

Chapter 5

Physical Processes Leading to Surface Inhomogeneities: The Case of Rotation

Michel Rieutord

Abstract In this lecture I discuss the bulk surface heterogeneity of rotating stars, namely gravity darkening. I especially detail the derivation of the ω -model of Espinosa Lara and Rieutord (Astron Astrophys 533:A43, 2011), which gives the gravity darkening in early-type stars. I also discuss the problem of deriving gravity darkening in stars owning a convective envelope and in those that are members of a binary system.

5.1 Introduction

As it has been much discussed in this school, surface inhomogeneities of stars are more and more frequently detected due to the increasing sensitivity of the instruments. If correctly understood, and therefore modeled, these data may open new windows on the interior or on the history of stars.

The purpose of this lecture is to first briefly review the processes that lead to such inhomogeneities and then to focus on a very fundamental one, namely rotation.

In the ancient times, when the eye was the only optical instrument to observe Nature, the Sun was thought as a pure uniform bright disc. The invention of the telescope by Galileo ruined this idea, showing that the Sun was spherical with spots on it. Presently, it suffices to have a look at images of the Sun taken at short wavelengths to understand that its surface is certainly not uniform. Such images actually reveal that the magnetic fields are a prominent cause of this non-uniformity. It looks like a mess which even impacts the distribution of surface flux and temperature. A closer look at the magnetic structures but also below, at the photospheric level, shows that all these heterogeneities evolve with time. On the photosphere, turbulent convection features the surface with two important scales: granulation and supergranulation (Rieutord and Rincon 2010). Even the bulk surface rotation is not uniform. This differential rotation, known since the nineteenth

M. Rieutord (✉)

Université de Toulouse, UPS-OMP, IRAP, Toulouse, France

CNRS, IRAP, 14, Avenue Edouard Belin, F-31400 Toulouse, France

e-mail: mrieutord@irap.omp.eu

century, with fast equatorial regions and slow polar regions, is now understood as driven by Reynolds stresses coming from the turbulence in the solar convection zone.

Hence, the surface of the Sun teaches us that we should expect non-uniform velocities, temperatures, flux and magnetic fields at the surface of all low-mass stars. But one consequence of the strong mixing imposed by turbulent convection and the ever changing magnetic fields is that the solar photosphere has a uniform chemical composition!

But uniform chemical composition is certainly not possible when turbulent convection disappears and no longer mixes the surface layers, that is when we consider stars of higher mass with an outer radiative envelope. There, combination of magnetic fields with microscopic diffusion processes (gravitational settling or radiative acceleration) may on the contrary lead to chemical surface inhomogeneities (Vauclair and Vauclair 1982; Alecian 2013; Korhonen et al. 2013). But even when magnetic fields are absent at their surface, early-type stars are still endowed with a non-uniform surface: absence of magnetic field is indeed correlated with fast rotation, a feature that makes the polar caps brighter than the equatorial regions.

From the foregoing presentation we see that three processes, intrinsic to each star may lead to surface inhomogeneities: rotation, convection and magnetic fields. There is a fourth one, but extrinsic to the star itself, namely binarity. A companion indeed raises tides, illuminates one side of the star or may even transfer mass.

Within these four physical processes that make the surface of stars not uniform, we shall concentrate on the most simple, namely rotation, which, a priori leads surface variations that only depend on the latitude. We shall discuss in detail this very basic physical effect, leaning on the recent works of Espinosa Lara and Rieutord (2011, 2012).

5.2 The Energy Flux in Radiative Envelopes of Rotating Stars

5.2.1 *von Zeipel 1924*

In a seminal paper, von Zeipel (1924) showed that a rotating star may be brighter at the poles than at the equator. This result is quite simple to derive if we assume that the star is barotropic and that the energy flux is given by Fourier's law. Indeed, if the star is barotropic, meaning that its equation of state can be simplified to

$$P \equiv P(\rho),$$

it implies that there exist a hydrostatic solution in the rotating frame and that all thermodynamic quantities can be expressed as a function of the total potential Φ

(i.e. gravitational plus centrifugal). Hence, one writes

$$\rho \equiv \rho(\Phi), \quad T \equiv T(\Phi), \quad \text{etc.}$$

Then, using Fourier's law to derive the heat flux, one finds

$$\mathbf{F}_{\text{rad}} = -\chi \nabla T = -\chi(\Phi) T'(\Phi) \nabla \Phi = K(\Phi) \mathbf{g}_{\text{eff}}$$

whence von Zeipel law

$$T_{\text{eff}} = K g_{\text{eff}}^{1/4} \quad \text{on the surface } \Phi = \text{Cst}$$

This result is simple but incorrect. Indeed, barotropic stars are realized in two cases: either the star is isentropic and thus fully convective or it is isothermal but this can hardly be the case.¹ So the closest case that may match the barotropic state is that of a fully convective star, but in such a case the flux cannot be derived from the Fourier's law. In fact, these hypothesis (barotropy and heat diffusion) lead to a contradiction. Indeed, we may note that the total potential and the temperature would verify in the envelope of the star,

$$\begin{cases} \text{Div}(\chi \nabla T) = 0 \\ \Delta \Phi = 4\pi G \rho + 2\Omega^2 \end{cases} \quad (5.1)$$

which leads to

$$\text{Div}(\chi(\Phi) T'(\Phi) \nabla \Phi) = 0 \quad \iff \quad 4\pi G \rho + 2\Omega^2 + (\ln(\chi T'))' g_{\text{eff}}^2 = 0$$

On an equipotential, ρ is constant as well as $(\ln(\chi T'))'$, but g_{eff} is not constant. Hence, the latter equation is impossible. The reason for that is that for a rotating star where heat is transported by diffusion, a barotropic state cannot be and should be replaced by a baroclinic state. In such a state, isobars, isotherms or equipotential are all different, not very different, but different. This is the normal state that comes from the fact (basically) that temperature, pressure, and gravitational potential all obeys different and independent equations. The barotropic state is therefore rather peculiar (but see Rieutord 2006, for a more detailed presentation).

Now one may wonder if it is possible to derive the dependence of the flux with latitude for a rotating star without computing the whole stellar structure and the associated flows as in Espinosa Lara and Rieutord (2013). Fortunately, this is indeed possible as Espinosa Lara and Rieutord (2011) have shown. It is not as simple as von Zeipel law, but it has the merit of relying on controllable hypothesis.

¹Some models of prestellar core use this hypothesis, sometimes.

5.2.2 The Idea of the ω -Model

In the following we shall first restrict ourselves to the case of early-type stars, that is to stars that have a radiative envelope around a convective core. We'll discuss the case of convective envelope in the next section.

Within the envelope of a star the flux just obeys:

$$\text{Div}\mathbf{F} = 0 \quad (5.2)$$

namely energy is conserved and there are no energy sources.

This is a single equation, not enough to determine the two components, (F_r, F_θ) , of the flux, but if we add a constraint to the flux we may find it. We thus assume that the flux is anti-parallel to the effective gravity

$$\mathbf{F} = -f(r, \theta)\mathbf{g}_{\text{eff}} \quad (5.3)$$

In order to avoid an additional unknown, we shall take the effective gravity \mathbf{g}_{eff} as given by the Roche model. In such a case we shall see that the flux function f can be determined and that the latitude variation of the flux depends on a single parameter ω defined as the ratio of the angular velocity to the keplerian angular velocity at the equator. In other words, the flux depends on

$$\omega = \frac{\Omega}{\Omega_k} = \Omega \left(\sqrt{\frac{GM}{R_e^3}} \right)^{-1} \quad (5.4)$$

Thus, we shall call this model the ω -model to emphasize the crucial role of the reduced angular velocity ω .

