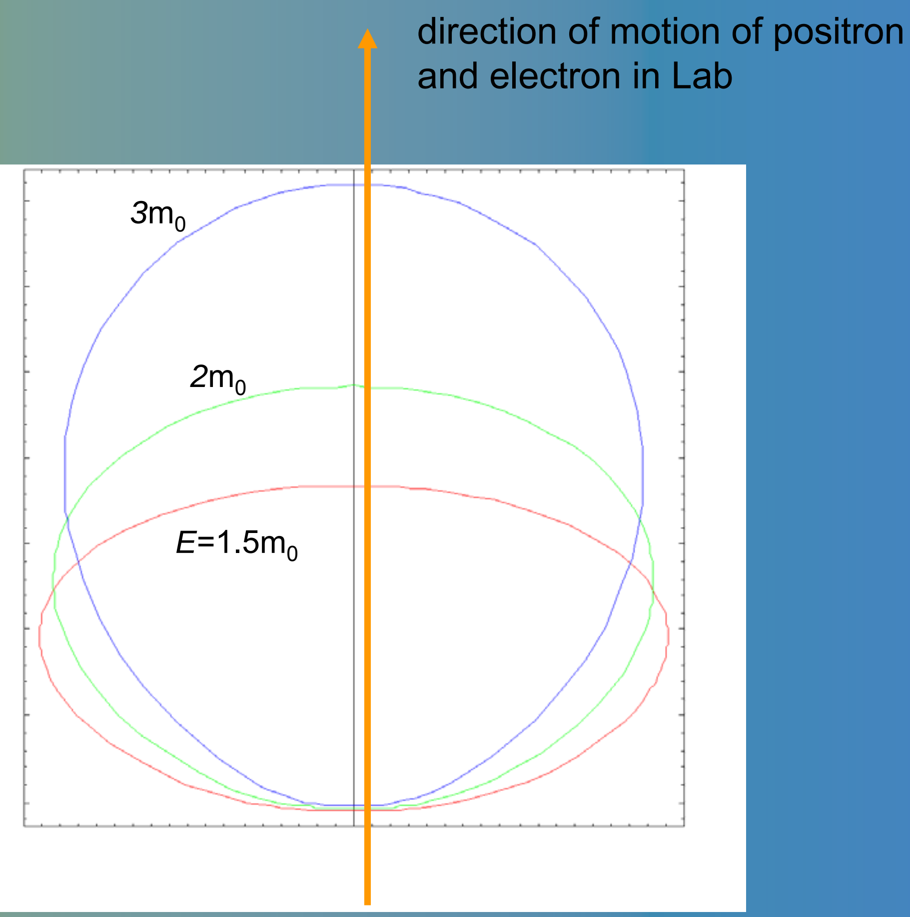
direction of magnetic line

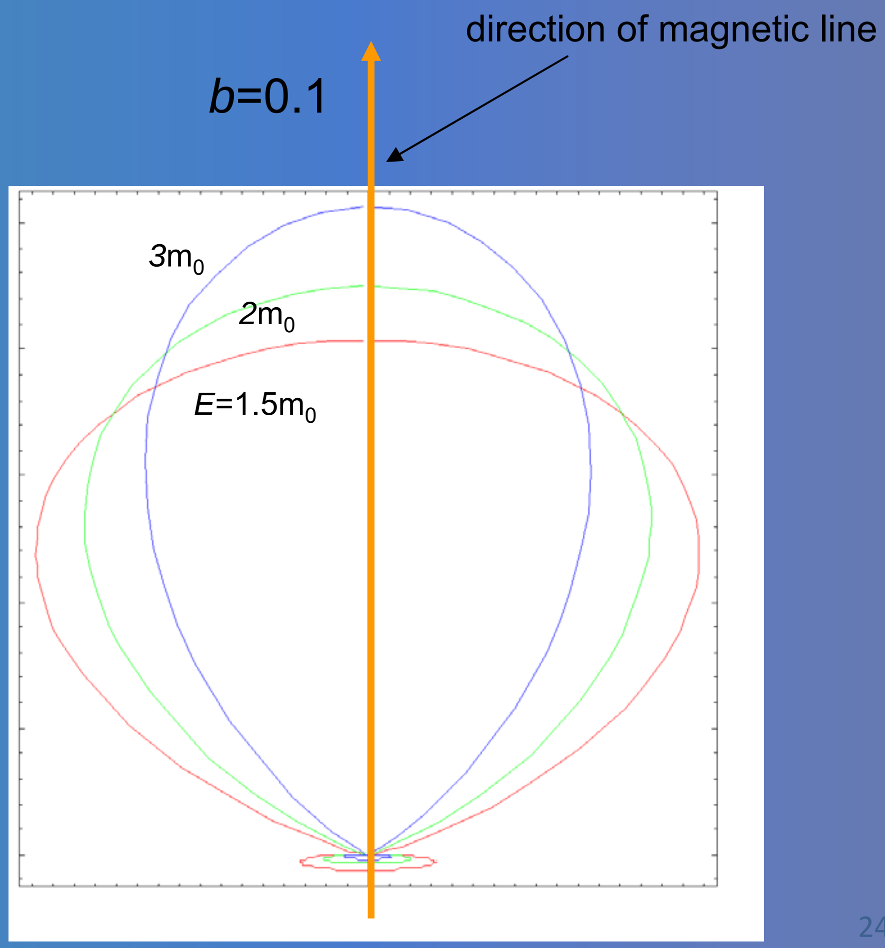
b=100



Differential cross section for two-photon annihilation in Laboratory system.

Dirac formula in Lab

