

Excitation of inertial modes in an experimental spherical Couette flow

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Spherical Couette flow (flow between concentric rotating spheres) is one of flows under consideration for the laboratory magnetic dynamos. Recent experiments have shown that such flows may excite Coriolis restored inertial modes. The present work aims to better understand the properties of the observed modes and the nature of their excitation. Using numerical solutions describing forced inertial modes of a uniformly rotating fluid inside a spherical shell, we first identify the observed oscillations of the Couette flow with nonaxisymmetric, retrograde, equatorially antisymmetric inertial modes, confirming first attempts using a full sphere model. Although the model has no differential rotation, identification is possible because a large fraction of the fluid in a spherical Couette flow rotates rigidly. From the observed sequence of the excited modes appearing when the inner sphere is slowed down by step, we identify a critical Rossby number associated with a given mode, below which it is excited. The matching between this critical number and the one derived from the phase velocity of the numerically computed modes shows that these modes are excited by an instability likely driven by the critical layer that develops in the shear layer, staying along the tangent cylinder of the inner sphere.

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I. INTRODUCTION

Large-scale flows in stars or planets in many circumstances take place in a spherical shell. Most astrophysical fluid flows are also under the dominating influence of a background rotation. This rotation leads to the presence of inertial oscillations for which the restoring mechanism comes from the Coriolis acceleration.

The properties of these modes of oscillation are not fully understood, in part because they obey a hyperbolic equation in the space variables and therefore do not easily comply with boundary conditions. The solutions of this equation, known as the Poincaré equation, have been studied in some detail in the recent years, using numerical and analytical tools (e.g., [1–6]). It has been shown that most of the eigenmodes of a rotating spherical fluid layer require viscosity to exist. Indeed, viscosity is necessary to regularize the singularities formed by the focusing of characteristics by the boundaries. Viscosity transforms these singularities into shear layers whose thickness scales with some fractional power of it (exponents $\frac{1}{4}$ or $\frac{1}{3}$ are the most common). In addition, very recent works [6,7] showed that the critical latitude singularity, where the characteristics of the hyperbolic equation are tangent to the inner core boundary, plays a crucial role in periodically forced flows. The reason for that is presently not clear.

However, all the aforementioned previous works are theoretical studies considering idealized situations and therefore should be compared to experimental studies. Observations of inertial modes are not extensive, either in nature or in the laboratory. A landmark in the experimental studies is the work of Aldridge [8,9]. More recently, attractors of characteristics triggering oscillatory shear layers have been investigated experimentally for the understanding of ocean dynamics. Such experiments were conducted both on stably stratified fluids [10–12] and rotating fluids [13–15] since internal modes (gravity waves) and inertial modes share many of the same mathematical properties. Other experiments have demonstrated the excitation of inertial waves in a fluid inside a precessing spheroidal cavity [16].

All these experiments have shown that inertial modes are robust features of rotating fluid flows. In a very recent experiment aimed at studying a fluid dynamo, inertial modes were detected through their coupling with an imposed magnetic field, in a spherical Couette flow [17]. This flow can indeed produce magnetic fields at sufficiently high magnetic Reynolds numbers [18]. In that experiment, the fluid was contained in a spherical shell with inner and outer radii equal to 10 and 30 cm respectively. The Ekman number, the nondimensional measure of viscosity (see below), was approximately 10^{-7} . In astrophysical or geophysical (Earth's core) situations this number is less than 10^{-12} . In fact, as was shown by recent numerical work (e.g., [3,6]), the asymptotic regime describing vanishingly small viscosities usually appears at Ekman numbers below 10^{-8} . A new experiment geometrically similar to the Earth's core, using a sphere with an outer radius of 1.46 m, offers a unique opportunity to observe some near-singular inertial

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modes close to their asymptotic regime since now Ekman numbers can be as low as 2.5×10^{-8} . Indeed, first results on this experiment using a precessionally forced flow [19] provided clear evidence of detached shear layers spawned by the critical latitude singularities.

In this paper we further consider the results of this experiment. Thanks to a simple model based on numerical solutions of forced inertial modes in a spherical shell, we find a scenario that explains the excitation of inertial modes in a nonlinear spherical Couette flow. For this, we first describe the experimental setup and the observational facts concerning the observed inertial modes (Sec. II). We then compute the response of an incompressible fluid inside a rigidly rotating spherical shell when some periodic forcing is applied (Sec. III). We use this simple model to identify the resonance peaks and to interpret their full width at half-maximum (FWHM). We then discuss a scenario that explains most of the experimental facts (Sec. IV). Conclusions end the paper.

II. THE EXPERIMENTAL SETUP AND OBSERVATIONS

The 3-m Geodynamo Experiment built at the University of Maryland consists of a rotating, stainless-steel spherical vessel with 1.46 m radius and an independently rotating 1.02-m-diameter inner sphere located at the center of the outer spherical vessel (see a schematic view in Fig. 1). This results in aspect ratio $\eta \simeq 0.348$. There is a 16.8-cm-diameter shaft supporting the inner sphere that extends along the axis of the

outer sphere and is connected to a motor. The outer sphere is driven by a motor via a geared ring attached to the top lid. The space between the spheres is filled here with water at room temperature (kinematic viscosity $\nu = 1.004 \times 10^{-6} \text{ m}^2/\text{s}$). The Rossby number is defined as $\text{Ro} = (\Omega_{\text{inner}}/\Omega) - 1$ where Ω_{inner} is the angular speed of the inner sphere and Ω is the angular speed of the outer sphere. The outer sphere can spin at a maximum of 4 Hz and a minimum of 0.05 Hz, although for the measurements presented here the maximum rotation rate corresponds to 1.5 Hz. Defining the Ekman number as $E = \nu/2\Omega R^2$, where $R = 1.46 \text{ m}$, the experimentally accessible range is $2.5 \times 10^{-8} < E < 7.5 \times 10^{-7}$. The inner sphere can reach a rotation rate up to 20 Hz with a minimum of 0.15 Hz, which translates into a Rossby number range such that $0.03 \lesssim |\text{Ro} + 1| \lesssim 400$.

During the experimental run corresponding to the measurements shown below, the outer sphere had a fixed rotation rate of 1.5 Hz ($E = 2.5 \times 10^{-8}$), while the inner sphere's angular speed was varied. We started with the inner sphere counter-rotating (near $\text{Ro} = -2$) and gradually reduced its angular speed in steps of 0.05 Hz until its minimum speed was reached ($\text{Ro} \simeq -1.1$). At that point the inner sphere's rotation direction was changed and its speed was increased from its minimum ($\text{Ro} \simeq -0.9$) until it reached almost the same speed as the outer sphere ($\text{Ro} \simeq -0.1$). Each speed step was maintained for about 15 min. Figure 2 shows a power spectral density $S(\omega)$ spectrogram computed from the pressure on one of the instrumentation ports, measured during counter-rotation. Figure 3 shows a spectrogram of the pressure measured during corotation.

