Modelling of rotating stars

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Outline

1. Introduction
    - Introduction
2. Modelling stellar structure: The ESTER code
    - Governing equations
    - Spheroidal coordinates
    - Spectral method
3. Results
    - Velocity field
    - Realistic model
    - Gravity darkening
    - Future work

Rotation

- Rapid rotation is very common in stars. Most of stars with spectral types earlier than the Sun rotate near its break-up velocity.
- Rotation influences:
  - Radiation emitted by a star
  - The abundance of elements in the photosphere
  - Its spectrum of oscillation
- Recent results of interferometry and asteroseismology require more precise models of rapidly rotating stars

Modelling rotating stars

- Modelling rotation in stars requires building a 2D model to deal with deformation due to centrifugal force.
- Fluid flows are present everywhere, even in radiative zones.
- The long term effect of these flows is the mixing of elements and the transport of angular momentum, which cannot be ignored in the evolution of rotating stars.

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Governing equations

$$
\begin{cases}
\Delta\phi = 4\pi G\rho \\
\nabla\cdot\mathbf{F} + \rho T\mathbf{v}\cdot\nabla s = \rho\varepsilon \\
\rho\left(2\boldsymbol{\Omega}\times\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p - \rho\nabla\left(\phi - \frac{1}{2}\Omega^2 r^2\sin^2\theta\right) + \mathbf{F}_v \\
\nabla\cdot(\rho\mathbf{v}) = 0
\end{cases}
$$

where

- Viscous force: $\mathbf{F}_v = \mu\left(\Delta\mathbf{v} + \frac{1}{3}\nabla\left(\nabla\cdot\mathbf{v}\right)\right)$
- Energy flux (Radiative): $\mathbf{F} = -\chi\nabla T$
- Equation of state: $\rho(p, T)$
- $\chi(\rho, T)$, $\varepsilon(\rho, T)$.

Boundary conditions & input parameters

Boundary conditions:

- Matching of the gravitational potential with the vacuum solution.
- Surface pressure and temperature from stellar atmosphere model.
- Stress-free boundary conditions for the velocity field.

Input parameters:

- Mass.
- Background rotation Ω_0.
- Chemical composition.

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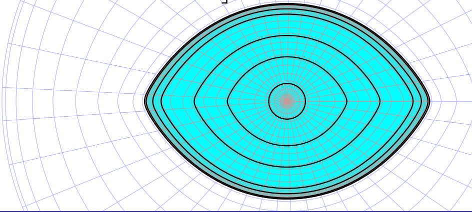
Spheroidal coordinates

- Use of non-orthogonal spheroidal coordinates ζ, θ adapted to the geometry of the star.
- We use a mapping of coordinates $r(\zeta,\theta)$ such that $\zeta = 1$ at the surface of the star.
- Need of a external domain to match the gravitational potential.
- Example: Lagrangian operator in spheroidal coordinates:

$$\Delta\phi = \quad g^{\zeta\zeta}\frac{\partial^2\phi}{\partial\zeta^2} + \left[\frac{2}{rr_\zeta}\left(1+\frac{r_\theta r_{\zeta\theta}}{rr_\zeta}\right) - g^{\zeta\zeta}\frac{r_{\zeta\zeta}}{r_\zeta} - \frac{r_{\theta\theta}+r_\theta\cot\theta}{r^2 r_\zeta}\right]\frac{\partial\phi}{\partial\zeta} - \\ -\frac{2r_\theta}{r^2 r_\zeta}\frac{\partial^2\phi}{\partial\zeta\partial\theta} + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}\right]$$



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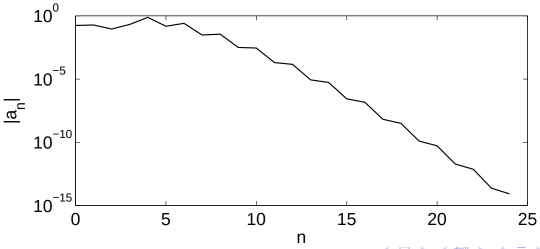
Spectral method

- Functions are approximated by a linear combination of orthogonal functions.