However, before going any further, we may wonder whether the assumptions are strong or not, especially (5.3).

In a radiative zone, the configuration is baroclinic so vectors are surely not aligned but fortunately we can now revert to 2D-models to get an idea of the misalignment. As shown in Fig. 5.1, the misalignment remains small, less than a degree, even if the star rotates close to criticality.

Thus, even if the envelope is the seat of baroclinic flows, the misalignment is small. Actually, the baroclinic torque $(\nabla P \times \nabla \rho)/\rho^2$ does not need a strong misalignment of the vectors to be efficient at driving baroclinic flows because the two gradients (of pressure and density) are already quite strong.

Let us pursue somewhat. From (5.2) and (5.3), we have

$$\text{Div}\mathbf{F} = 0 \quad \iff \quad \text{Div}(f \nabla \Phi) = 0$$

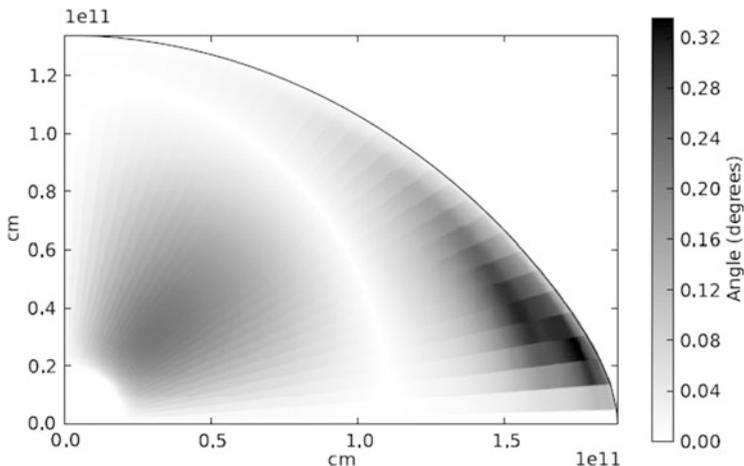


Fig. 5.1 Misalignment between pressure gradient and flux for a configuration with a flatness $\sim 30\%$ (from Espinosa Lara and Rieutord 2011)

thus

$$\mathbf{g}_{\text{eff}} \cdot \nabla \ln f = -2\Omega^2 \tag{5.5}$$

because $\Delta\Phi = -2\Omega^2$; hence,

$$\frac{\partial \ln f}{\partial \xi} = -\frac{2\Omega^2}{g_{\text{eff}}} \tag{5.6}$$

where we introduced the local vertical coordinate ξ . Equality (5.6) shows that $\frac{\partial \ln f}{\partial \xi}$, and therefore f , has latitudinal variations similar to those of g_{eff} . Hence the horizontal variations of the flux cannot be given by von Zeipel law. In other words $T_{\text{eff}}/g_{\text{eff}}^{1/4}$ cannot be constant.

The second hypothesis is the use of the Roche model. This model assumes that the whole mass of the star is concentrated at the centre thus leading to a gravitational potential in $1/r^2$ everywhere. For the regions we are interested in, namely the envelope of early-type stars, this is a rather good approximation since these stars are usually said to be “centrally condensed”. In Fig. 5.2 we show the density profile of a non-rotating $5 M_{\odot}$ star along with the profile of two polytropes. We see that the $n = 3$ polytrope represents fairly well the density profile of the star and that the $n = 3/2$ -polytrope, which is a very good model for fully convective stars, is much less “centrally condensed”. Hence, gravity in the outer envelope of an early-type star is well represented by the Roche model (see Fig. 5.2 right). The interior discrepancy with the Roche model has no consequence for our purpose.

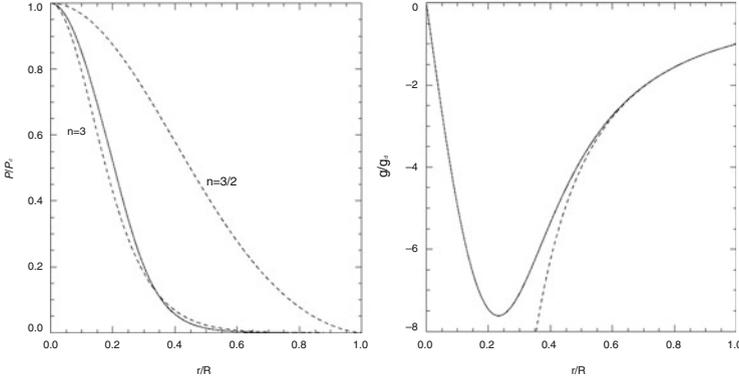


Fig. 5.2 *Left:* Density profile of a $M = 5 M_{\odot}$ ZAMS, non rotating star (*solid line*) together with that of a $n = 3/2$ and $n = 3$ polytropes. *Right:* The interior gravity of the same stellar model (*solid line*) together with the $-1/r^2$ Roche model (*dashed line*). The ZAMS model is an ESTER model (e.g. Espinosa Lara and Rieutord 2013)

5.2.3 The Derivation of $f(r, \theta)$

f is given by (5.5) but we first need to scale this function so as to introduce a non-dimensional function F that accounts for the radial and latitudinal variation of the flux. This is easily done if we observe that near the star's centre

$$\mathbf{F} \sim \frac{L}{4\pi r^2} \mathbf{e}_r \quad \mathbf{g}_{\text{eff}} \sim -\frac{GM}{r^2} \mathbf{e}_r$$

So that we may set

$$f(r, \theta) = \frac{L}{4\pi GM} F(r, \theta) \quad (5.7)$$

with

$$\lim_{r \rightarrow 0} F(r, \theta) = 1$$

Then, we scale the gravity with GM/R_e^2 and the length scale with the equatorial radius R_e . The scaled angular velocity is therefore given by

$$\omega = \frac{\Omega}{\Omega_k} = \Omega \left(\sqrt{\frac{GM}{R_e^3}} \right)^{-1}$$

At this point we should underline that the angular velocity is scaled by the keplerian angular velocity given by the equatorial radius. It is often the case in the literature that the scale of angular velocity is the critical velocity associated with the Roche model of the considered mass M (e.g. Monnier et al. 2012, for instance). This gives a different ω (i.e. fraction of critical velocity). We give in the Appendix the relation between these two ways of appreciating angular velocity.

We now proceed to the derivation of F the scaled version of f . From (5.7) and (5.5) we get

$$\left(\frac{1}{\omega^2 r^2} - r \sin^2 \theta \right) \frac{\partial F}{\partial r} - \sin \theta \cos \theta \frac{\partial F}{\partial \theta} = 2F$$

With $F(0, \theta) = 1$ we have all the elements for solving the equation for F .