The flow is monitored by pressure probes and an ultrasound velocimetry transducer (see Fig. 1), all fixed to the outer bounding sphere. The pressure spectrograms show the

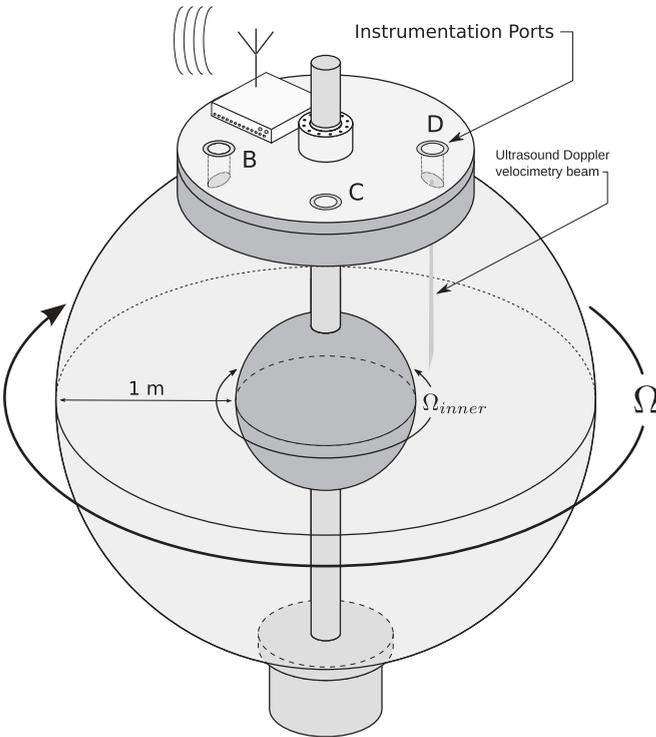


FIG. 1. Schematic of the 3-m spherical Couette apparatus. Inner and outer spheres rotate independently, driven by two 250-kW motors (not shown). There are three pressure sensors (azimuthally 90° apart) on the top lid ports, and an ultrasound velocimetry transducer measuring vertical components of fluid velocities (ultrasound beam depicted coming from port D).

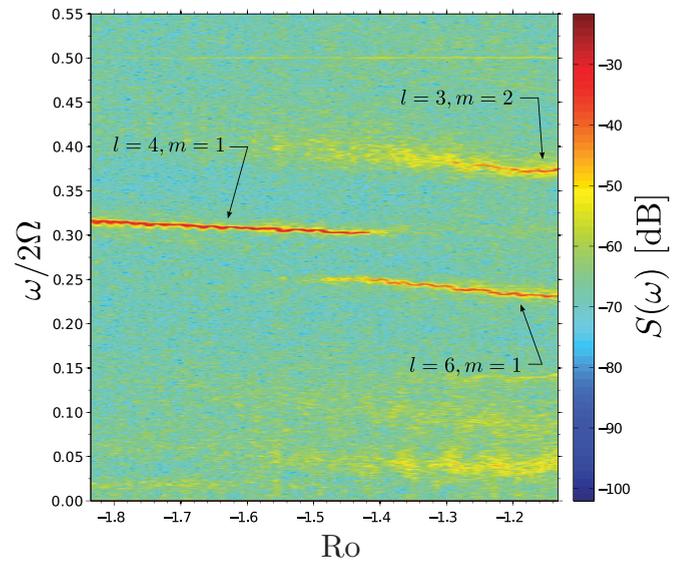


FIG. 2. (Color online) Spectrogram from pressure measurements as the inner sphere rotates with different speeds in counter-rotation (inner sphere rotating in the opposite direction as the outer sphere). Each vertical line in the spectrogram is the power spectral density of the pressure using $\rho \text{Ro} \Omega^2 L^2$ as the unit pressure. Outer sphere rotation rate is 1.5 Hz, corresponding to $E = 2.5 \times 10^{-8}$. Modes indicated correspond to full sphere modes characterized by $(l, m, \omega/2\Omega)$.

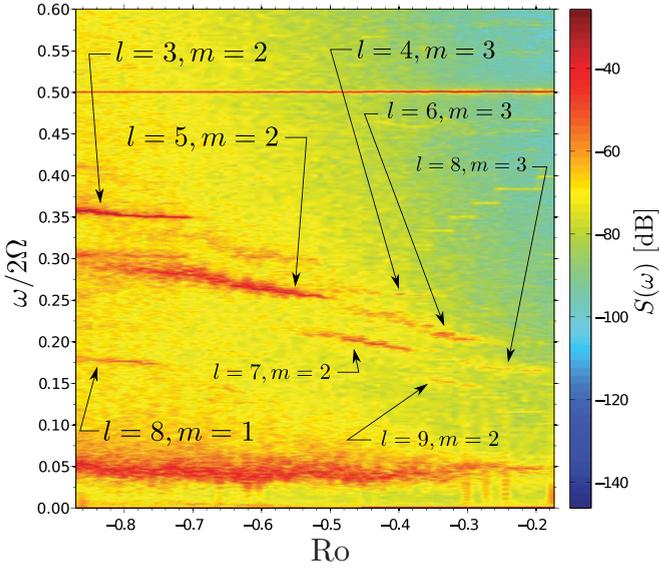


FIG. 3. (Color online) Same as Fig. 2 but for a Ro range corresponding to corotation (inner sphere rotating in the same direction as the outer sphere). The band near $\omega/2\Omega = 0.05$, which has an azimuthal number $m = 1$, is possibly not a single inertial mode. Its signature in the experiment reported by Kelley *et al.* [17] is weak, perhaps a consequence of its quasigeostrophic character (see the discussion at the end of Sec. III B).

signature of inertial modes excited in a sequence as Ro is varied. The azimuthal wave numbers m could be determined up to $m = 4$ by comparing the relative phases of the pressure signals from different ports. These modes match very closely the frequencies and azimuthal wave numbers of the inertial modes excited in the hydromagnetic experiment performed by Kelley *et al.* [20] using a 60-cm diameter sphere, which is geometrically similar but smaller than the 3-m sphere. We use the spatial pattern data from the 60-cm experiment in order to identify the modes in the 3-m experiment. Those modes, as evidenced by lines in the spectrogram, correspond approximately to full sphere inertial modes [21,22], which

can be characterized by a pair of indices (l, m) and a dimensionless frequency $\hat{\omega} = \omega/2\Omega$. We also noticed the retrograde propagation of the modes, thanks to the pressure probes distributed in longitude over the outer shell. This was also the case for the previous (smaller) experiment reported by Kelley *et al.* [17].