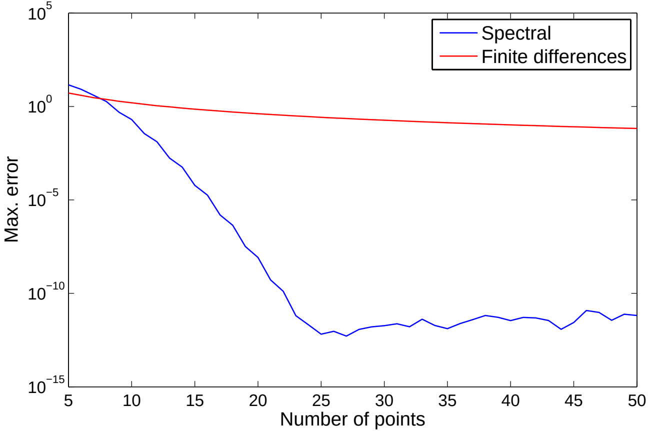
$$\phi(\zeta,\theta) = \sum_{n=0}^{N-1} \sum_{l=0}^{L-1} a_{nl} T_n(\zeta) P_l(\cos\theta)$$

  - Radial ($T_n(\zeta)$): Chebyshev polynomials (Gauss-Lobatto)
  - Latitudinal ($P_l(\cos\theta)$): Legendre polynomials
- For smooth functions, spectral methods exhibit exponential convergence.



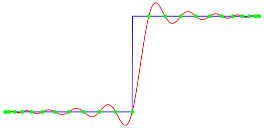
Error in the derivative of a function using spectral methods

- The error in the calculation of the derivative using spectral methods decreases exponentially with the number of points used in the discretization.



Multi-domain spectral method

- Stars have discontinuities (convective-radiative interface layer).
- Spectral methods have very good properties for smooth functions, but they fail for discontinuous functions because of the **Gibbs phenomenon**.



- The solution is to divide the domain in several regions, avoiding the discontinuities. $\longrightarrow$ Multi-domain method
- Multi-domain also helps to deal with different length scales in the model.

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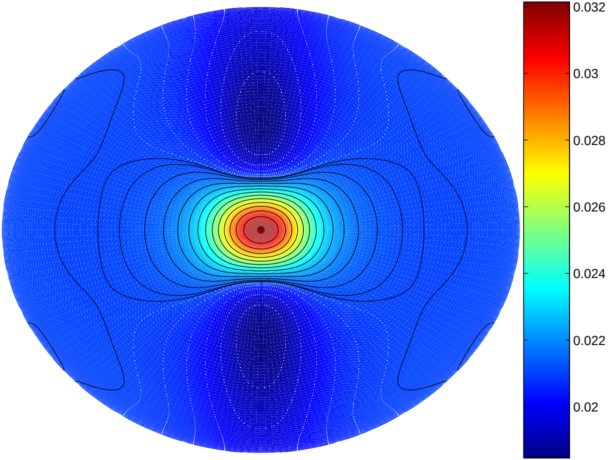
Simplified model with viscosity

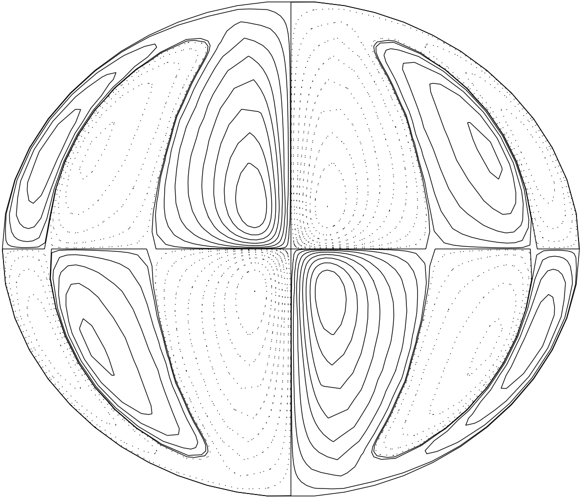
Rieutord & Espinosa 2009

- Low pressure contrast: 10^{-5}
- Simplified physics (Kramer opacities, Ideal gas).
- Fully radiative (No convection).
- Viscosity: Constant dynamic viscosity (μ).
- M=1.105 $M_{\odot}$, $\Omega \approx 0.55\Omega_c$.