First, we solve for $\ln F$, namely,

$$\left(\frac{1}{\omega^2 r^2} - r \sin^2 \theta \right) \frac{\partial \ln F}{\partial r} - \sin \theta \cos \theta \frac{\partial \ln F}{\partial \theta} = 2. \quad (5.8)$$

If we set $\ln F = \ln G + A(\theta)$, so that $A(\theta)$ removes the RHS of (5.8), we immediately find that

$$A'(\theta) = -2 / \sin \theta \cos \theta \implies A(\theta) = -\ln(\tan^2 \theta).$$

But we still have to solve the homogeneous equation, namely

$$\left(\frac{1}{\omega^2 r^2} - r \sin^2 \theta \right) \frac{\partial \ln G}{\partial r} - \sin \theta \cos \theta \frac{\partial \ln G}{\partial \theta} = 0 \quad (5.9)$$

Such a first order partial differential equations is solved by the method of characteristics. We therefore look for places where $\ln G$ is constant. These places are called the characteristics curves of G . They are such that

$$\frac{\partial \ln G}{\partial r} dr + \frac{\partial \ln G}{\partial \theta} d\theta = 0$$

but G also verifies (5.9) so that we can eliminate $\frac{\partial \ln G}{\partial r}$ and $\frac{\partial \ln G}{\partial \theta}$ and get

$$\left(\frac{1}{\omega^2 r^2} - r \sin^2 \theta \right) d\theta + \sin \theta \cos \theta dr = 0 \quad (5.10)$$

which is the equation of characteristics.

We first observe that we may multiply this equation by any function $H(r, \theta)$ without changing anything. So we may also look for h such that

$$\begin{cases} \frac{\partial h}{\partial r} = H \sin \theta \cos \theta \\ \frac{\partial h}{\partial \theta} = H \left(\frac{1}{\omega^2 r^2} - r \sin^2 \theta \right) \end{cases} \quad (5.11)$$

where H needs to be chosen so that this system can be integrated. After trial and error, we find that $H = \omega^2 r^2 \cos \theta \cot \theta$ is the right function. Thus

$$\begin{cases} \frac{\partial h}{\partial r} = \omega^2 r^2 \cos^3 \theta \\ \frac{\partial h}{\partial \theta} = \frac{\cos^2 \theta}{\sin \theta} - \omega^2 r^3 \cos^2 \theta \sin \theta \end{cases} \quad (5.12)$$

and the solution is

$$h(r, \theta) = \frac{1}{3} \omega^2 r^3 \cos^3 \theta + \cos \theta + \ln \tan(\theta/2)$$

The curves $h(r, \theta) = \text{Cst}$ are the characteristics. Note that the polar equation of a characteristic, namely the dependence $r \equiv r(\theta)$, is just implicitly known, and depends on the chosen constant.

Now, we know that $\ln G$ or G is constant on the curves where $h(r, \theta) = \text{Cst}$. So we can write

$$G \equiv G(h) . \quad (5.13)$$

It means that the variations of G with (r, θ) are through those of $h(r, \theta)$ only. So we find that

$$\ln F = \ln G(h) - \ln \tan^2 \theta \quad \text{or} \quad F = \frac{G(h(r, \theta))}{\tan^2 \theta}$$

This is the solution of the partial differential equation, but it is up to an arbitrary function $G(h)$ that we should determine. For that, we need to revert to the boundary conditions, namely that $F(0, \theta) = 1$. We thus need to impose

$$\frac{G(h(0, \theta))}{\tan^2 \theta} = 1 \quad (5.14)$$

or

$$G(\cos \theta + \ln \tan(\theta/2)) = \tan^2 \theta \quad (5.15)$$

for all θ . This is certainly a weird expression of G , but actually it is sufficient. Let's introduce the function h_0 such that

$$h_0(\theta) = \cos \theta + \ln \tan(\theta/2) \quad (5.16)$$

Hence, we have

$$(G \circ h_0)(\theta) = \tan^2 \theta$$

or

$$G \circ h_0 = \tan^2 \quad \implies \quad G = \tan^2 \circ h_0^{-1}$$

so formally, the solution for G is

$$G(r, \theta) = \tan^2(h_0^{-1}(h(r, \theta)))$$

To make it more understandable, we set

$$\psi = h_0^{-1}(h(r, \theta)) \quad (5.17)$$

so that

$$h_0(\psi) = \frac{1}{3}\omega^2 r^3 \cos^3 \theta + \cos \theta + \ln \tan(\theta/2)$$

or

$$\cos \psi + \ln \tan(\psi/2) = \frac{1}{3}\omega^2 r^3 \cos^3 \theta + \cos \theta + \ln \tan(\theta/2) \quad (5.18)$$

which is a transcendental equation for ψ . However, it is not difficult to solve numerically (we know that when r or ω are small $\psi \simeq \theta$). So finally we find

$$F(r, \theta) = \frac{\tan^2(\psi(r, \theta))}{\tan^2 \theta} \quad (5.19)$$

where $\psi(r, \theta)$ is given by (5.18).

5.2.4 Two Interesting Latitudes

F seems to be singular at the pole ($\theta = 0$) and at the equator ($\theta = \pi/2$). Let us explore these two latitudes.

Starting with the pole, we see that if $\theta \ll 1$, then, from (5.18), we find that $\psi \ll 1$ as well. Indeed, for small values of the angles we have

$$1 + \ln \tan(\psi/2) \simeq \frac{1}{3}\omega^2 r^3 + 1 + \ln \tan(\theta/2)$$

so that

$$\psi \simeq \theta e^{\omega^2 r^3/3} \quad (5.20)$$

and

$$F(r, 0) = e^{2\omega^2 r^3/3} \quad (5.21)$$

which gives the values of F along the rotation axis.

The equator is more complicated. We need to know that if $\varepsilon \ll 1$ then

$$\ln \left(\tan \left[\frac{\pi}{4} - \varepsilon \right] \right) = -\varepsilon - \frac{1}{6}\varepsilon^3 - \dots$$

With this asymptotic expansion we find

$$F(r, \pi/2) = (1 - \omega^2 r^3)^{-2/3}$$

5.2.5 The Final Solution of the ω -Model

Back to the definitions we started with, we can express the flux with the effective gravity in the following way:

$$\mathbf{F} = -\frac{L}{4\pi GM} F(\omega, r, \theta) \mathbf{g}_{\text{eff}} \quad (5.22)$$

so that we also get the effective temperature

$$T_{\text{eff}} = \left(\frac{L}{4\pi\sigma GM} \right)^{1/4} \sqrt{\frac{\tan \psi}{\tan \theta}} g_{\text{eff}}^{1/4} \quad (5.23)$$

From this expression, we see that the function $\sqrt{\tan \psi / \tan \theta}$ shows the deviation from the von Zeipel law.

Noting that

$$\mathbf{g}_{\text{eff}} = \frac{GM}{R_e^2} \left(-\frac{\mathbf{e}_r}{r^2} + \omega^2 r \sin \theta \mathbf{e}_s \right)$$

for the Roche model (\mathbf{e}_s is the unit radial vector of cylindrical coordinates and \mathbf{e}_r that of spherical coordinates). We find that the flux is given by

$$\mathbf{F} = -\frac{L}{4\pi R_e^2} \left(-\frac{\mathbf{e}_r}{r^2} + \omega^2 r \sin\theta \mathbf{e}_s \right) F(\omega, r, \theta) \quad (5.24)$$

which shows that it depends only on ω and a scaling factor $\frac{L}{4\pi R_e^2}$, hence the name “ ω -model”.