The spectrograms show that the frequency of excited modes varies with the Rossby number. In Figs. 4 and 5 we plot these variations for five modes and two values of the Ekman number. The variations are approximately linear, showing an increasing frequency with a decreasing (negative) Rossby number or, equivalently, with an increasing differential rotation. A change of the Ekman number induces only a mild change of the frequencies. However, we may notice that the first of the $m = 1$ modes (Fig. 4) shows a steeper dependence with Ro for the lowest value of the Ekman number.

The pressure spectrograms are complemented by velocity measurements displayed in Fig. 6. Those show the vertically averaged velocity amplitude over a fluid layer located at $0.8325 < z/R < 0.8599$, as a function of the Rossby number (z is the coordinate along the rotation axis, with $z = 0$ corresponding to the equatorial plane). The various curves show that some modes, already identified in the pressure spectrograms, dominate the dynamics of the sampled layer for specific ranges of the Rossby number. Especially, we observe that the spin-over mode ($m = 1$ and $\omega = -\Omega$) is always present and dominates the oscillations for $Ro \gtrsim -0.4$.

III. NUMERICAL MODEL

The foregoing description of the experiment shows that despite a steady forcing, the generated flows are unsteady [17]. This means that the spherical Couette flow is unstable to time dependence and the excitation of inertial modes.

From the observed features of spherical Couette flow, which is shown in Fig. 7 in the limit $|Ro| \ll 1$, we expect that most of the time-dependent forcing occurs in the fluid lying near or inside the tangent cylinder. This cylinder is clearly marked by the vertical Stewartson shear layer that exhibits large viscous dissipation. Most of the shear is localized inside the Stewartson

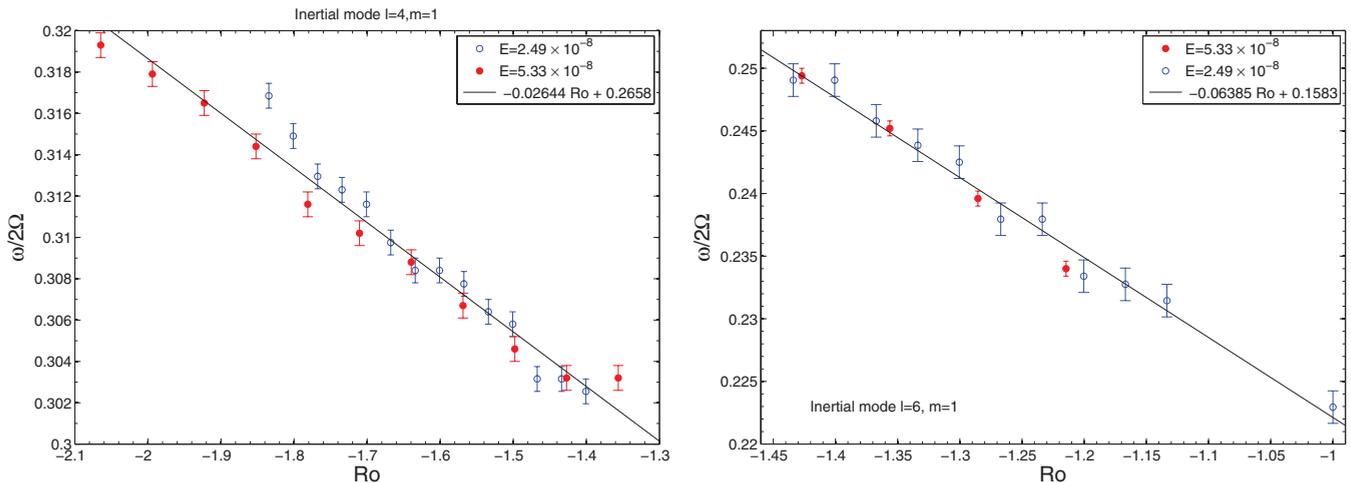
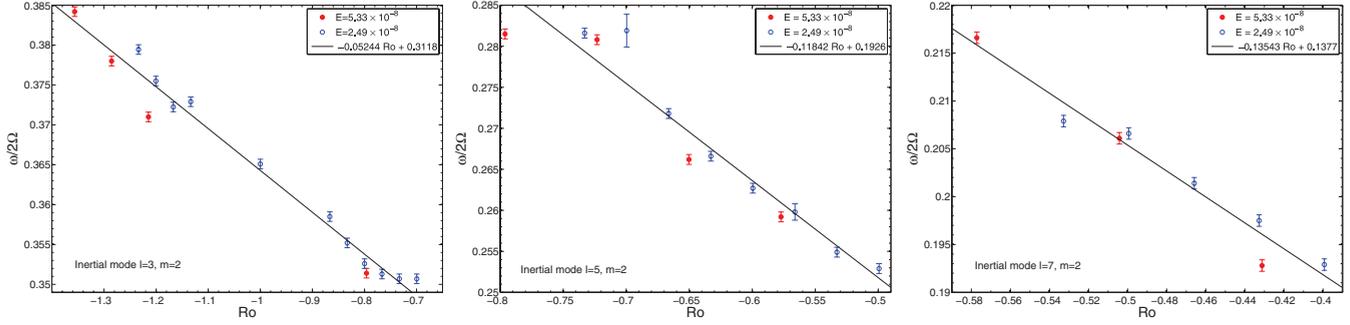


FIG. 4. (Color online) Rossby number dependence of the frequency of the two most prominent $m = 1$ modes. The inset gives the equation of the linear best fit.


 FIG. 5. (Color online) Same as Fig. 4 but for the three most prominent $m = 2$ modes.

layer. This is also approximately the case at finite Rossby numbers as shown by Matsui *et al.* [23]. However, the volume inside the tangent cylinder is a rather small fraction of the total volume of the fluid. This fraction reads

$$1 - \frac{(1 - \eta^2)^{3/2}}{1 - \eta^3} \simeq 3\eta^2/2 - \eta^3 \quad \text{if } \eta \ll 1.$$

In the actual experiment where $\eta = 0.348$, this fraction is 0.14. Thus, 86% of the fluid lies outside the tangent cylinder and rotates near the solid body with the angular velocity of the outer shell (e.g., [24]). This remark suggests that oscillations of a rigidly rotating spherical fluid shell may not be very far from those observed in the aforementioned experiment and that, as a first step, differential rotation can be ignored.