Differential rotation



Meridional circulation



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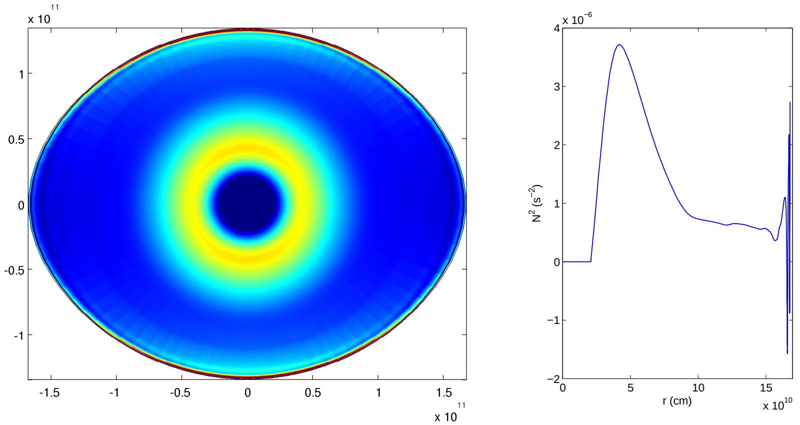
Realistic model

- Tabulated opacity and equation of state (OPAL).
- Stellar atmosphere: $p_s \propto \dfrac{\mathbf{n}\cdot\mathbf{g}_{\rm eff}}{\kappa}$, $\mathbf{n}\cdot\mathbf{F} \propto \sigma T_s^4$
- Convectic core (isentropic).
- Neglect the effect of viscosity in pressure stratification.
- Need to impose surface rotation profile ($\Omega_s(\theta) = \Omega_0$).

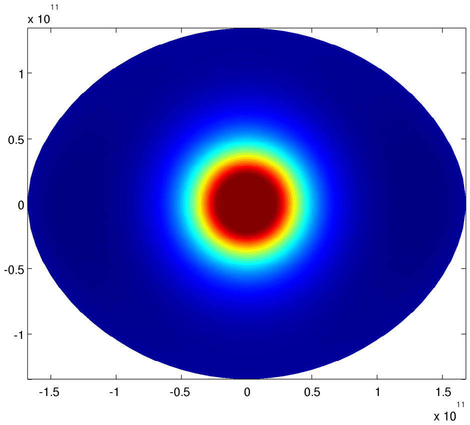
Square of the Brunt-Vaisälä frequency

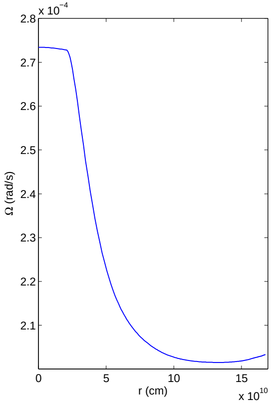
$M = 3M_\odot$, $R_{\rm pol} = 1.934R_\odot$ $(\epsilon = 0.20)$, $L = 76.48L_\odot$, $\Omega = 0.7\Omega_c$, $X = 0.7$, $Z = 0.02$



Differential rotation (v_{eq}=341 km/s)

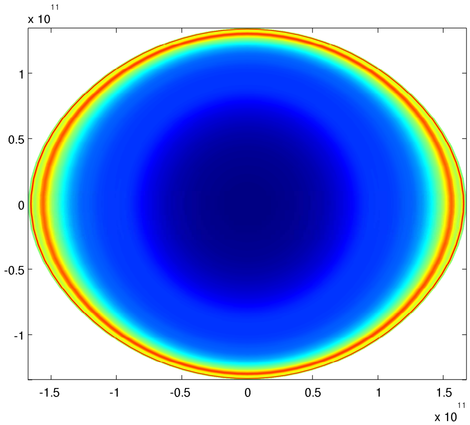
$M = 3M_\odot,\ R_{pol} = 1.934R_\odot\ (\epsilon = 0.20),\ L = 76.48L_\odot,\ \Omega = 0.7\Omega_c,\ X = 0.7,\ Z = 0.02$

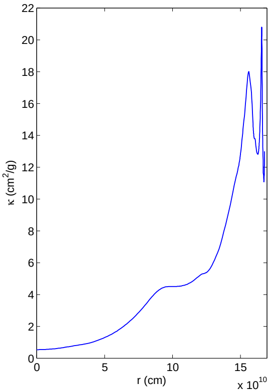




Mean Rosseland Opacity

$M = 3M_\odot$, $R_{\rm pol} = 1.934R_\odot$ ($\epsilon = 0.20$), $L = 76.48L_\odot$, $\Omega = 0.7\Omega_c$, $X = 0.7$, $Z = 0.02$