5.2.6 Comparison with 2D Models: A Test of the β - and ω -Models

After the foregoing mathematical developments we certainly would like to compare the results of this modeling to more elaborated models. For this purpose, we compared the latitude variations of the flux with the prediction of fully two-dimensional ESTER models (Espinosa Lara and Rieutord 2013). We recall that ESTER models give a full solution of the internal structure of a rotating early-type star including the differential rotation and the meridional circulation driven by the baroclinicity of the envelope. They also include the full microphysics (opacity and equation of state) from OPAL tables. Figures 5.3 and 5.4 show that the ω -model matches very well the output of the full ESTER models. Moreover, we also note

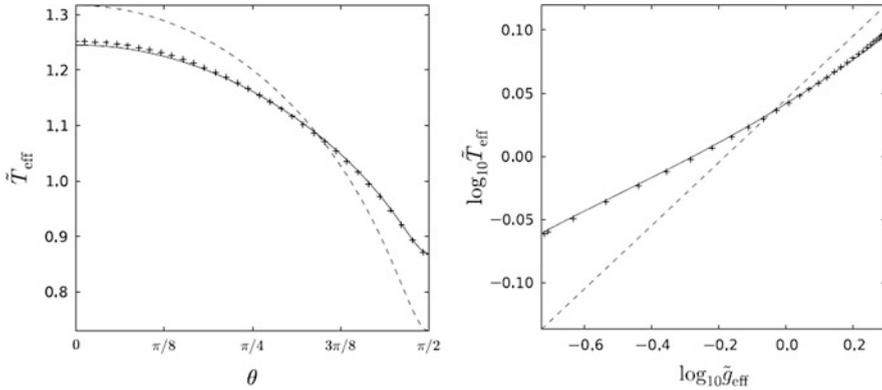
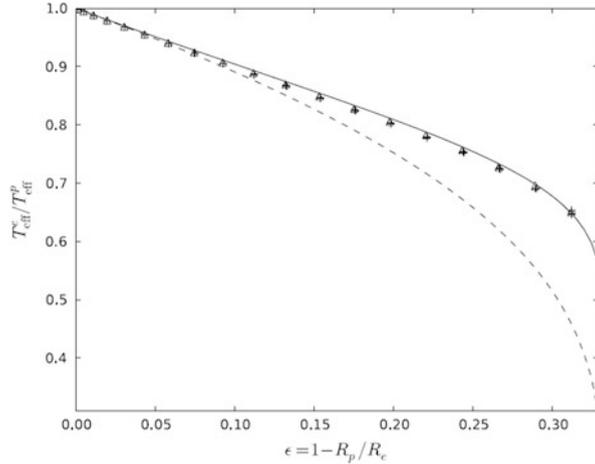


Fig. 5.3 *Left:* Scaled effective temperature as a function of colatitude for a $3 M_{\odot}$ model at $\Omega = 0.9\Omega_k$. The *solid line* shows the prediction of the simplified model, ‘*pluses*’ show the prediction of a fully 2D ESTER model including differential rotation (see Espinosa Lara and Rieutord 2013, for details), while the *dashed line* is for the von Zeipel law. *Right:* With the same *symbols* as on *left*, the effective temperature as a function of the effective gravity (plots from Espinosa Lara and Rieutord 2011)

Fig. 5.4 Variation of the ratio of effective temperature at pole and equator as a function of the flatness of the star. *Symbols* are the same as in Fig. 5.3 (plots from Espinosa Lara and Rieutord 2011)



that the dependence of the effective temperature versus gravity is close to but not exactly a power law.

Observational data often show the polar-equator contrast in effective temperature in terms of the exponent β defined as

$$T_{\text{eff}} \propto g_{\text{eff}}^{\beta} \quad (5.25)$$

We shall call this approximate modeling the “ β -model”. Actually, note that (5.25) demands that

$$\beta = \left. \frac{\partial \ln T_{\text{eff}}}{\partial \ln g_{\text{eff}}} \right|_{r=R(\theta)} \quad (5.26)$$

where $R(\theta)$ is the radius of the star at colatitude θ . Since the relation between T_{eff} and g_{eff} is not a power law, β is not constant on the surface of a rotating star. It varies between two extreme values that we can also compute.

To make things simpler we therefore define the b -exponent as follows:

$$T_e = T_p \left(\frac{g_e}{g_p} \right)^b \quad \text{or} \quad b = \frac{\ln(T_e/T_p)}{\ln(g_e/g_p)} \quad (5.27)$$

where the indices e and p refer to the equator and pole respectively. T and g designate the effective temperature and effective surface gravity.

From the polar and equatorial expression of the flux, we get

$$F_e = (1 - \omega^2)^{-2/3} g_e \quad \text{and} \quad F_p = e^{2\omega^2 r_p^3/3} g_p$$

for the ω -model, while, from the Roche model,

$$\frac{g_e}{g_p} = r_p^2(1 - \omega^2) \quad \text{with} \quad r_p = \frac{1}{1 + \omega^2/2}$$

where r_p is the polar radius. So we find

$$\left(\frac{T_e}{T_p}\right)^4 = \frac{(1 - \omega^2)^{1/3}}{(1 + \omega^2/2)^2} e^{-2\omega^2 r_p^3/3}$$

and

$$b = \frac{1}{4} - \frac{1}{6} \frac{\ln(1 - \omega^2) + \omega^2 r_p^3}{\ln(1 - \omega^2) - 2 \ln(1 + \omega^2/2)} \quad (5.28)$$

We plotted in Fig. 5.5 the values of b with increasing values of the flatness (namely with increasing rotation). In this figure we see that the b -exponent is close to a linear dependence $b = \frac{1}{4} - \frac{1}{3}\varepsilon$ up to $\varepsilon = 0.3$. However, note that since the true dependence is not a power law, β , as given by (5.26), varies at the surface of a given star. We also show in Fig. 5.5 its range of variation and it is clearly not negligible when ε is larger than ~ 0.15 . This means that if we had access to a very high spatial resolution of the stellar surface we would find different β 's whether we look at the pole (large values) or at the equator (low values).

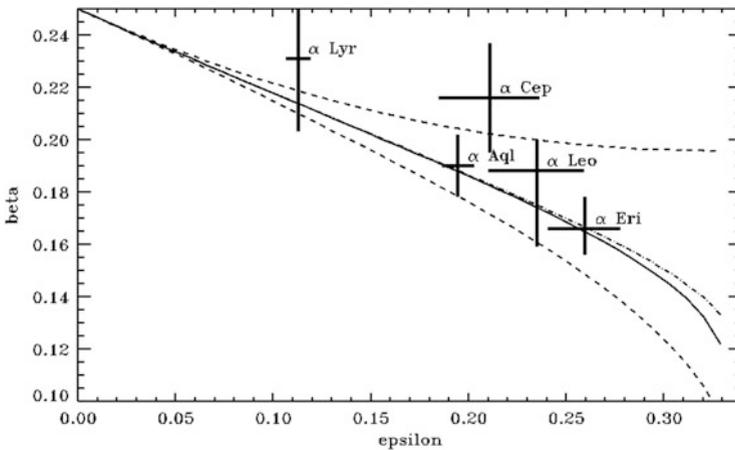


Fig. 5.5 The β -values from various models: The *solid line* shows the b -exponent of the ω -model, while the *dot-dashed line* shows the corresponding ESTER model. The extra *dashed lines* give the range of β values spawned at the stellar surface by a β -model. Data from interferometric observations of some early-type stars are shown (from Domiciano de Souza et al. 2014)

If the β -model is a poor representation of the latitudinal variation of the flux, can we devise a better one? Surely, a decomposition of the effective temperature on the spherical harmonics basis, namely

$$T_e(\theta) = \sum_{l,m} t_m^l Y_\ell^m$$

has the advantage of being model independent. The coefficients of the expansion are the results of observations. Such an expansion is already used for the description of spotted stars for the reconstruction of their magnetic fields (e.g. Donati et al. 2006 but see also the lecture of Kochukhov in this volume).