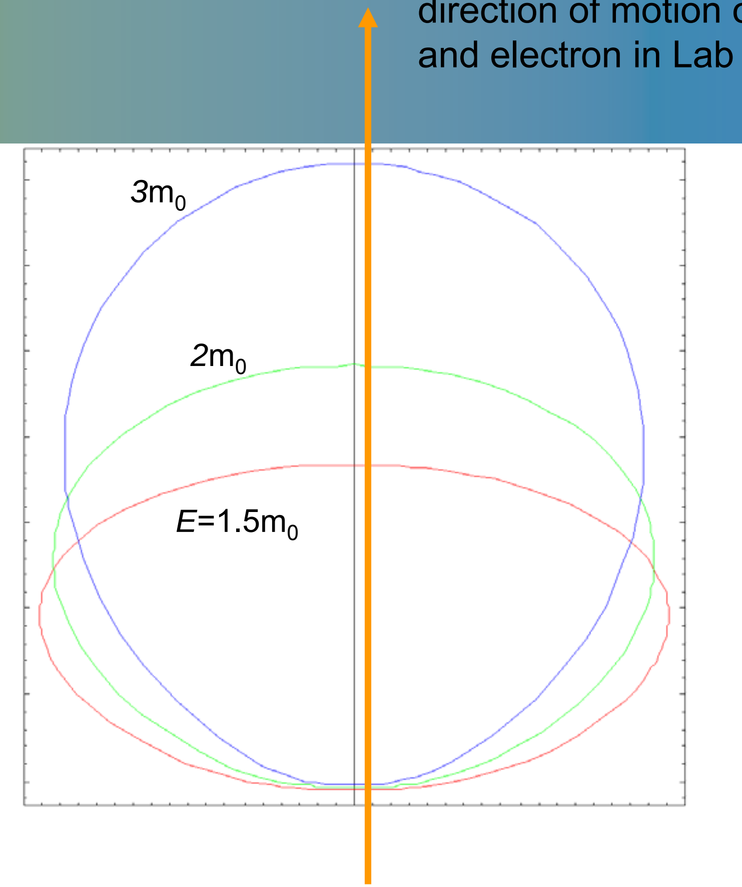




Differential cross section for two-photon annihilation in laboratory system

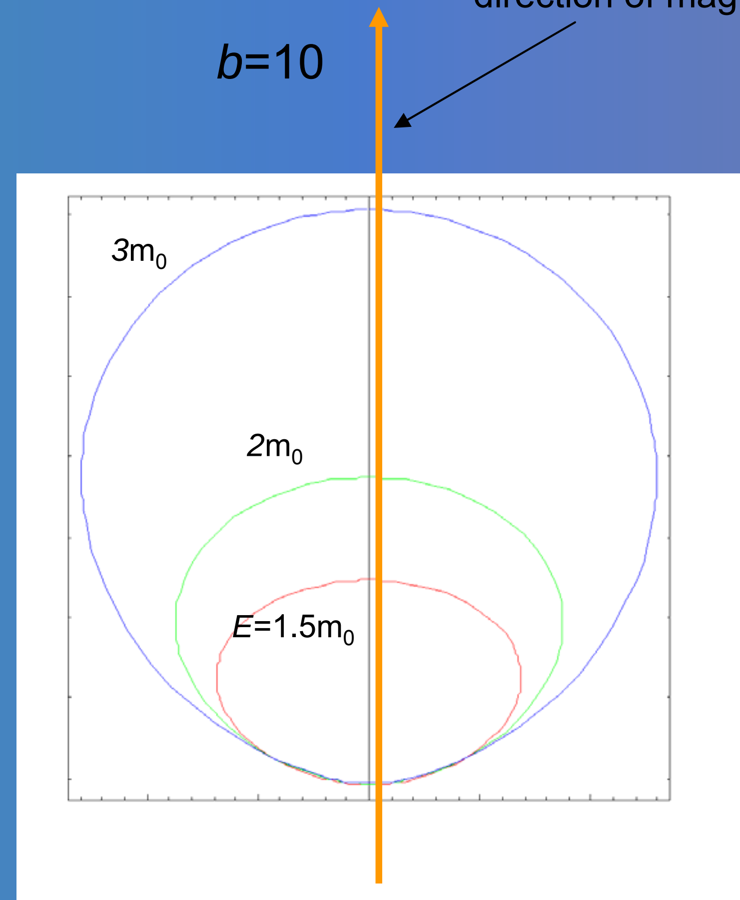
Free particles in **Lab (Dirac formula)**