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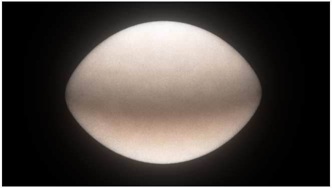
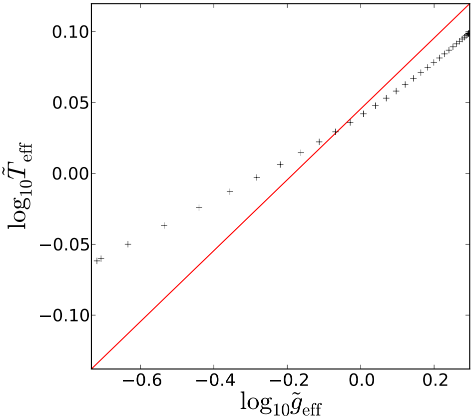
Gravity darkening

- The effective temperature of rotating stars is not uniform across its surface.
- Poles are hotter and brighter than the equator.
- Von Zeipel's law (Barotropic model, 1924):

$$T_{\text{eff}} \propto g_{\text{eff}}^{1/4}$$



Gravity darkening



$$\Omega = 0.9\Omega_k$$

— : Von Zeipel's law

++ : ESTER model ($M = 3M_\odot$)

A new analytic model for gravity darkening

Espinosa Lara & Rieutord, Astronomy & Astrophysics 533, A43 (2011)

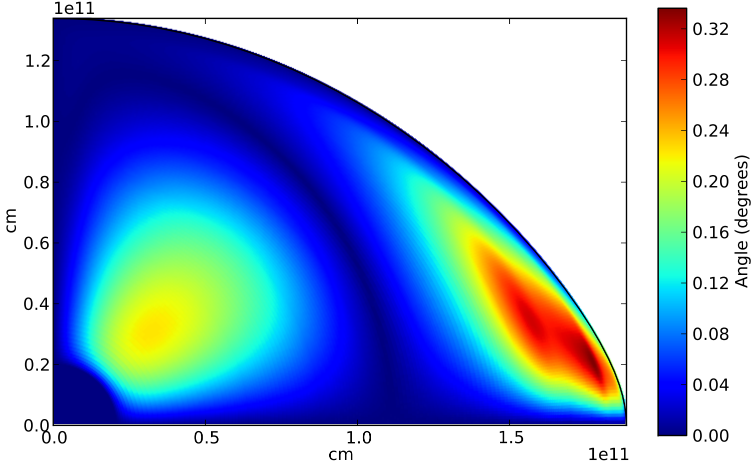
Hypothesis

- Deviation from barotropicity is small.
- Energy flux is antiparallel to the local effective gravity.

$$\mathbf{F} = -f(r,\theta)\mathbf{g}_{\rm eff}$$

  - Convection: Energy transport driven by bouyancy.
  - Radiation: Angle between ∇T and $\mathbf{g}_{\rm eff}$ remains small ($< 1^\circ$).

Angle between ∇T and ∇p $(\Omega = 0.9\Omega_k)$



A new analytic model for gravity darkening

In the envelope of a star, where no heat is generated:

$$\nabla \cdot \mathbf{F} = 0$$

and then

$$\mathbf{g}_{\text{eff}} \cdot \nabla f + f \nabla \cdot \mathbf{g}_{\text{eff}} = 0$$

Energy flux depends only on the shape of the equipotential surfaces and hence on mass distribution.

Rapidly rotating stars are usually intermediate or high mass stars, and thus centrally condensed.