In Fig. 5.5, we also show the observationally derived values for a few early-type stars observed with interferometers. The matching is quite remarkable, even if some cases like α Cep certainly need a more detailed study.

To finish with the case of early-type stars, let us consider the case of small rotations. We may first derive the linear dependence of the b -exponent with ε . From (5.28) we get

$$b = \frac{1}{4} - \frac{1}{6}\omega^2 + \mathcal{O}(\omega^4) \quad \text{or} \quad b = \frac{1}{4} - \frac{1}{3}\varepsilon + \mathcal{O}(\varepsilon^2) \quad (5.29)$$

where we observed that $\varepsilon = 1 - r_p$. This expression shows that in the limit of small rotation we recover von Zeipel law. To understand the origin of this property, it is useful to reconsider the ω -model and the expression of the function $F(r, \theta)$. Let us first solve (5.18) in the limit $\omega \ll 1$. This yields

$$\psi = \theta + \frac{1}{3}\omega^2 \sin \theta \cos \theta + \mathcal{O}(\omega^4) \quad (5.30)$$

From this relation, we derive the asymptotic expression of $F(r, \theta)$ at low ω , namely

$$F(r, \theta) = 1 + \frac{2}{3}\omega^2 r^3$$

The latitudinal dependence has disappeared. Hence the latitudinal variations of the flux are those of the effective gravity. Therefore, von Zeipel law applies at low rotation rates. We can understand this result, if we recall that in the limit of zero rotation, the star is spherical and all surfaces of constant pressure, temperature, etc. are spheres so that we can consider the gravitational potential or the pressure as the independent variable. Thus we recover a kind of barotropic situation where one can use a relation between pressure and density, and derive a von Zeipel law.

5.3 The Case of Convective Envelopes

5.3.1 Lucy's Problem

In the 1960s it was realized that gravity darkening was very important for the interpretation of light curves of contact binaries (like the W UMa-type stars). But most of these stars are low-mass stars, thus with a convective envelope. The use of von Zeipel law, which is based on heat diffusion, was therefore doubtful.

So Lucy (1967) asked: "What is the gravity-darkening law appropriate for late-type stars whose subphotospheric layers are convective?" Lucy's reasoning was the following.

In the convective envelope of a rotating star, if we go deep enough, we should reach a medium of constant entropy. This value should be the same whatever the latitude. 1D models show that the entropy jumps from a minimum near the surface (where the convective driving ceases) to a plateau in the deep layers where convective mixing is efficient (see Fig. 5.6). Lucy argues that the value of the entropy s on this plateau is a function of the surface gravity g_s and effective temperature T_{eff} . He thus writes

$$s \equiv s(g_s, T_e) \quad (5.31)$$

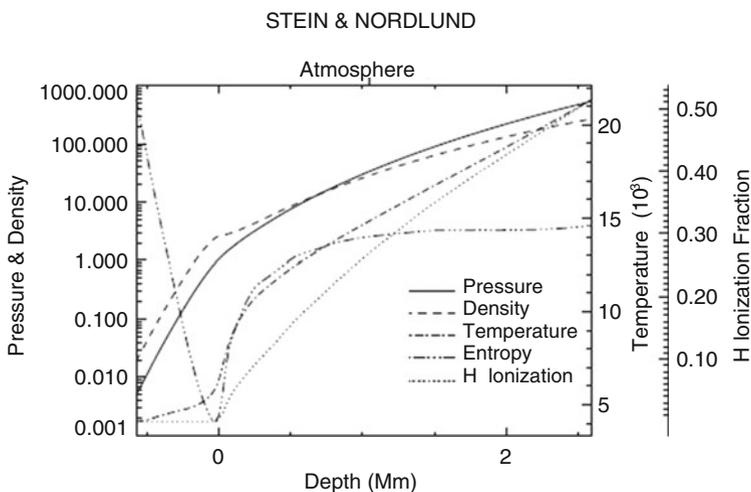


Fig. 5.6 Thermodynamic profile of the Sun according to Stein and Nordlund 1998

In the case of a rotating star, where g_s and T_{eff} vary, we must have

$$s(g_s, T_e) = s_0 \quad (5.32)$$

where s_0 is the entropy on the plateau. If we differentiate this expression with respect to g_s and T_e , we find that

$$\frac{\partial s}{\partial g_s} dg_s + \frac{\partial s}{\partial T_e} dT_e = 0$$

in the deep layers of the rotating star. Since we admit that $T_e \propto g_s^\beta$, then we have

$$\frac{\partial s}{\partial \ln g_s} + \beta \frac{\partial s}{\partial \ln T_e} = 0 \quad (5.33)$$

Thus, if we are able to evaluate the values of the above partial derivatives, we can obtain β . For that, Lucy considered various 1D neighbouring models (we do not know how the variations were made), and evaluated the partial derivatives so as to find β . Using five stellar models (three with $M = 1 M_\odot$, two with $M = 1.26 M_\odot$), he found that

$$0.069 \leq \beta \leq 0.088$$

Lucy adopted $\beta = 0.08$ as a representative value.

5.3.2 A New Derivation of Lucy's Result

It is interesting to note that Lucy's results may be derived from simple considerations on one dimensional stellar models in the solar mass range.

Let us first recall that the surface of a star is usually determined by a surface pressure given by

$$P = \frac{2g_s}{3\kappa} \quad (5.34)$$

where g_s is the surface gravity and κ an average opacity. This boundary condition comes from the assumption of hydrostatic equilibrium of the atmosphere, namely

$$\frac{\partial P}{\partial z} = -\rho g \quad \Longleftrightarrow \quad \frac{1}{\rho \kappa} \frac{\partial P}{\partial z} = -\frac{g}{\kappa} \quad \Longleftrightarrow \quad \frac{\partial P}{\partial \tau} = \frac{g}{\kappa}$$

where the last relation is integrated from the zero optical depth down to $\tau = 2/3$. In the range of density and temperatures typical of the solar type stars, opacity may be approximated by a power law of the form:

$$\kappa = \kappa_0 \rho^\mu T^{-s} \quad (5.35)$$

For instance Christensen-Dalsgaard uses $\mu = 0.408$ and $s = -9.283$ for the Sun (e.g. Christensen-Dalsgaard and Reiter 1995).

Now in convective envelopes, the variation of pressure and density are related to temperature through

$$P \propto T^{n+1} \quad \text{and} \quad \rho \propto T^n .$$

namely with a polytropic law with $n = 3/2$.

Using the foregoing power laws for the opacity, pressure and density, we can express gravity as a function of temperature. We find that

$$g \propto T^{n(\mu+1)+1-s}$$

Identifying temperature and effective temperature, we find a gravity darkening exponent which reads:

$$\beta = \frac{1}{n(\mu + 1) + 1 - s} \quad (5.36)$$

Using Christensen-Dalsgaard's solar values and $n = 3/2$, the foregoing expression yields

$$\beta \simeq 0.0807$$

which is precisely the value found by Lucy. This is no surprise since Lucy used models similar to solar models, so the power law fit of Christensen-Dalsgaard is appropriate.