direction of motion of positron and electron in Lab

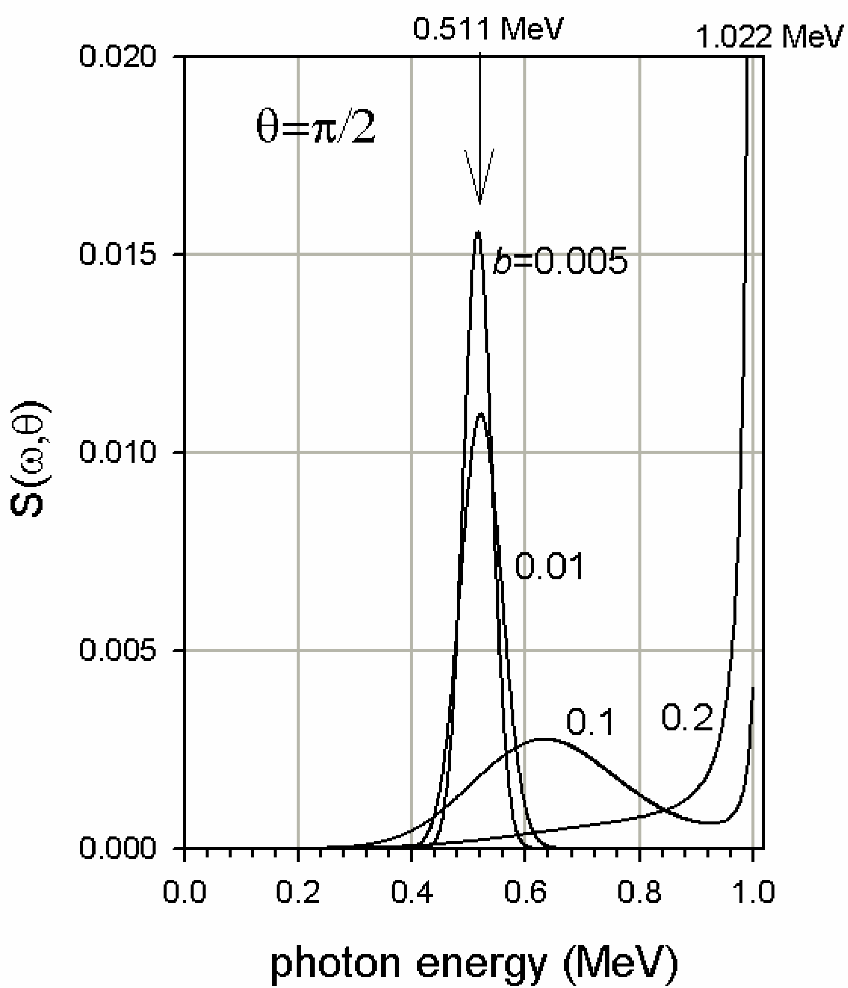


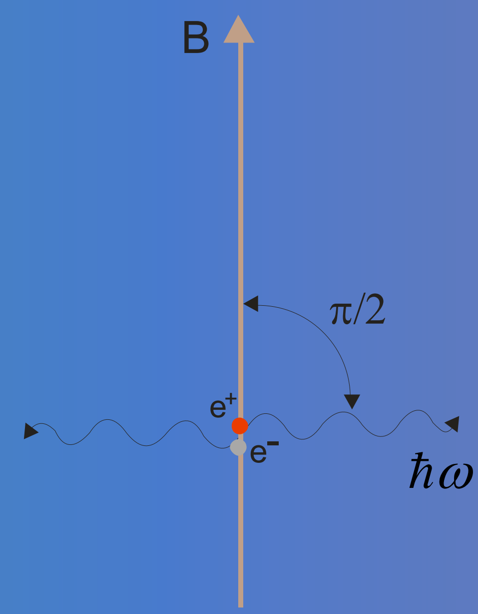
direction of magnetic line

b=10

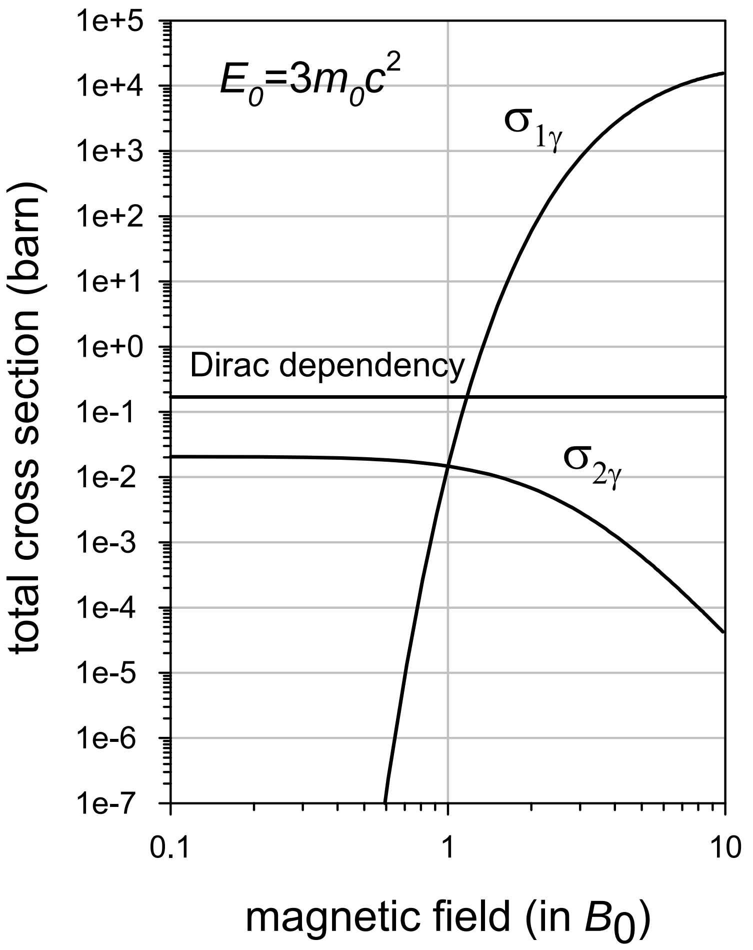


The spectrum of the annihilation line (511 keV) for annihilation at rest as seen perpendicular to the magnetic field direction.

