Contents

1 The Foundations of Fluid Mechanics 1
1.1 A Short Historical Perspective 1
1.2 The Concept of a Fluid 2
  1.2.1 Introduction 2
  1.2.2 Continuous Media 2
1.3 Fluid Kinematics 3
  1.3.1 The Concept of Fluid Particle 3
  1.3.2 The Lagrangian View 3
  1.3.3 The Eulerian View 4
  1.3.4 Material Derivatives 4
  1.3.5 Distortion of a Fluid Element 5
  1.3.6 Incompressible Fluids 8
  1.3.7 The Stream Function 9
  1.3.8 Evolution of an Integral Quantity Carried by the Fluid 10
1.4 The Laws of Fluid Motion 11
  1.4.1 Mass Conservation 11
  1.4.2 Momentum Conservation 14
  1.4.3 Energy Conservation 17
  1.4.4 The Constitutive Relations 19
1.5 The Rheological Laws 19
  1.5.1 The Pressure Stress 19
  1.5.2 The Perfect Fluid 21
  1.5.3 Newtonian Fluids 22
1.6 The Thermal Behaviour 26
  1.6.1 The Heat Flux Surface Density 26
  1.6.2 The Equations of Internal Energy and Entropy 27

1.7 Thermodynamics 29
  1.7.1 The Ideal Gas 30
  1.7.2 Liquids 31
  1.7.3 Barotropic Fluids 31
1.8 Boundary Conditions 32
  1.8.1 Boundary Conditions on the Velocity Field 32
  1.8.2 Boundary Conditions on Temperature 35
  1.8.3 Surface Tension 35
  1.8.4 Initial Conditions 37
1.9 More About Rheological Laws: Non-Newtonian Fluids ♠ 37
  1.9.1 The Limits of Newtonian Rheology 37
  1.9.2 The Non-Newtonian Rheological Laws 38
  1.9.3 Linear Viscoelasticity 39
  1.9.4 The Nonlinear Effects 40
  1.9.5 Extensional Viscosities 41
  1.9.6 The Solid–Fluid Transition 45
1.10 An Introduction to the Lagrangian Formalism ♠ 45
  1.10.1 The Equations of Motion 46
  1.10.2 An Example of the Use of the Lagrangian Formulation 47
1.11 Exercises 48
Further Reading 49
References 49

2 The Static of Fluids 51
2.1 The Equations of Static 51
2.2 Equilibrium in a Gravitational Field 52
  2.2.1 Pascal Theorem 53
  2.2.2 Atmospheres 54
  2.2.3 A Stratified Liquid Between Two Horizontal Plates 56
  2.2.4 Rotating Self-gravitating Fluids ♠ 57
2.3 Some Properties of the Resultant Pressure Force 60
  2.3.1 Archimedes Theorem 61
  2.3.2 The Centre of Buoyancy 62
  2.3.3 The Total Pressure on a Wall 63
2.4 Equilibria with Surface Tension 63
  2.4.1 Some Specific Figures of Equilibrium 64
  2.4.2 Equilibrium of Liquid Wetting a Solid 65
2.5 Exercises 66
Further Reading 70
References 70

3 Flows of Perfect Fluids 71
3.1 Equations of Motions 71
  3.1.1 Other Forms of Euler's Equation 72

3.2 Some Properties of Perfect Fluid Motions ... 72
  3.2.1 Bernoulli's Theorem ... 72
  3.2.2 The Pressure Field ... 74
  3.2.3 Two Examples Using Bernoulli's Theorem ... 75
  3.2.4 Kelvin's Theorem ... 77
  3.2.5 Influence of Compressibility ... 79
3.3 Irrotational Flows ... 80
  3.3.1 Definition and Basic Properties ... 80
  3.3.2 Role of Topology for an Irrotational Flow ... 81
  3.3.3 Lagrange's Theorem ... 82
  3.3.4 Theorem of Minimum Kinetic Energy ... 83
  3.3.5 Electrostatic Analogy ... 84
  3.3.6 Plane Irrotational Flow of an Incompressible Fluid ... 85
  3.3.7 Forces Exerted by a Perfect Fluid ... 88
3.4 Flows with Vorticity ... 95
  3.4.1 The Dynamics of Vorticity ... 95
  3.4.2 Flow Generated by a Distribution of Vorticity: Analogy with Magnetism ... 97
  3.4.3 Examples of Vortex Flows ... 99
3.5 Problems ... 105
Further Reading ... 109
References ... 109