This derivation clearly shows that this β -exponent, as defined by (5.33), depends on the chemical properties of the surface through the opacities.

5.3.3 *Can Lucy's Law Represent a Gravity Darkening Effect?*

The foregoing derivation of Lucy's result enlightens us on the origin of Lucy's value of the β exponent. We see that it is essentially due to the strong dependence of opacity with temperature in the surface layers. Since the values for μ and s are chosen to fit the table values in some range of density and temperature, we understand that Lucy's result applies only to stars similar to the Sun, in terms of gravity and

effective temperature. We may note, as Espinosa Lara and Rieutord (2012), that if the opacity law extend in the deep layers so as to control the structure of the envelope and leads to a radiative one, then $\beta = 1/4$ because the polytropic index is $n = (s + 3)/(\mu + 1)$. We recover the previous result for non rotating radiative envelopes. We see that when the opacity is such that the polytropic index is less than $3/2$, and the envelope is convective, the β is governed by the opacity of the surface layers, those which are assumed to be transparent and fixing the atmosphere. The structure of the envelope is close to the adiabatic index $n = 3/2$.

Now we wish considering the case of rotating stars. The question is whether we can use the foregoing value of the exponent, if we consider a fast rotating star of solar type. A first obstacle is the validity of the boundary condition (5.34), which relies on a hydrostatic equilibrium. When rotation is present such an equilibrium is impossible because of baroclinicity (for the same reason as the origin of the so-called von Zeipel paradox, see Rieutord 2006). The proper boundary condition, replacing (5.34) should be derived from

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi$$

where \mathbf{v} is the fluid velocity in an inertial frame. Basically the flow is a differential rotation plus some weak meridional currents. The important point is that the differential rotation is latitude dependent. Hence, if we were to use some pressure boundary condition like (5.34), we should expect some extra variations from this latitudinal differential rotation. But this is likely not the whole story as we shall discuss it now.

If we consider the deep convective envelope of a rapidly rotating star, we might consider too contradicting effects. First the Coriolis effect: analysis of a linear stability of a convectively unstable layer shows that polar regions are less unstable than equatorial ones. This a consequence of the presence of the Coriolis force. This force indeed prevents variations of the velocity field along the rotation axis (the so-called Taylor–Proudman theorem). It shows up in numerical simulations as convective rolls parallel to the rotation axis near the equatorial regions (Glatzmaier and Olson 1993). For stars this may imply that the convective flux is larger in the equatorial plane than in the polar region, thus meaning a negative β . However, in the equatorial plane the effective gravity is less, and so is the buoyancy force. This is the effect of centrifugal force, which therefore points to more flux in the polar region (thus for a positive β). The conclusion of the foregoing argument is that nothing is clear. We may only guess that if Lucy’s law applies, it is for slowly rotating stars of solar type only. This is a deceptive conclusion since we may have interesting data only on fast rotators or evolved stars with weak self-gravity. In addition, the previous remarks do not mention the magnetic fields that are almost unavoidable in late-type stars.

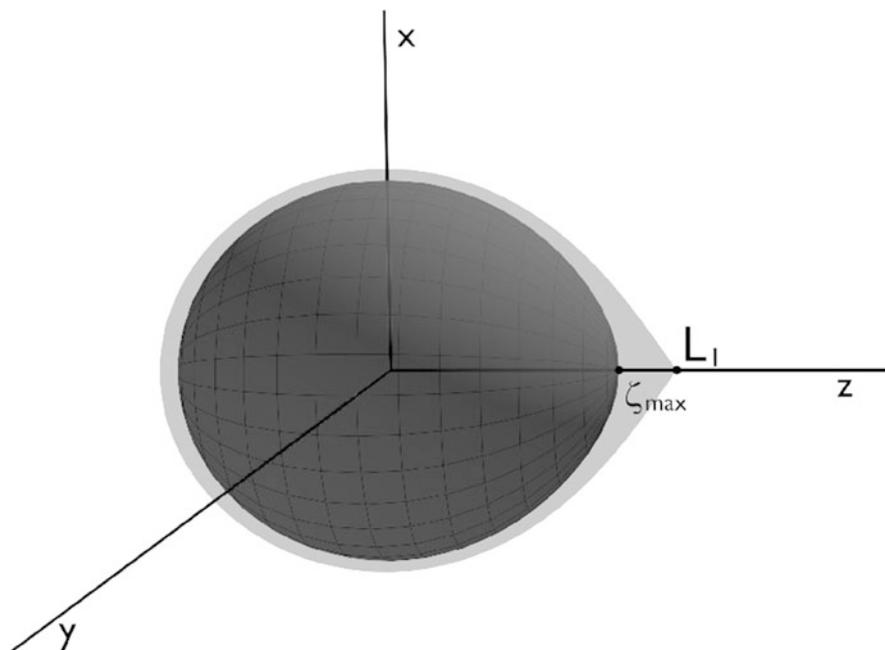


Fig. 5.7 Schematic representation of the primary star with filling factor $\rho = 0.8$. The position of the Lagrange point L_1 is shown on the z -axis that joins the centre of the two stars (from Espinosa Lara and Rieutord 2012)

5.4 Binary Stars

Binary stars is another domain where gravity darkening has been considered, mainly for reproducing the light curves of eclipsing binaries. We may wonder if the ω -model can be generalized to predict the gravity darkening of a star belonging to a binary system. It does but without any (known) analytic solution.

Let us follow the work of Espinosa Lara and Rieutord (2012). In the radiative envelope of an early-type star member of a binary system we can still write the conservation of the flux and assume the anti-parallelism of flux and effective gravity:

$$\text{Div}\mathbf{F} = 0 \quad \text{and} \quad \mathbf{F} = -f\mathbf{g}_{\text{eff}}$$

but now the effective gravity comes from the 3D potential:

$$\phi = -\frac{GM_1}{r} - \frac{GM_2}{\sqrt{a^2 + r^2 - 2ar \cos \theta}} - \frac{1}{2}\Omega^2 r^2 (\sin^2 \theta \sin^2 \varphi + \cos^2 \theta) + a \frac{M_2}{M_1 + M_2} \Omega^2 r \cos \theta, \quad (5.37)$$

where M_1 and M_2 are the masses of the two stars, 'a' is the distance between the two stellar centres and Ω is the orbital angular velocity. The orbit is assumed circular.

Let us write $\text{Div}(f\mathbf{g}_{\text{eff}}) = 0$ as

$$\mathbf{n} \cdot \nabla \ln f = \frac{\nabla \cdot \mathbf{g}_{\text{eff}}}{g_{\text{eff}}}, \quad (5.38)$$

where we set $\mathbf{g}_{\text{eff}} = -g_{\text{eff}}\mathbf{n}$.

We consider the three-dimensional curve $\mathcal{C}(\theta_0, \varphi_0)$ that starts at the centre of the star with the initial direction given by (θ_0, φ_0) , and that is tangent to \mathbf{n} at every point. $\mathcal{C}(\theta_0, \varphi_0)$ is therefore a field line of the effective gravity field.

The value of f at a point \mathbf{r} along the curve can be calculated as a line integral

$$f(\mathbf{r}) = f_0 \exp\left(\int_{\mathcal{C}(\theta_0, \varphi_0)} \frac{\nabla \cdot \mathbf{g}_{\text{eff}}}{g_{\text{eff}}} dl\right) \quad \text{for } \mathbf{r} \in \mathcal{C}(\theta_0, \varphi_0). \quad (5.39)$$

Despite much efforts no analytical expression could be found for f . Expression (5.39) is thus integrated numerically.