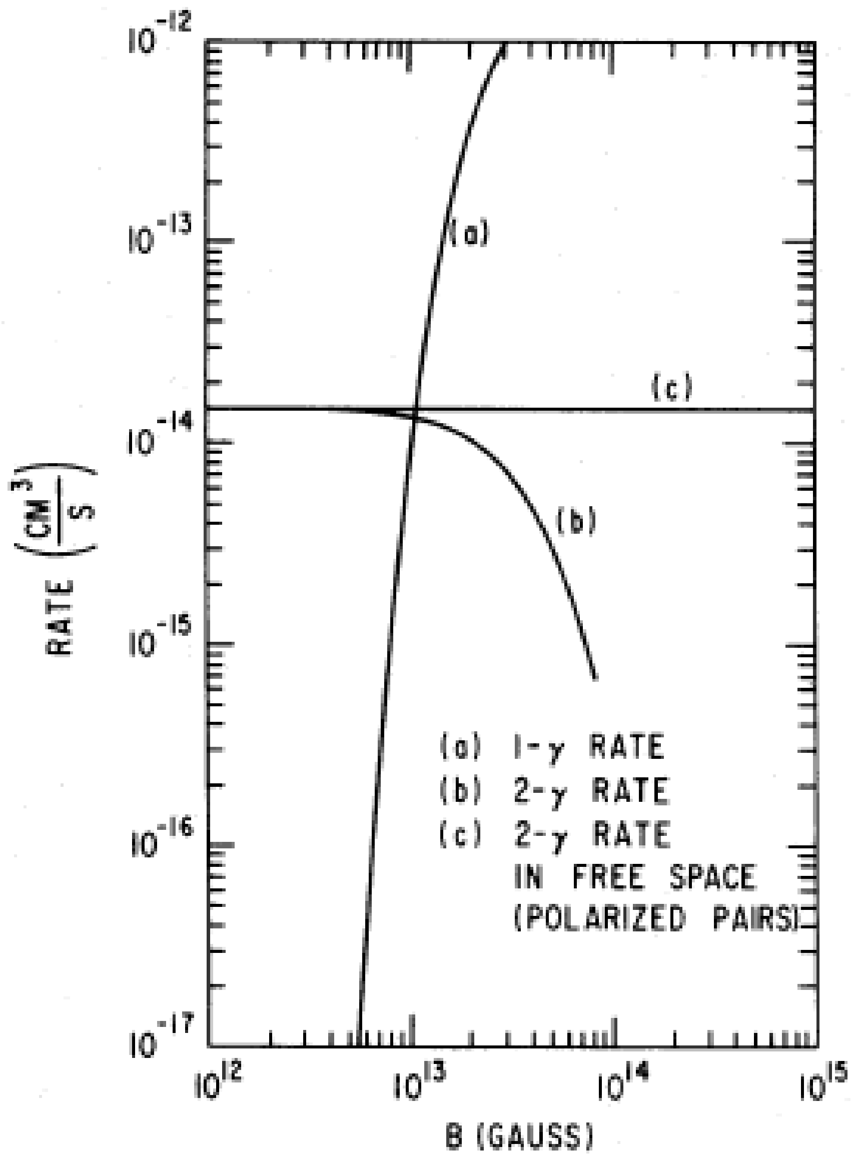




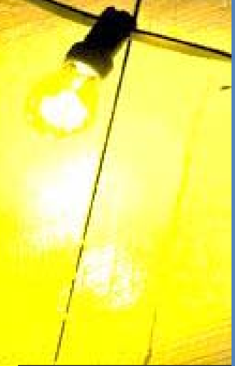
Comparison



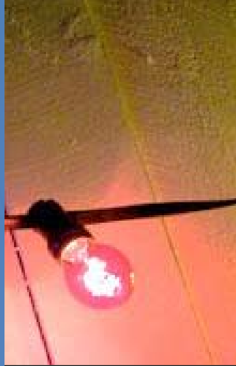
$E_0=3m_0c^2$
$\sigma_{1\gamma}$
$\sigma_{2\gamma}$
Dirac dependency
total cross section (barn)
1e+5
1e+4
1e+3
1e+2
1e+1
1e+0
1e-1
1e-2
1e-3
1e-4
1e-5
1e-6
1e-7
0.1
1
10
magnetic field (in B_0)



Daugherty and Bussard, The Astrophysical Journal, 238 (1980) 296



In „annihilation in flight” process positron or electron contributs to the energy of emitted photons

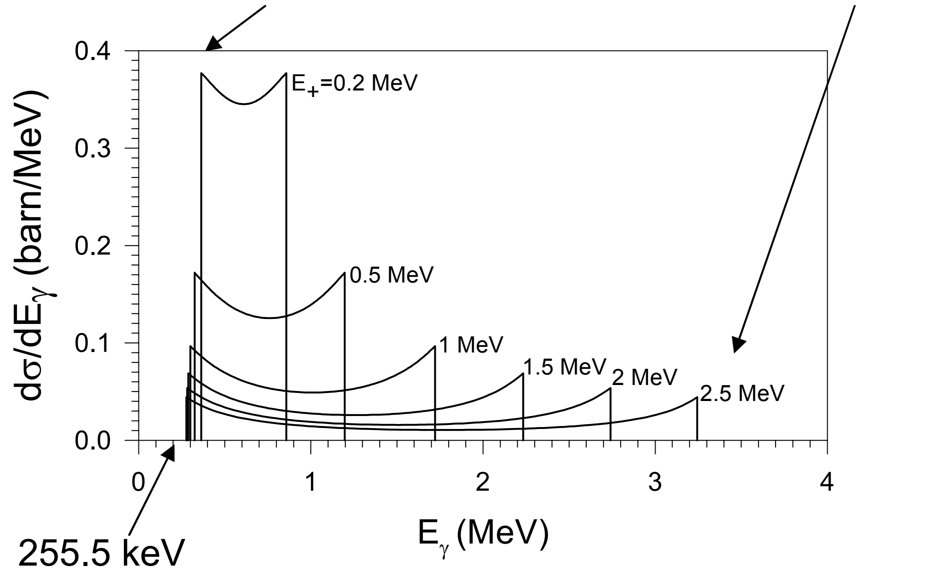


Free particles in Lab (From Dirac formula)

$$\frac{d\sigma_{2\gamma}}{dE_\gamma} = -\frac{2\pi r_o^2 m_0}{E-m_0}\left[\frac{1}{E+m_0} - \frac{1}{2}\frac{E+3m_0}{E_\gamma(E+m_0-E_\gamma)} + \frac{1}{2}\frac{E+m_0}{E_\gamma{}^2(E+m_0-E_\gamma)^2}\right]$$

$$E_\gamma^{(\min)} = \frac{(E+m_0)m_0}{E+m_0+\sqrt{E^2-m_0^2}} \simeq \frac{1}{2}m_0 \qquad E_\gamma^{(\max)} = \frac{(E+m_0)m_0}{E+m_0-\sqrt{E^2-m_0^2}} \simeq E+\frac{1}{2}m_0$$

,



Spectra of annihilation in flight photon

Kontrovich et al., The 10th International Symposium on Origin of Matter and Evolution of Galaxis (OMEG-2010)