4 Flows of Incompressible Viscous Fluids ... 111
4.1 Some General Properties ... 111
  4.1.1 The Equations of Motion ... 111
  4.1.2 Law of Similarity ... 112
  4.1.3 Discussion ... 114
4.2 Creeping Flows ... 114
  4.2.1 Stokes' Equation ... 114
  4.2.2 Variational Principle ♠ ... 115
  4.2.3 Flow Around a Sphere ... 117
  4.2.4 Oseen's Equation ... 121
  4.2.5 The Lubrication Layer ... 121
4.3 Boundary Layer Theory ... 125
  4.3.1 Perfect Fluids and Viscous Fluids ... 125
  4.3.2 Method of Resolution ... 127
  4.3.3 Flow Outside the Boundary Layer ... 127
  4.3.4 Flow Inside the Boundary Layer ... 128
  4.3.5 Separation of the Boundary Layer ... 130
  4.3.6 Example of the Laminar Boundary Layer: Blasius' Equation ... 131
4.4 Some Classic Examples ... 134
  4.4.1 Poiseuille's Flow ... 134
  4.4.2 Head Loss in a Pipe ... 137
  4.4.3 Flows Around Solids ... 139

4.5 Forces Exerted on a Solid 141
  4.5.1 General Expression of the Total Force 141
  4.5.2 Coefficient of Drag and Lift 142
  4.5.3 Example: Stokes' Force 142
4.6 Exercises 146
Further Reading 147
References 147

5 Waves in Fluids 149
5.1 Ideas on Disturbances 149
  5.1.1 Equation of a Disturbance 149
  5.1.2 Analysis of an Infinitesimal Disturbance 150
  5.1.3 Disturbances with Finite Amplitude 152
  5.1.4 Waves and Instabilities 153
5.2 Sound 153
  5.2.1 Equation of Propagation 153
  5.2.2 The Dispersion Relation 154
  5.2.3 Examples of Acoustic Modes in Wind Instruments 155
5.3 Surface Waves 157
  5.3.1 Surface Gravity Waves 157
  5.3.2 Capillary Waves 160
5.4 Internal Gravity Waves 161
5.5 Waves Associated with Discontinuities 163
  5.5.1 Propagation of a Disturbance as a Function of the Mach Number 164
  5.5.2 Equations for a Finite-Amplitude Sound Wave 165
  5.5.3 The Equations of Characteristics 166
  5.5.4 Example: The Compression Wave 167
  5.5.5 Interface and Jump Conditions 169
  5.5.6 Relations Between Upstream and Downstream Quantities in an Orthogonal Shock 171
  5.5.7 Strong and Weak Shocks 173
  5.5.8 Radiative Shocks 174
  5.5.9 The Hydraulic Jump 175
5.6 Solitary Waves ♠ 178
  5.6.1 The Korteweg and de Vries Equation 178
  5.6.2 The Solitary Wave 182
  5.6.3 Elementary Analysis of the KdV Equation 183
  5.6.4 Examples 186
5.7 Exercises 187
Appendix: Jump Conditions 188
Further Reading 189
References 189

6 Flows Instabilities 191
6.1 Local Analysis of Instabilities 191
6.1.1 Definitions 191
6.1.2 The Gravitational Instability 192
6.1.3 Convective Instability 193
6.2 Linear Analysis of Global Instabilities 195
6.2.1 Centrifugal Instability: Rayleigh's Criterion 195
6.2.2 Shear Instabilities of Parallel Flows 198
6.2.3 Rayleigh's Equation 200
6.2.4 The Orr–Sommerfeld Equation 202
6.3 Some Examples of Famous Instabilities 203
6.3.1 Example: The Kelvin–Helmholtz Instability 203
6.3.2 Instabilities Related to Kelvin–Helmholtz Instability 204
6.3.3 Disturbances of the Plane Couette Flow 206
6.3.4 Shear and Stratification 207
6.3.5 The Bénard-Marangoni Instability ♠ 210
6.4 Waves Interaction ♠ 216
6.4.1 The Energy of a Wave 217
6.4.2 Application to the Kelvin–Helmholtz Instability 218
6.5 The Nonlinear Development of an Instability 219
6.5.1 Amplitude Equations 220
6.5.2 A Short Introduction to Bifurcations 221
6.5.3 Finite Amplitudes Instabilities ♠ 223
6.6 Optimal Perturbations ♠ 226
6.6.1 Introduction 226
6.6.2 Plane-Parallel Flows 226
6.6.3 A Simplified Model 228
6.6.4 Back to Fluids: Algebraic Instabilities 230
6.6.5 Non-Normal Operators 230
6.6.6 Spectra, Pseudo-Spectra and the Resolvent of an Operator 232
6.6.7 Examples of Optimal Perturbations in Flows 236
6.7 Exercises 237
Further Reading 239
References 239

7 Thermal Convection 241
7.1 Introduction 241
7.2 The Conductive Equilibrium 242
7.2.1 Equilibrium of an Ideal Gas Between Two Horizontal Plates 242
7.2.2 The Adiabatic Gradient 243
7.2.3 The Potential Temperature 244