A. Mathematical formulation

Assuming that the inertial modes are of relatively small amplitude, they solve the linearized equations governing a

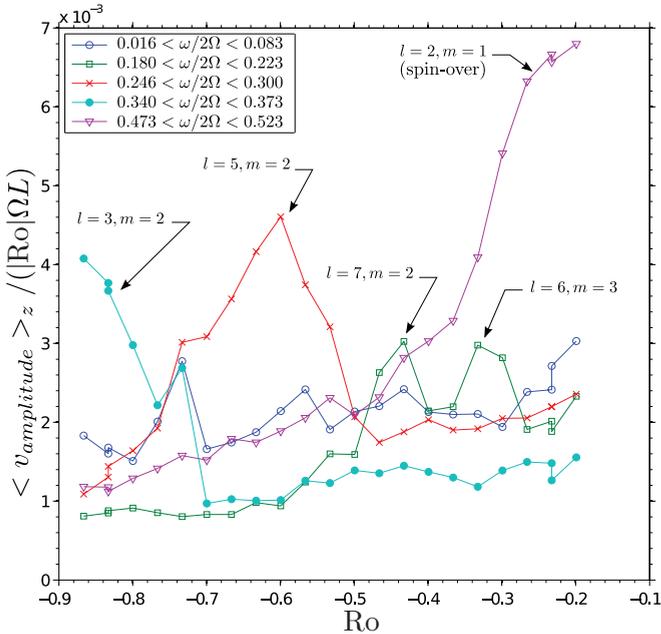


FIG. 6. (Color online) Velocity amplitude averaged over a thin layer, perpendicular to the rotation axis, located around $z = 0.846$ as a function of the Rossby number. Each curve corresponds to the averaged velocity amplitude restricted to the frequency range indicated in the legend. Labels correspond to identified inertial modes.

viscous rotating fluid with constant density. We force periodic perturbations in the numerical simulations to drive the modes. With a length scale of the outer radius of the shell R , the time scale as $(2\Omega)^{-1}$, the equations of the nondimensional pressure (p) and velocity perturbation (\vec{u}) may be written as

$$\begin{aligned} i\hat{\omega}\vec{u} + \vec{e}_z \times \vec{u} &= -\vec{\nabla}p + E\Delta\vec{u} \\ \text{div } \vec{u} &= 0, \end{aligned} \quad (1)$$

where $\hat{\omega} = \omega/2\Omega$ is a dimensionless frequency, $E = \nu/2\Omega R^2$ is the Ekman number, and ν is the kinematic viscosity.

We complete these equations with no-slip boundary conditions, which assists in forcing the flow. Since the real forcing is not known, we drive oscillations by setting a toroidal motion on one of the boundaries. Namely, we assume that on either the inner or outer boundary

$$\vec{u}(\theta, \varphi) = \left(\frac{1}{\sin\theta} \frac{\partial Y_m^m}{\partial\varphi} \vec{e}_\theta - \frac{\partial Y_m^m}{\partial\theta} \vec{e}_\varphi \right) e^{i\hat{\omega}\tau} \quad (2)$$

and

$$\vec{u} = 0 \quad (3)$$

on the other boundary. We note that forcing by boundaries has also been used to force axisymmetric inertial modes [8,25,26].

We solve these equations numerically, following the same spectral method as in [6]. The fields are first expanded into

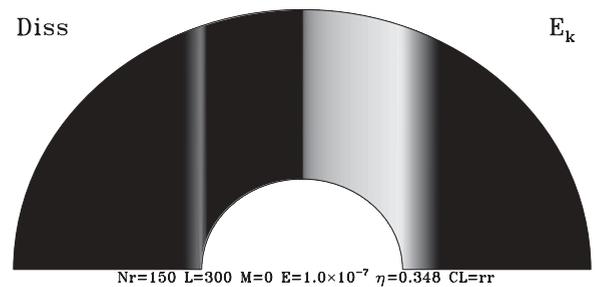


FIG. 7. Numerically computed spherical Couette flow at very low Rossby number $|\text{Ro}| \ll 1$. This solution has been obtained by solving steady, linear, axisymmetric equations of the spherical Couette flow using a spectral method with spherical harmonics (horizontally) and Chebyshev polynomials (radially). Left: the viscous dissipation emphasizing the Stewartson shear layer. Right: the kinetic energy of the flow in the reference frame of the outer shell.

spherical harmonics as follows:

$$\vec{u} = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} u_m^\ell(r) \vec{R}_\ell^m + v_m^\ell(r) \vec{S}_\ell^m + w_m^\ell(r) \vec{T}_\ell^m,$$

with

$$\vec{R}_\ell^m = Y_\ell^m(\theta, \varphi) \vec{e}_r, \quad \vec{S}_\ell^m = \vec{\nabla} Y_\ell^m, \quad \vec{T}_\ell^m = \vec{\nabla} \times \vec{R}_\ell^m,$$

where gradients are taken on the unit sphere. The radial u_m^ℓ and w_m^ℓ function thus verify

$$\begin{aligned} E \Delta_\ell w_m^\ell - i \hat{\omega} w_m^\ell &= -A_\ell r^{\ell-1} \frac{\partial}{\partial r} \left(\frac{u_m^{\ell-1}}{r^{\ell-2}} \right) - A_{\ell+1} r^{-\ell-2} \frac{\partial}{\partial r} \left(r^{\ell+3} u_m^{\ell+1} \right) \\ E \Delta_\ell \Delta_\ell (r u_m^\ell) - i \hat{\omega} \Delta_\ell (r u_m^\ell) &= B_\ell r^{\ell-1} \frac{\partial}{\partial r} \left(\frac{w_m^{\ell-1}}{r^{\ell-1}} \right) + B_{\ell+1} r^{-\ell-2} \frac{\partial}{\partial r} \left(r^{\ell+2} w_m^{\ell+1} \right) \end{aligned} \quad (4)$$

for $\ell \in [m, L]$. L is the order of the truncation in the spherical harmonics expansion. v_m^ℓ is eliminated using mass conservation, and we introduce

$$A_\ell = \frac{1}{\ell^2} \sqrt{\frac{\ell^2 - m^2}{4\ell^2 - 1}}, \quad B_\ell = \ell^2(\ell^2 - 1)A_\ell,$$

$$\Delta_\ell = \frac{1}{r} \frac{d^2}{dr^2} r - \frac{\ell(\ell+1)}{r^2}.$$

The radial functions are then discretized on Gauss-Lobatto collocations nodes. Equations are completed by boundary conditions (2) and (3) leading to a linear system like $[A]x = b$, which is solved by classical numerical methods. Note that these numerical solutions do not take into account the shaft bearing the inner core.