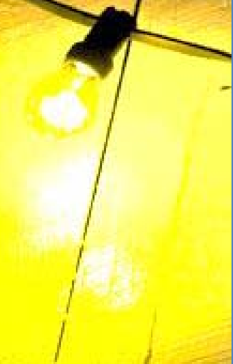
Positronium Ps in a strong magnetic field. The problem can be attacked by solution of the Bethe-Salpeter equation.
(Leinson and Perez: JHEP11(2000)039 **)**

The Positronium wave function:

$$\hat{\phi}_{00l}^{P_2^0}(\mathbf{r}) = f_{\downarrow\uparrow}^{(0,l)}(z,x_0)\sqrt{\frac{m_0^2 b}{2\pi}}\exp\left\{-\frac{m_0^2 b}{4}[(x+x_0)^2+y^2]\right\}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}$$

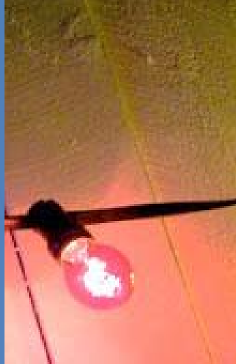
$$-\frac{1}{2\mu_{00}}\frac{\partial^2}{\partial z^2}f_{\downarrow\uparrow}^{(0,l)}(z,x_0)+V_{\downarrow\uparrow}^{\downarrow\uparrow}(z,x_0)\,f_{\downarrow\uparrow}^{(0,l)}(z,x_0)=\varepsilon_{00}^{l}(x_0)\,f_{\downarrow\uparrow}^{(0,l)}(z,x_0)$$

where $V_{00}(z, x_0)$ is an effective potential expressed by the elliptic integral of the first kind and $x_0 = P_2^0 / m_0^2 b$ (P_2^0 - is the transverse momentum characterizing the motion of the mass center across the magnetic field)



The case of small x_0 means the small distances between the centers of Landau orbits in the plane orthogonal to the magnetic field, then the effective potential reads:

$$V_{00}(z) \cong \frac{e^2}{|z| + \sqrt{\dfrac{2b}{\pi m_0^2}}}$$



$$f(z) \equiv f_{\downarrow\uparrow}^{(0,0)}(z,0) = \delta \exp\left(-\delta^2 |z|\right)$$

$$\rho(P_x, P_y, P_u) = \left|\hat{\Phi}(P_x, P_y, P_u)\right|^2 = \frac{4\delta^6\alpha^2}{\pi^2 b} \frac{\exp\left(-2(P_x^2 + P_y^2)/m_0^2 b\right)}{((P_u\, m\, \alpha)^2 + \delta^4)^2}$$

$$\delta \equiv \sqrt{\alpha \frac{m_0}{2} \ln\left(\sqrt{\frac{2}{\pi}} \frac{2\sqrt{b}}{\alpha}\right)}$$