7.3 Two Approximations 245
7.3.1 The Boussinesq Approximation: A Qualitative Presentation 245
7.3.2 The Asymptotic Expansions 247
7.3.3 Anelastic Approximation ♠ 251
7.4 Baroclinicity or the Impossibility of Static Equilibrium 253
7.4.1 Thermal Convection Between Two Vertical Plates 253
7.5 Rayleigh–Bénard Instability 256
7.5.1 Qualitative Analysis of Stability: Schwarzschild's Criterion 256
7.5.2 Evolution of Disturbances 258
7.5.3 Expression of the Solutions 260
7.5.4 Criterion of Stability 261
7.5.5 The Other Boundary Conditions ♠ 263
7.6 Convection Patterns 267
7.6.1 Three-Dimensional Disturbances 267
7.6.2 Convection Rolls 268
7.6.3 Other Patterns of Convection 268
7.7 The Weakly Nonlinear Amplitude Range 270
7.7.1 Periodic Boundary Conditions 270
7.7.2 Small Amplitudes 270
7.7.3 Derivation of the Amplitude Equation 273
7.7.4 Heat Transport: The Nusselt Number 277
7.8 Fixed Flux Convection ♠ 278
7.8.1 Introduction 278
7.8.2 Formulation 279
7.8.3 The Chapman–Proctor Equation 279
7.8.4 Properties of the Small-Amplitude Convection 282
7.9 The Route to Turbulent Convection 284
7.9.1 The Lorenz Model 284
7.9.2 The Domain of Very Large Rayleigh Numbers 285
7.10 Exercises 288
Further Reading 289
References 289

8 Rotating Fluids 291
8.1 Introduction 291
8.1.1 The Equation of Motion 291
8.1.2 New Numbers 292
8.2 The Geostrophic Flow 293
8.2.1 Definition 293
8.2.2 The Taylor–Proudman Theorem 294
8.2.3 The Expression of the Geostrophic Flow 294
8.2.4 Examples 296

8.3 Waves in Rotating Fluids ... 298
  8.3.1 Inertial Waves ... 298
  8.3.2 Inertial Modes ... 299
  8.3.3 The Poincaré Equation ... 301
  8.3.4 Rossby Waves ... 303
8.4 The Effects of Viscosity ... 306
  8.4.1 The Method ... 306
  8.4.2 The Boundary Layer Solution ... 307
  8.4.3 Ekman Pumping and Ekman Circulation ... 310
  8.4.4 An Example: The Spin-Up Flow ... 311
8.5 Hurricanes ... 316
  8.5.1 A Qualitative Presentation ... 316
  8.5.2 The Steady State: A Carnot Engine ... 317
  8.5.3 The Birth of Hurricanes ... 320
8.6 Exercises ... 321
Further Reading ... 321
References ... 321

9 Turbulence ... 323
9.1 The Fundamental Problem of Turbulent Flows ... 323
  9.1.1 How Can We Define Turbulence? ... 323
  9.1.2 The Closure Problem of the Averaged Equations ... 324
9.2 The Tools ... 325
  9.2.1 Ensemble Averages ... 325
  9.2.2 Probability Distributions ... 326
  9.2.3 Moments and Cumulants ... 326
  9.2.4 Correlations and Structure Functions ... 327
  9.2.5 Symmetries ... 327
9.3 Two-Points Correlations ... 328
  9.3.1 The Reynolds Stress ... 328
  9.3.2 The Velocity Two-Point Correlations ... 330
  9.3.3 Vorticity and Helicity Correlations ... 332
  9.3.4 The Associated Spectral Correlations ... 333
  9.3.5 Spectra ... 335
  9.3.6 The Isotropic Case ... 336
  9.3.7 Triple Correlations ... 338
9.4 Length Scales in Turbulent Flows ... 340
  9.4.1 Taylor and Integral Scales ... 340
  9.4.2 The Dissipation Scale ... 341
9.5 Universal Turbulence ... 341
  9.5.1 Kolmogorov Theory ... 342
  9.5.2 Dynamics in the Spectral Space ... 345
  9.5.3 The Dynamics in Real Space ... 347
  9.5.4 Some Conclusions on Kolmogorov Theory ... 351

9.6 Intermittency ... 351
9.6.1 Presentation ... 351
9.6.2 The Scaling Laws of Structure Functions ... 353
9.7 Theories for the Closure of Spectral Equations ... 357
9.7.1 The EDQNM Theory ... 357
9.7.2 The DIA ... 358
9.7.3 The Renormalization Group Approach ... 358
9.8 Inhomogeneous Turbulence ... 359
9.8.1 A Short Review of the Closure Models ... 359
9.8.2 Examples: Turbulent Jets and Turbulent Plumes ... 364
9.9 Two-Dimensional Turbulence ... 367
9.9.1 Spectra and Second Order Correlations ... 368
9.9.2 Enstrophy Conservation and the Inverse Cascade ... 369
9.9.3 Turbulence with Rotation or Stratification ... 371
9.10 Some Conclusions on Turbulence ... 372
9.11 Exercises ... 372
Appendix: Complements for the K-ε Model ... 375
Further Reading ... 377
References ... 377