B. Results from the model

The above model is very simple since it just reproduces the geometry of the container, the background rotation outside the tangent cylinder, and the effects of viscosity. No differential rotation is included. Thus, from the comparison of its results with the data we expect only an identification of the modes and a comparison of the width of the resonances (i.e., of the dissipative processes).

Solving Eq. (1) with Eqs. (2) and (3), we computed the resonance curves for various azimuthal symmetries, namely $m = 1, 2, 3, 4$. Figures 8 and 9 show the total kinetic energy of the responding oscillations in some arbitrary unit as a function of the forcing frequency. We show the response of the fluid when the excitation is imposed either on the inner or outer boundaries. Obviously, the outer boundary forcing is much more efficient than the inner boundary one. The hatched bands correspond to the observed range of frequencies of the oscillations when the Rossby number is varied. As shown by the spectrograms (Figs. 2 and 3 and Figs. 4 and 5), the frequency of the modes systematically decreases when $|\text{Ro}|$ decreases, that is, when the background flow gets closer to the solid body rotation. Therefore, only the left boundary of the frequency bands should be compared to the frequencies

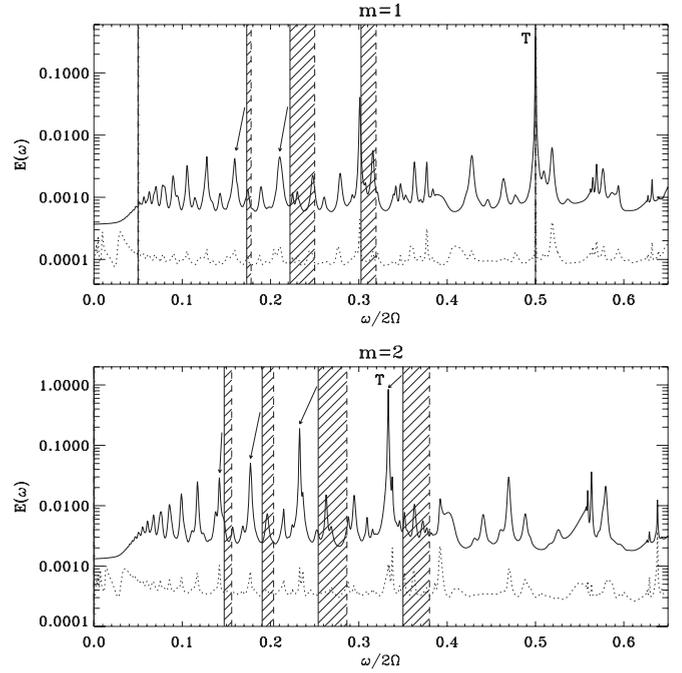


FIG. 8. Resonance curves showing the kinetic energy (in arbitrary units) of the model oscillating flow as a function of the scaled frequency of the forcing. The solid line is for the forcing on the outer boundary, the dotted line for a forcing on the inner shell. The hatched bands of frequencies show the range of frequencies of the observed modes, when the Rossby number is varied. The lower bound of the bands, materialized by a solid vertical line, corresponds to the lowest $|\text{Ro}|$ value. The “T” marks the toroidal mode resonance. Arrows show the identifications we suggest between observed oscillations frequencies and resonances in the model. Top figure is for the $m = 1$ forcing, while the bottom figure is for an $m = 2$ forcing. The Ekman number is set to $E = 2.5 \times 10^{-8}$.

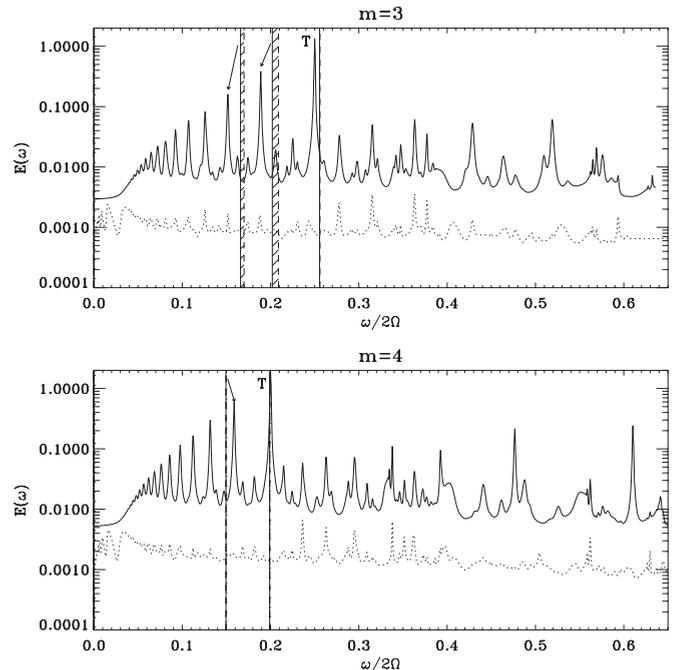


FIG. 9. Same as Fig. 8 but for $m = 3$ and $m = 4$.

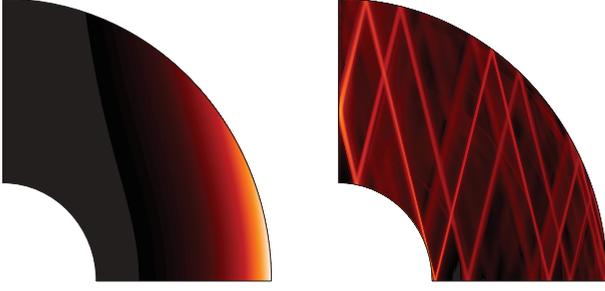


FIG. 10. (Color online) Left: the kinetic energy distribution in a meridional plane of the flow associated with the resonance of the $m = 3$ mode at $\hat{\omega} = 0.25$. The Ekman number used for this calculation is 2.5×10^{-9} . The numerical resolution is $L = 780$ (highest spherical harmonic order) and $N_r = 300$ (highest Chebyshev polynomial order). Right: the same as left but when the large-scale part of the flow field described by Eq. (5) is filtered out.

of the model. Thus doing, we can identify clearly the purely toroidal modes for $m = 1, 2, 3, 4$ whose frequencies are

$$\hat{\omega} = \frac{\omega}{2\Omega} = \frac{1}{m+1}.$$

For these frequencies, an analytical solution exists [2] and reads

$$\begin{aligned} v_\theta &= Ar^m (\sin \theta)^{m-1} \sin(m\phi + \omega_m t) \\ v_\phi &= Ar^m (\sin \theta)^{m-1} \cos \theta \cos(m\phi + \omega_m t) \end{aligned} \quad (5)$$

if viscosity is neglected. When viscosity is included the structure of these modes is more complex since shear layers are excited by the critical latitude singularity. In Fig. 10 we display the kinetic energy of the flow at $\omega/2\Omega = 1/4$ ($m = 3$) and when the large-scale part expressed in Eq. (5) is filtered out. One clearly sees the dissipative structures which are associated with the viscous stress.