One interesting result of this approach, is that there is not a one-to-one relation between effective gravity and effective temperature. Indeed, because of the absence of symmetry of the star (except of the equatorial one if the obliquity is zero), two different points of the stellar surface may have the same effective gravity but a different effective temperature. This property comes from expression (5.39): the path integrals that lead to two points of identical effective gravity are not necessarily the same and can lead to different values of the flux. This property is illustrated in Fig. 5.8. In this figure we see that the curve $T_{\text{eff}} = f(g_{\text{eff}})$ is not smooth because

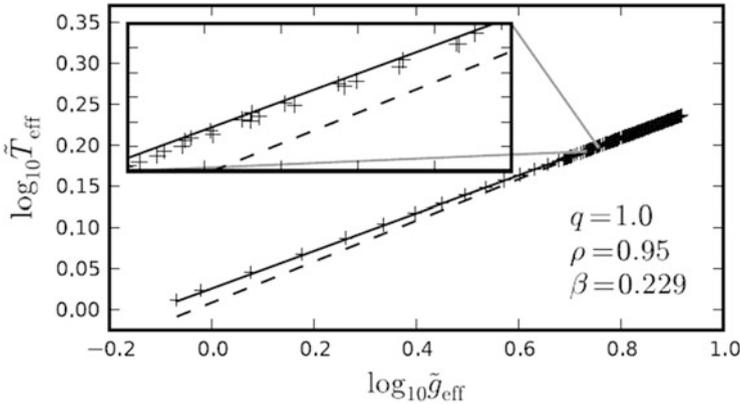


Fig. 5.8 From Espinosa Lara and Rieutord (2012), correlation of effective temperature and effective gravity in the primary early-type star of a binary system. Mass ratio is unity and the star fills the Roche lobe at 95 % (see text for our definition). The correlation may be represented by a β -exponent of 0.229. The *solid line* shows a linear fit, the *dashed line* the von Zeipel law, and pluses are from our generalized ω -model

similar values of g_{eff} lead to different values of T_{eff} . Fortunately, these variations are small.

As in Espinosa Lara and Rieutord (2012), we define q as the mass ratio, and evaluate the filling of the Roche lobe by the radius ρ of the star along the line joining the stellar centres, taking the distance between the star centre and the Lagrange L_1 point as unity. Hence, a star filling its Roche lobe has $\rho = 1$ while the one filling it at 95 % has $\rho = 0.95$ (see Fig. 5.7). The different positions where the same effective temperature are found, is illustrated in the two cases shown in Fig. 5.9. There we see that the curves of isoflux are not simple curves over the stellar surface.

As shown by Djurašević et al. (2003), the determination of the β -exponent from the light curves of semi-detached binaries is almost impossible since magnetic spots induce similar variations (see Figs. 5.10 and 5.11).

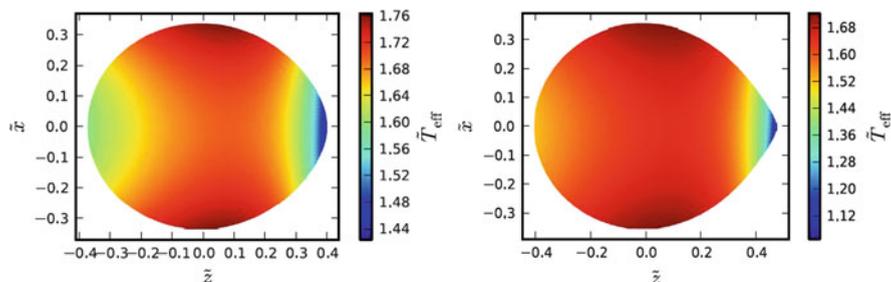


Fig. 5.9 From Espinosa Lara and Rieutord (2012): distribution of the effective temperature at the surface of a tidally distorted star. *Left:* $q = 1$ and $\rho = 0.8$. *Right:* $q = 1$ and $\rho = 0.95$

TV Cas

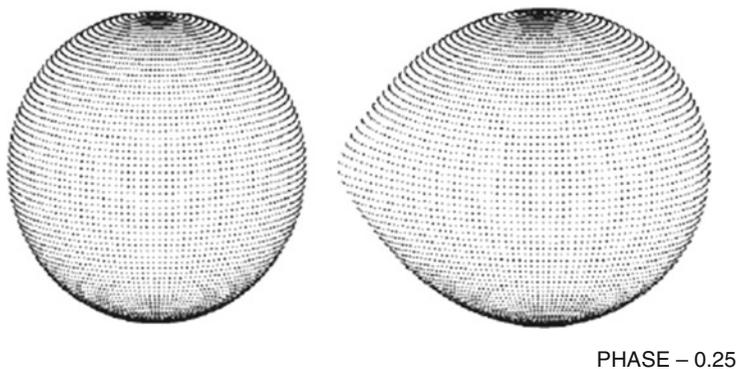


Fig. 5.10 A model of TV Cas that leads to $\beta = 0.15$ from fitting the light curve (from Djurašević et al. 2003)

TV Cas

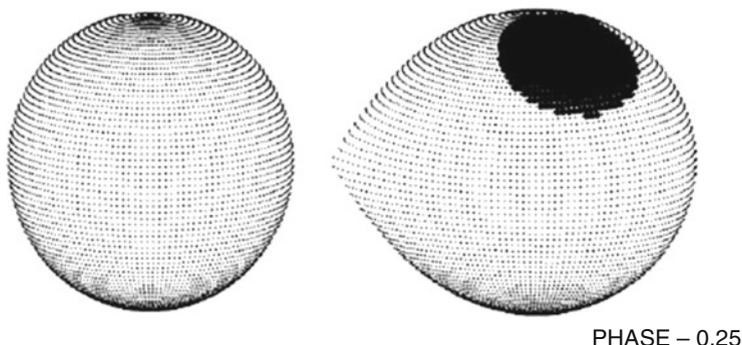


Fig. 5.11 The second model with a spot and $\beta = 0.25$ for TV Cas (from Djurašević et al. 2003); the difference between the calculated and observed light curve is the same as with the model of Fig. 5.11

5.5 Conclusions

To conclude these notes, I would like to stress a few points on gravity darkening:

- As far as non-magnetic early-type stars are concerned, gravity darkening has no longer to be proved. The use of the β -model, which is not physically sound can be left aside and replaced by the ω -model, which has the advantage of giving a direct estimate of the ω parameter.
- As far as late-type stars or giant stars are concerned, the situation is much more uncertain. The problem is indeed more difficult both on the theoretical and observational sides. On the theoretical side, the absence of any universally accepted model of turbulent rotating convection impedes any serious prediction on the latitude dependence of the convective flux. Observational constraints are therefore most welcome. However, this is not a simple matter either. Convective envelopes are usually harboring magnetic fields which can disturb the flux distribution. Ideally, the surface of these stars should be constrained by both interferometers and Zeeman-Doppler Imaging so as to disentangle the effects.
- Finally, for both type of (single) stars, we may recommend the following scheme of hypothesis and measurements. First assume the axi- and equatorial symmetry of the star. Then, if the star is centrally condensed (like an early-type or a giant one), adopt the Roche model. If the star is not centrally condensed (like a late-type star of the main sequence), a bipolytropic model is fine. Such a model, which fits the radiative and convective zones with a polytrope, just depends on three parameters, mass, equatorial radius and ω , just like the Roche model. Then, the flux or the effective temperature $T_{\text{eff}}(\theta)$ can be derived after an expansion on the spherical harmonic basis along with an atmosphere model used for the

determination of the limb darkening effect. The gravity darkening law can then be evaluated from the curve (or correlation) $T_{\text{eff}}(\theta)$ versus $g_{\text{eff}}(\theta)$.