The momentum spectrum of annihilation radiation

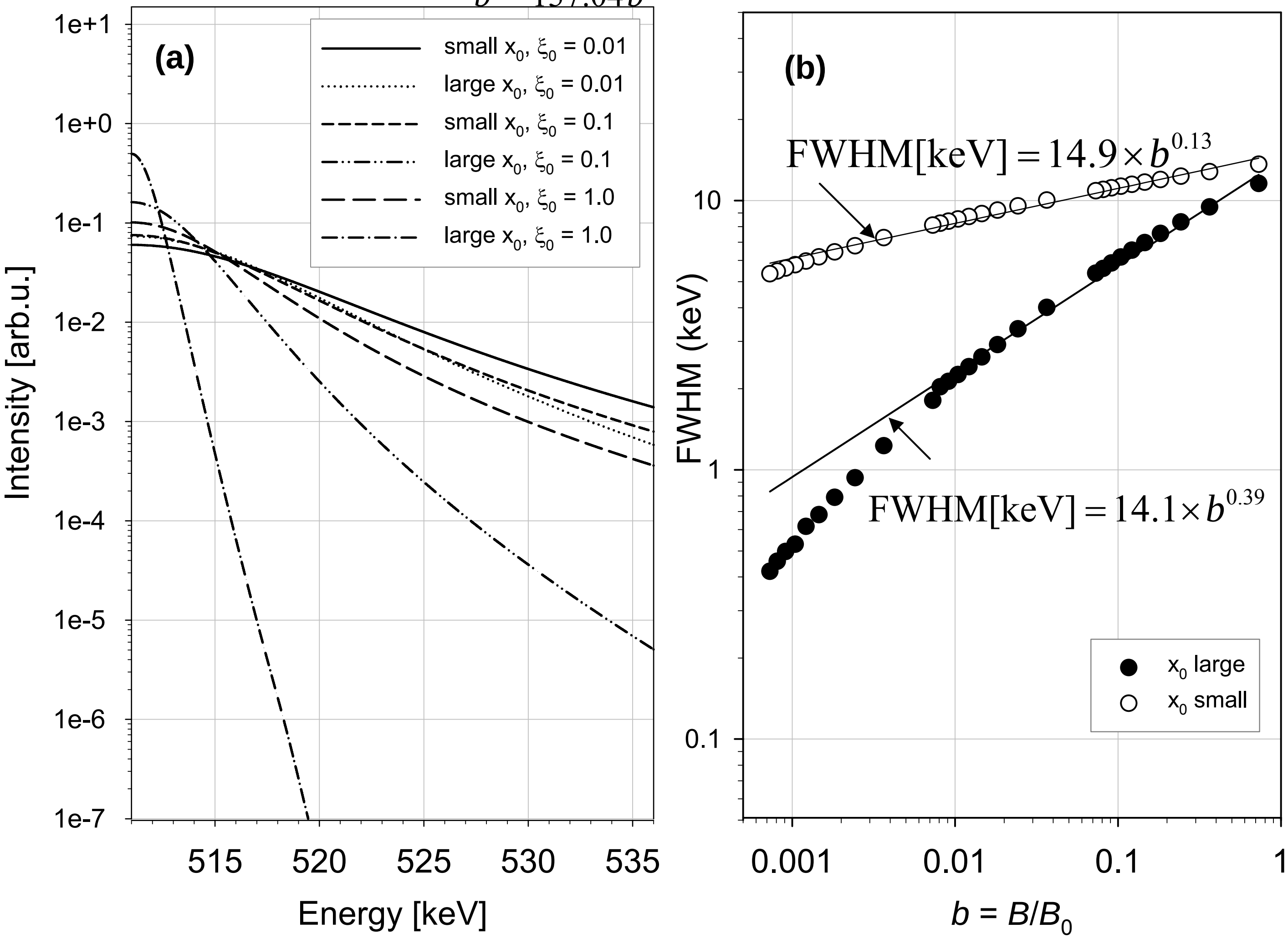
$$N(P_i) = \int dP_{j\neq i,k} \int dP_{k\neq i,j}\, \rho(P_i, P_j, P_k); \quad i, j, k \in \{x, y, z\}$$

$$N(P_{x,y}) = \sqrt{\frac{2}{\pi m_0^2 b}} \exp\left\{-\frac{2P_{x,y}^2}{m_0^2 b}\right\}, \qquad N(P_z) = \frac{2\delta^6}{\pi(\delta^4 + P_z^2)^2}.$$

Intensity of the annihilation line versus the photon energy

c

$$\xi_0 = \frac{\alpha}{b} = \frac{1}{137.04\,b}$$



Lewicka and Dryzek, Materials Sciene Forum, 666 (2011) 31

The case of large x_0. The created Positronium has a quasi momentum of the order of $2m_0$, moreover, a typical pulsar magnetic field $b \leq 0.1$ giving the limit of large x_0. In such a case the effective potential $V_{00}(z)$ takes on the analytical form:

$$V_{00}\left(u = m\alpha\, z\right) = -\frac{1}{\sqrt{(x_0\, m_0\alpha)^2 + u^2}}$$

Ground states of Positronium in strong magnetic field as a function of *b*, calculated from the Schrödinger-like equation in the case of large x_0.

Magnetic field in *b* unit	Bounding energy E_b (eV)
0 (vacuum)	6.80
0.0073	9.13
0.073	51.9
0.73	197.2

Annihilation rate of positronium dependent on the magnetic field *b* (compared to the values obtained by Wunner and Herold [3] in the non-relativistic regime).

B [G]	**$\lambda[s^{-1}]$** λ (for small x_0)	**$\lambda[s^{-1}]$ (Wunner and Herold results)**	**$b = B/B_0$**	**$\lambda[s^{-1}]$** λ (for large x_0)
$1 \cdot 10^{12}$	$7.40 \cdot 10^{11}$	$7.2 \cdot 10^{12}$	0.729927 (ξ_0=0.01)	$3.8457 \cdot 10^{11}$ $\tau = 0.003$ ns
$3 \cdot 10^{12}$	$2.53 \cdot 10^{12}$	$2.5 \cdot 10^{13}$	0.226552 ($B=10^{13}$G)	$1.81645 \cdot 10^{8}$ $\tau = 5.5$ ns
$5 \cdot 10^{12}$	$4.46 \cdot 10^{12}$	$4.2 \cdot 10^{13}$	0.243309 (ξ_0=0.03)	$3.57567 \cdot 10^{8}$ $\tau = 2.8$ ns
$1 \cdot 10^{13}$	$9.56 \cdot 10^{12}$	$7.9 \cdot 10^{13}$	0.145985 (ξ_0=0.05)	$7.22707 \cdot 10^{5}$ $\tau = 1.4\ \mu s$