Back to Figs. 8 and 9, we note that the frequency mismatch between model and experiment becomes less as m increases for the purely toroidal modes. This is because modes of high m are more concentrated in the equatorial region of the outer shell [as may be seen from Eq. (5)] and therefore less affected by the differential rotation.

Other modes with $m = 2$ or $m = 3$ may also be identified with the resonance peaks, which stay on the left of the band. Indeed, the resonances of the model are valid for the asymptotic case $\text{Ro} \rightarrow 0$, while the observed frequencies decrease with $|\text{Ro}|$. For other nontoroidal modes (with $m = 1$ and $m = 4$) identification is only tentative.

Besides, the model gives the width of the resonances. Table I shows the FWHM of the identified modes and the corresponding value of the model. We note that for half of the observed modes, the comparison is good. For the remaining modes, the peaks are either much wider or much narrower than those of the model. In a linear model, the width of the peaks is related to the dissipative process at work or to a casual blend of another neighboring peak. Wider experimental peaks mean a higher dissipation in the experiment than in the model, which is understandable: The experimental flow may contain small-scale turbulence that can increase dissipation. The opposite is more surprising. It may mean either a misidentification or a real change of the important scales of the modes when

TABLE I. Comparison of the experimental FWHM ($\delta\hat{\omega}$) of the mode at frequency $\hat{\omega}$ of azimuthal wave number m , measured at Rossby number Ro , with the FWHM predicted by the model. The last column gives the ratios of the experimental and numerical FWHMs.

m	$\hat{\omega}$	$\delta\hat{\omega} \times 10^4$	Ro	$\delta\hat{\omega}_{\text{model}} \times 10^4$	$\delta\hat{\omega}/\delta\hat{\omega}_{\text{model}}$
1	0.31	29	-1.75	12	2.4
1	0.23	8	-1.13	56	0.14
1	0.17	9	-0.80	38	0.24
2	0.3526	8	-0.80	8	1.0
2	0.2628	12	-0.60	10	1.2
2	0.1980	23	-0.43	29	1.26
2	0.1667	20	-0.33	22	0.93
3	0.2573	20	-0.4	10	2.0
3	0.2035	12	-0.3	12	1.0
3	0.1662	9	-0.23	16	0.55

the differential rotation is strong. We actually note that the mismatch is larger for the $m = 1$ resonances, which exist at very negative Ro .

In Fig. 11, we show two resonances which appear at $\text{Ro} = -0.75$. One is associated with $m = 1$. The model curve has been shifted so that peaks frequencies coincide. We see that the widths poorly match. The other mode visible at this Rossby number is the $m = 2$ purely toroidal one. Its observed frequency is $\hat{\omega} \sim 0.35$, expected at $\hat{\omega} = \frac{1}{3}$ for a solid body rotation. Here too we shift the model curve and note that the widths of the peaks better match. Moreover, we also notice a

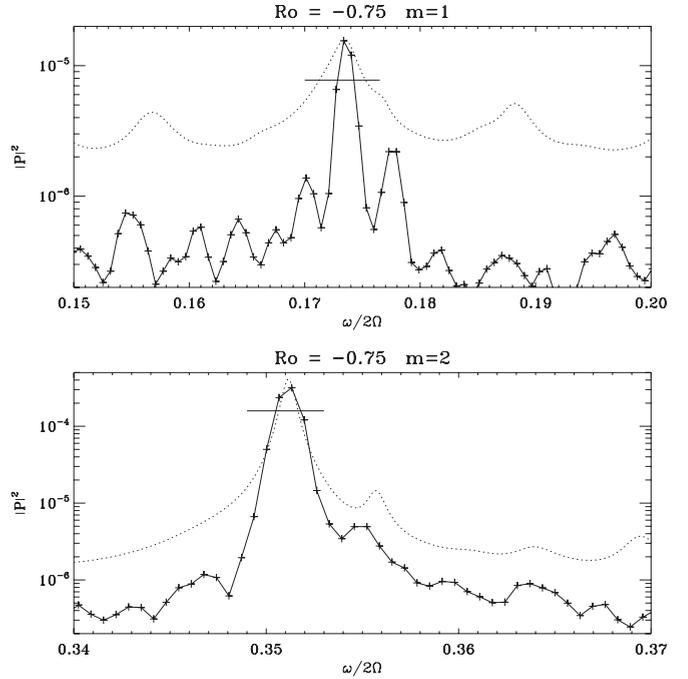


FIG. 11. Comparison of the width of two observed resonances at $\text{Ro} = -0.75$ (solid lines with pluses) with the prediction of the model (dotted lines). The model curve has been shifted in frequency so as to match the observed frequency. The top figure is centered on the $m = 1$ mode at $\hat{\omega} \sim 0.17$, while the bottom one shows the resonance of the purely toroidal $m = 2$ mode. The horizontal line gives the position of the half-maximum. $E = 2.510^{-8}$.