10 Magnetohydrodynamics ... 379
10.1 Approximations Leading to Magnetohydrodynamics ... 379
10.2 The Flow Equations ... 381
10.2.1 **j** and **B** Equations ... 381
10.2.2 Boundary Conditions on the Magnetic Field ... 383
10.2.3 The Energy Equation with a Magnetic Field ... 385
10.3 Some Properties of MHD Flows ... 387
10.3.1 The Frozen Field Theorem ... 387
10.3.2 Magnetic Pressure and Magnetic Tension ... 387
10.3.3 Force-Free Fields ... 388
10.3.4 The Equipartition Solutions and Elsässer Variables ... 390
10.4 The Waves ... 391
10.4.1 Alfvén Waves ... 391
10.4.2 Magnetosonic Waves ... 392
10.5 The Dynamo Problem ... 394
10.5.1 The Kinematic Dynamo ... 395
10.5.2 The Amplification of the Magnetic Field ... 395
10.5.3 Some Anti-Dynamo Theorem ... 397
10.5.4 An Example: The Ponomarenko Dynamo ... 398
10.5.5 The Turbulent Dynamo ... 399
10.5.6 The Alpha Effect ... 401
10.6 Exercises ... 402
Appendix: Equations of the Axisymmetric Field ... 403
Further Reading ... 405
References ... 405

11 Beyond Fluid Mechanics: An Introduction to the Statistical Foundations of Gas Dynamics 407
11.1 Introduction 407
11.2 A Qualitative Approach 408
11.2.1 Back to the Continuous Medium 408
11.2.2 Particles Interactions, Collisions and the Mean Free Path 409
11.2.3 The Velocity of Particles 411
11.2.4 Energy Transport 411
11.2.5 Momentum Transport 414
11.2.6 The Prandtl Number 415
11.2.7 Comparing with Experimental Results 415
11.3 Concepts and Questions for a Statistical Approach 417
11.3.1 The Distribution Function 417
11.3.2 An Equation Governing the Distribution Function 418
11.4 Boltzmann Equation ♠ 421
11.4.1 The Collision Integral 423
11.4.2 Thermodynamic Equilibrium 426
11.4.3 The Mean Free Path 427
11.5 Equations of Fluid Flow as Mean-Field Equations 429
11.5.1 Mean Quantities 429
11.5.2 Equation for a Quantity Conserved by Collisions 430
11.5.3 Equation for Momentum 431
11.5.4 Kinetic Energy 433
11.6 Continuous Media, Perfect Fluids and Ideal Gases 434
11.7 Gas Dynamics in a Newtonian Regime 435
11.7.1 Towards Navier–Stokes 435
11.7.2 The BGK54 Model and the Theory of Chapman–Enskog 436
11.7.3 Expression of the Heat Flux and of Thermal Conductivity 438
11.7.4 Viscosity 439
11.7.5 Comparison with Experiments 442
11.8 Conclusions 444
11.9 Exercises 445
Appendix: Maxwell–Boltzmann Distribution 446
Further Reading 451
References 451

12 Complements of Mathematics 453
12.1 A Short Introduction to Tensors 453
12.1.1 Definitions 454
12.1.2 ϵ_{ijk} 455

12.2 The Divergence Theorem 456
12.2.1 Statement and Demonstration 456
12.2.2 Corollary 457
12.2.3 A Few Consequences 457
12.3 Radius of Curvature 458
12.3.1 For a Plane Curve 458
12.3.2 For a Curve in Space 459
12.4 The Boundary Layer Theory Viewed from Differential Equation Theory 460
12.5 The Sturm–Liouville Problem 462
12.6 Second Order Partial Differential Equations 464
12.6.1 The Different Types 464
12.6.2 An Introduction to Characteristics 464
12.6.3 A Hyperbolic Equation: The Wave Equation 465
12.6.4 A Parabolic Equation: The Diffusion Equation 466
12.6.5 An Elliptic Equation: The Laplace Equation 468
12.7 Exercises 472
Appendix: Formulae 473
Further Reading 477
References 477

13 The Solutions of Exercises 479
Chapter 1 479
Chapter 2 481
Chapter 3 486
Chapter 4 489
Chapter 5 491
Chapter 6 493
Chapter 7 495
Chapter 8 496
Chapter 9 498
Chapter 10 501
Complements of mathematics 501

Index 503