in vaccum

$$\lambda_{exp}\left(1^1S_0 \rightarrow 2\gamma\right) = \left(7.9909 \pm 0.0017\right) 10^9$$

$$\tau = 0.125 \text{ ns}$$

Wunner and Herold: Astrophys. Space Sci. 63 (1979) p. 503

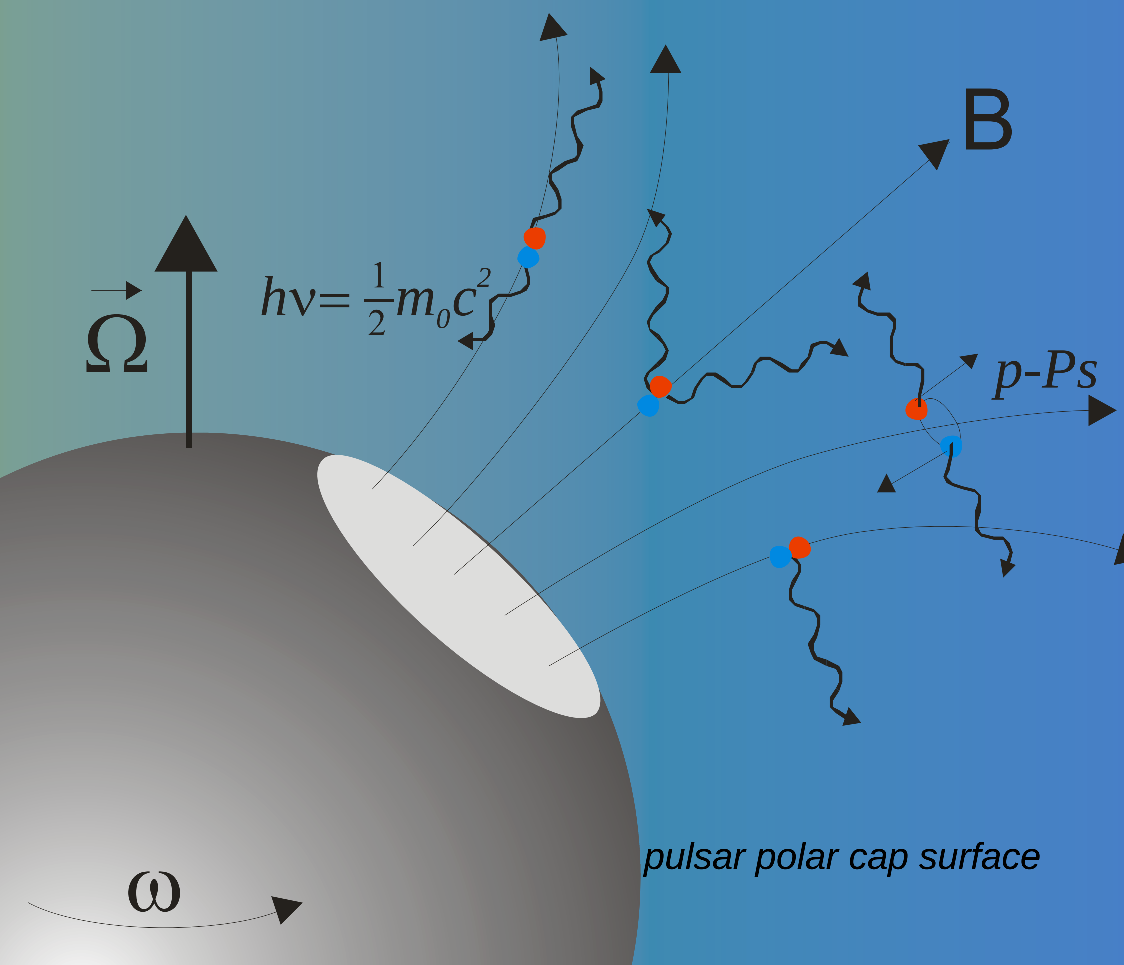
Daugherty and Bussard suggested to using the narrow widths of the annihilation line (two photons annihilation) to place an upper limit on the magnetic field strength.

$$\left.\frac{\Delta E_{\gamma}}{E_{\gamma}}\right|_{FWHM} \sim 0.35\,b$$

In the case of Positronium formation it could be as follows:

$$\mathrm{FWHM[keV]} = 14.9 \times b^{0.13} \qquad \mathrm{FWHM[keV]} = 14.1 \times b^{0.39}$$

Summary



Possible annihilation channels
in a strong magnetic field:

-single photon (most important signature the line of 1022 keV)
-two-photon (not important for *b*>1)
-annihilation in flight
(**signature the line of 255,5 keV**)
-Positronium lifetime is much longer than free particle.

Cross sections depends
on the magnetic field strength.

Possible use for determination
of the upper limit of the magnetic
field strenght.

Thank you for your attention