For simplicity we use $\mathbf{g}_{\text{eff}}$ given by the **Roche model**.

$$\mathbf{g}_{\text{eff}} = -\frac{GM}{r^2}\mathbf{e}_r + \Omega^2 r \sin\theta \mathbf{e}_s$$

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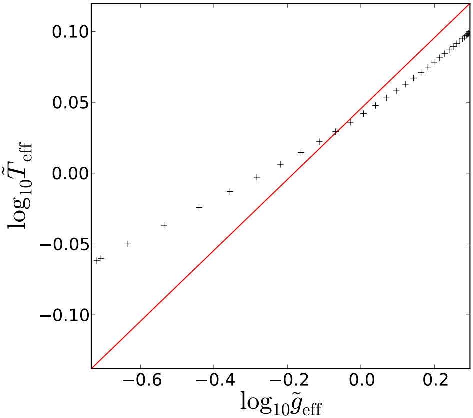
$$T_{\mathrm{eff}} = \left(\frac{L}{4\pi\sigma GM}\right)^{1/4} \sqrt{\frac{\tan\theta_0}{\tan\theta}}\, g_{\mathrm{eff}}^{1/4}$$

where

$$\cos\theta_0 + \ln\tan\frac{\theta_0}{2} = \frac{1}{3}\omega^2\tilde{r}^3\cos^3\theta + \cos\theta + \ln\tan\frac{\theta}{2}$$

- Gravity darkening depends only on $\omega = \dfrac{\Omega}{\Omega_k}$. $\left(\Omega_k = \sqrt{\dfrac{GM}{R_e^3}}\right)$
- For slow rotation $\theta_0 \approx \theta$ and we recover von Zeipel's law.

Gravity darkening



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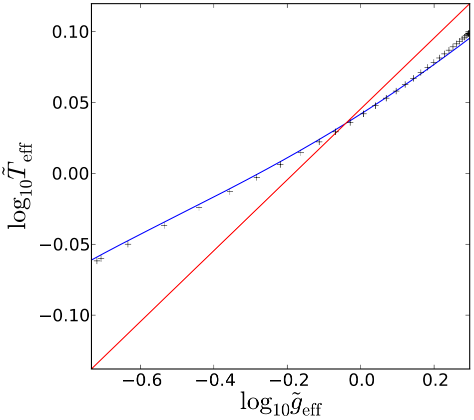

$$\Omega = 0.9\Omega_k$$

— : Analytic model
— : Von Zeipel's law
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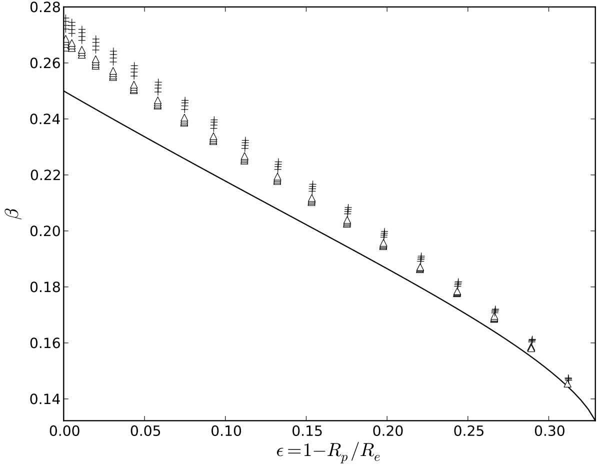
$(M = 3M_\odot)$

Gravity darkening exponent



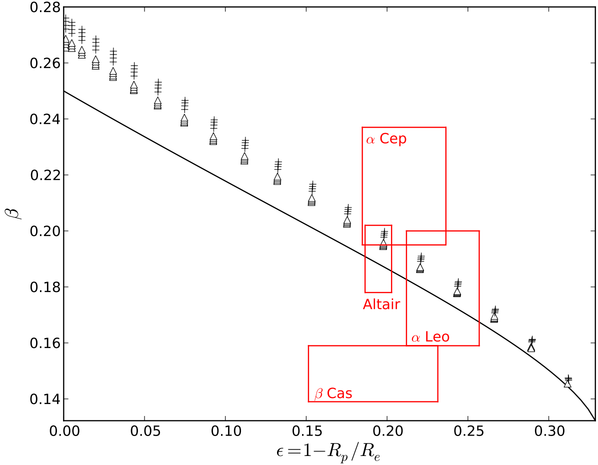
Fitted values of β in $T_{\rm eff} \propto g_{\rm eff}^{\beta}$

— : Analytic model

+, △ : ESTER models ($M = 2.5 - 4\,M_\odot$)

Von Zeipel's value: $\beta = 0.25$

Gravity darkening exponent



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Future work

- Make a flexible velocity solver which can easily deal with turbulence models
- Low mass stars with an outer convection zone
- Take into account temporal evolution: $P(r,\theta,t)$,$\vec{v}(r,\theta,t)$