Acknowledgements I am grateful to the organizers of the Besançon school for their invitation, and the opportunity to present in more details the recent work I did with Francisco Espinosa Lara on gravity darkening. This school triggered many stimulating discussions that helped me deepen this subject. Finally, I would like to stress that this work owes much to Francisco who had the original idea of the ω -model.

Appendix: Angular Velocity with Respect to Critical Rotation

In this appendix we discuss the correspondence between two definitions of the scaled angular velocity. The first one is the one we used in the text, namely

$$\omega = \frac{\Omega}{\Omega_k} = \Omega \left(\sqrt{\frac{GM}{R_e^3}} \right)^{-1}$$

where Ω_k is the orbital angular velocity for an orbit at the actual equatorial radius of the star.

The other definition is based on the Roche model and considers the angular velocity Ω_c such that the rotation on the equatorial radius is keplerian. This latter definition is a true critical angular velocity, while the previous one is a keplerian velocity at the actual equatorial radius. However, the critical angular velocity is model dependent, this is why we have to mention Roche's model for the definition of Ω_c . The first definition does not need any model, but it is not the exact critical angular velocity. This latter quantity cannot in general be computed a priori with just a given spherical model of a star. It needs a full computation of the structure at the actual critical velocity and is thus an output of 2D models like ESTER ones (e.g. Espinosa Lara and Rieutord 2013).

So we now only consider Roche models where all quantities can be derived in a simple manner. We recall that the polar R_p and equatorial R_e of an equipotential of a star rotating at angular velocity Ω are related by

$$\frac{GM}{R_p} = \frac{GM}{R_e} + \frac{1}{2}\Omega^2 R_e^2 \quad (5.40)$$

Then, the critical angular velocity Ω_c and the critical equatorial radius R_{ec} are related by

$$\Omega_c^2 = \frac{GM}{R_{ec}^3} \quad (5.41)$$

and hence

$$R_{ec} = \frac{3}{2}R_p \quad (5.42)$$

at critical rotation.

If the rotation is subcritical, the Roche model gives the following relation between R_e and R_p

$$R_p = R_e \left(1 + \frac{\omega^2}{2}\right)^{-1} \quad (5.43)$$

But we may take Ω_c as the scale of the rotation rate and set

$$\tilde{\omega} = \frac{\Omega}{\Omega_c} \quad (5.44)$$

From the preceding definitions we get the relation between $\tilde{\omega}$ and ω , namely

$$\tilde{\omega} = \omega \sqrt{\frac{27}{8}} (1 + \omega^2/2)^{-3/2} \quad (5.45)$$

We note that if $\omega = 1$ then $\tilde{\omega} = 1$ as expected. We also note that if Ω is subcritical, then $R_e < 3R_p/2$ and therefore $\Omega_k > \Omega_c$, which implies that we always have

$$\tilde{\omega} \geq \omega \quad (5.46)$$

Equation (5.45) shows that it is easy to compute $\tilde{\omega}$ from ω but the opposite is a little more complicated since a cubic equation must be solved. Setting $\chi = \arcsin \tilde{\omega}$, we find

$$\omega = \sqrt{\frac{6}{\tilde{\omega}} \sin(\chi/3) - 2} \quad (5.47)$$

References

- Alecian, G. (2013). Atomic diffusion in the atmospheres of upper main-sequence stars. In *New advances in stellar physics: From microscopic to macroscopic processes. EAS Publications Series* (Vol. 63, pp. 219–226).
- Christensen-Dalsgaard, J., & Reiter, J. (1995). A comparison of precise solar models with simplified physics. In R. K. Ulrich, E. J. Rhodes, Jr., & W. Dappen (Eds.), *ASP Conf. Ser. 76: GONG 1994. Helio- and Astro-Seismology from the Earth and Space* (p. 136).
- Djurašević, G., Rovithis-Livaniou, H., Rovithis, P., Georgiades, N., Erkapić, S., & Pavlović, R. (2003). Gravity-darkening exponents in semi-detached binary systems from their photometric observations: Part I. *Astronomy and Astrophysics*, 402, 667–682.

- Domiciano de Souza, A., Kervella, P., Moser Faes, D., Dalla Vedova, G., Mérand, A., Le Bouquin, J.-B., et al. (2014). The environment of the fast rotating star Achernar. III. Photospheric parameters revealed by the VLTI. *Astronomy and Astrophysics*, 569, A10.
- Donati, J.-F., Howarth, I. D., Jardine, M. M., Petit, P., Catala, C., Landstreet, J. D., et al. (2006). The surprising magnetic topology of τ Sco: Fossil remnant or dynamo output? *Monthly Notices of the Royal Astronomical Society*, 370, 629–644.
- Espinosa Lara, F., & Rieutord, M. (2011). Gravity darkening in rotating stars. *Astronomy and Astrophysics*, 533, A43.
- Espinosa Lara, F., & Rieutord, M. (2012). Gravity darkening in binary stars. *Astronomy and Astrophysics*, 547, A32.
- Espinosa Lara, F., & Rieutord, M. (2013). Self-consistent 2D models of fast rotating early-type stars. *Astronomy and Astrophysics*, 552, A35.
- Glatzmaier, G., & Olson, P. (1993). Highly supercritical thermal convection in a rotating spherical shell: Centrifugal vs. radial gravity. *Geophysical and Astrophysical Fluid Dynamics*, 70, 113–136.
- Korhonen, H., González, J. F., Briquet, M., Flores Soriano, M., Hubrig, S., Savanov, I., et al. (2013). Chemical surface inhomogeneities in late B-type stars with Hg and Mn peculiarity. I. Spot evolution in HD 11753 on short and long time scales. *Astronomy and Astrophysics*, 553, A27.
- Lucy, L. B. (1967). Gravity-darkening for stars with convective envelopes. *Zeitschrift für Astrophysik*, 65, 89.
- Monnier, J., Che, X., Zhao, M., Ekström, S., Maestro, V., Aufdenberg, J., et al. (2012). Resolving vega and inclination controversy with CHARA/MIRC. *The Astrophysical Journal*, 761, L3.
- Rieutord, M. (2006). On the dynamics of radiative zones in rotating star. In M. Rieutord, & B. Dubrulle (Eds.), *Stellar fluid dynamics and numerical simulations: From the sun to neutron stars*. EAS (Vol. 21, pp. 275–295).
- Rieutord, M., & Rincon, F. (2010). The sun's supergranulation. *Living Reviews in Solar Physics*, 7, 1–70.
- Stein, R. F., & Nordlund, Å. (1998). Simulations of solar granulation. I. General properties. *The Astrophysical Journal*, 499, 914.
- Vauclair, S., & Vauclair, G. (1982). Element segregation in stellar outer layers. *Annual Review of Astronomy and Astrophysics*, 20, 37–60.
- von Zeipel, H. (1924). Zum Strahlungsgleichgewicht der Sterne. In *Probleme der Astronomie (Festschrift für H. von Seeliger)* (pp. 144–152). Heidelberg: Springer.