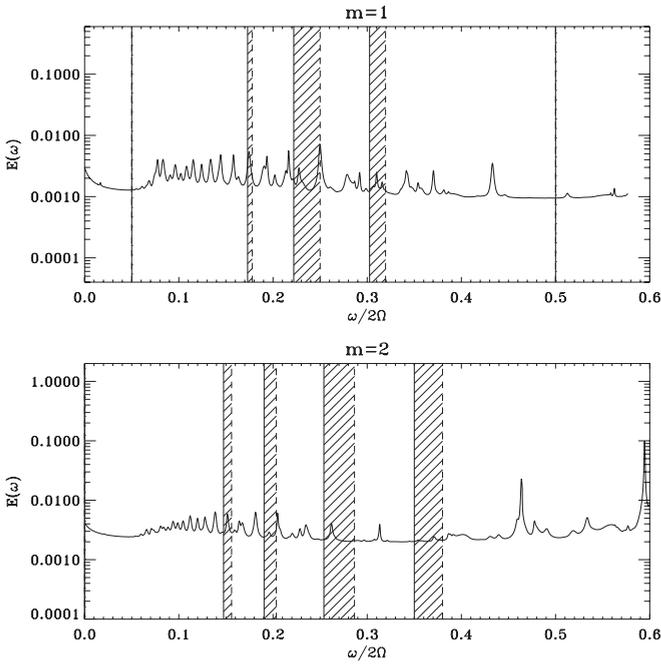


FIG. 12. Same as Fig. 8 but for equatorially symmetric modes with $m = 1$ and $m = 2$.

tiny peak on the high-frequency side of the main resonance that appears both in the experimental data and in the model, suggesting that we have correctly identified this mode.

To further secure mode identification, we also investigated the other possible symmetry of the modes. Indeed, in equatorially symmetric containers, two independent set of modes coexist for a given azimuthal wave number m : modes for which the pressure function is symmetric with respect to equator and those for which it is antisymmetric. For these latter modes the pressure and the azimuthal velocity component v_ϕ vanish at equator. We thus computed the analog of Figs. 8 and 9 but for equatorially symmetric modes. Results are shown in Figs. 12 and 13. Quite clearly, none of the $m = 3$ or $m = 4$ modes can be related to experimental resonances. For the $m = 1$ and $m = 2$ azimuthal wave numbers, the two lowest frequencies may possibly correspond to symmetric modes.

To conclude on mode identifications, we may say that the vicinity of frequencies, together with the matching of the width of the resonances, allows us to identify almost all the observed frequencies with the eigenfrequencies of equatorially antisymmetric retrograde inertial modes of spherical shell in solid rotation. Doubts may remain about two modes actually: $m = 1$, with $\hat{\omega} = 0.17$ and $\hat{\omega} = 0.23$, because they show experimental FWHMs which are significantly less than those predicted by the model. For these two modes the background shear may have more important consequences, for instance changing the length scales that control viscous dissipation (in addition, two equatorially symmetric modes have similar frequencies, thus possibly adding some confusion). Hence, in this experimental spherical Couette flow, we should remember that excited inertial waves are generally (if not systematically) equatorially antisymmetric.

Finally, let us note that the spectrograms show that excited inertial modes live in a definite interval of Rossby numbers.

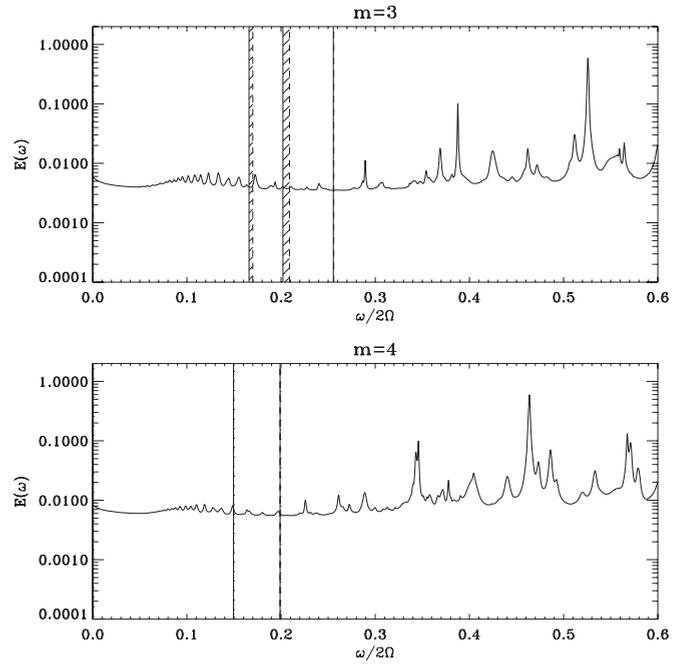


FIG. 13. Same as Fig. 12 but for $m = 3$ and $m = 4$.

We may even observe that for a given m , when the Rossby number is decreased, modes appear one by one, the “old” one disappearing when a new one appears. The series ends when the purely toroidal one associated with the given m [see Eq. (5)] is excited. We note, however, that the spin-over mode at $\hat{\omega} = 0.5$ and a (or a set of) low-frequency resonance(s) with $m = 1$ around $\hat{\omega} = 0.05$ does not follow this rule, being permanently excited when $Ro > -1$.

The resonance near $\hat{\omega} = 0.05$ may be associated with the broad peak appearing on the $m = 1$ curve at $\hat{\omega} \sim 0.03$, when the forcing is set on the inner core. We show in Fig. 14 the

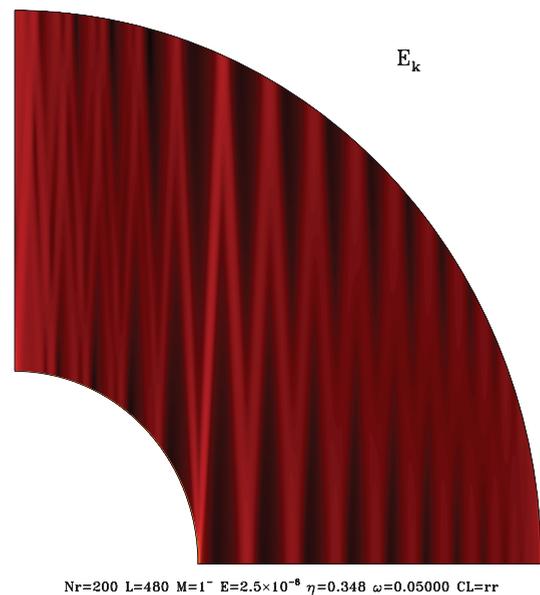


FIG. 14. (Color online) The kinetic energy distribution in a meridional plane of the flow associated with the resonance at $\hat{\omega} = 0.05$ for $m = 1$.

shape of the oscillating flow at this frequency for a solid body rotation. It looks like a set of shear layers spawned by the critical latitude singularity on the inner boundary.

IV. DISCUSSION

Let us first summarize the foregoing conclusions derived from the combination of experimental and numerical results. The spherical Couette flow with $E = 2.5 \cdot 10^{-8}$ and $-1.8 < \text{Ro} = \frac{\Omega_i}{\Omega} - 1 < 0$ (the inner sphere rotating more slowly than the outer sphere) displays a series of excited inertial modes which have the following properties:

- (1) They are all nonaxisymmetric, with an azimuthal wave number $m \in \{1, 2, 3, 4\}$.
- (2) They are likely all antisymmetric with respect to equator.
- (3) They all propagate azimuthally in the opposite direction of the outer shell rotation (i.e., they are retrograde).
- (4) Their highest observed frequency is that of the purely toroidal mode [see Eq. (5)] associated with the given azimuthal wave number m , namely $\omega = 2\Omega/(m + 1)$.
- (5) When the differential rotation is increased from zero (i.e., when decreasing the Rossby number from zero), inertial modes of a given m turn on at a specific Rossby number and turn off when the next mode of the series turns on. The spin-over mode is an exception, being always excited when $\text{Ro} > -1$.

We understand the emergence of these inertial modes in the experimental Couette flow as a consequence of the existing shear layer staying on the tangential cylinder, namely the layer which replaces the Stewartson layer when Ro is not small. Our scenario is as follows.

Let us first recall that at very small Rossby numbers (in the linear regime), and in the limit of vanishing Ekman numbers, the spherical Couette flow may be represented by a solid body rotation outside the tangent cylinder and a differential rotation inside the tangent cylinder. Weak circulations and the Stewartson layer complete these basic azimuthal flows [27]. Proudman [28] has shown that inside the tangent cylinder

$$\Omega(s) = \Omega_i + (\Omega - \Omega_i) \frac{(1 - s^2/\eta^2)^{1/4}}{(1 - s^2/\eta^2)^{1/4} + (1 - s^2)^{1/4}}, \quad (6)$$

where s is the radial cylindrical coordinate. The main point shown by this formula is that the fluid inside the tangent cylinder rotates almost rigidly at an intermediate angular velocity:

$$\Omega(0) = \frac{\Omega_i + \Omega}{2}.$$

Figure 15 further illustrates this point, showing that the angular velocity does not vary much except near the Stewartson layer.

Equation (6) has been derived in the asymptotic limit of small numbers and it is certainly approximate when applied to the experiment. However, it seems reasonable to represent the actual flow by a solid body rotation at angular velocity Ω outside the tangent cylinder and at angular velocity $(\Omega_i + \Omega)/2$ inside it. A shear layer, generalizing the Stewartson one, connects these two rotations.

Using the $(2\Omega)^{-1}$ time scale, the angular velocity of fluid $\Omega(s)$ thus varies in the interval $[\text{Ro}/4 + 1/2, 1/2]$. Hence, in the foregoing experiment where $\text{Ro} < 0$, the flow is retrograde

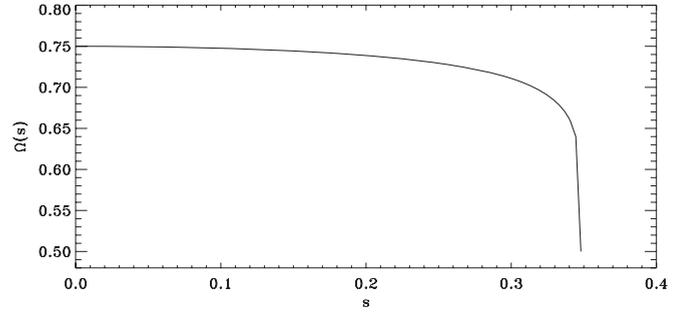


FIG. 15. View of the differential rotation inside the tangential cylinder as given by the linear solution of [28]. Here $\Omega_i = 0.5$, $\Omega = 1$, and $\eta = 0.348$. Note that the angular velocity decrease is a feature of the inviscid solution that is compensated by the rapid rise in the Stewartson layer (see [27] for a complete view with viscous effect).

inside the tangent cylinder as viewed from the outer shell. This implies that inertial waves propagating outside the tangent cylinder and with a retrograde angular velocity face a critical layer if their angular phase velocity $-\hat{\omega}/m$ is less (in absolute value) than $\text{Ro}/4$. In other words, if

$$\text{Ro} < -\frac{4\hat{\omega}}{m} < 0 \quad (7)$$

is verified, then there exists a place within the shear layer where the phase velocity of the wave equals that of the fluid. This is the place where a critical layer develops. In Fig. 16 we sketch out the position of the critical layer. Such a layer may over reflect the inertial waves and be at the origin of the selection of excited modes as proposed by Kelley *et al.* [17]. Critical layers are indeed known to play a crucial part in the dynamics of the Rossby waves in the Earth's atmosphere [29,30]. Recently, Baruteau and Rieutord [31] found that some inertial modes propagating over a differentially rotating fluid in a spherical shell can be unstable when a critical layer exists. We shall now examine this scenario.

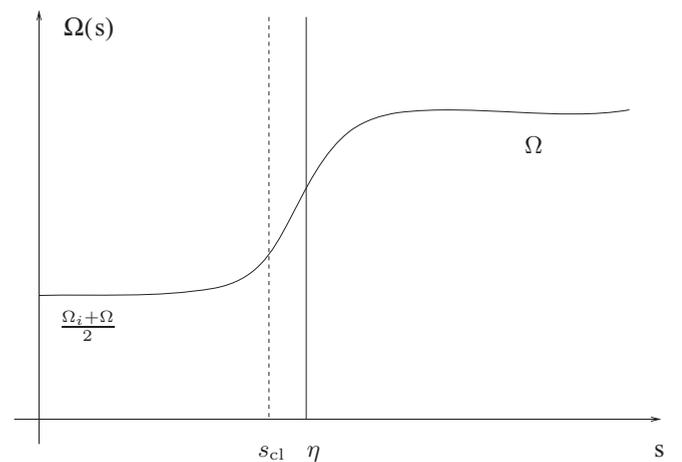


FIG. 16. Schematic view of the bulk rotation of the fluid in the spherical Couette flow for $\eta < z < \sqrt{1 - \eta^2}$ ($0.348 < z < 0.937$). The transition between the outer solid body rotation and the inner (approximate) solid body rotation is along the tangent cylinder at $s = \eta$. When Ω_i is decreased, the critical layer of a given mode first appears on the left side of the shear layer and moves toward larger s .

TABLE II. Table of the critical Rossby numbers (Ro_{crit}) as derived from the experimental spectrograms compared to the predictions of the numerical model $-4\hat{\omega}/m$. α is a parameter controlling the frequency drift with Ro and is derived from Figs. 4 and 5.

m	$\hat{\omega}$	$-4\hat{\omega}/m$	Ro_{crit}	α
1	0.1595	-0.638	-0.74	
1	0.210	-0.840	> -1.14	0.27
1	0.301	-1.204	-1.39	0.11
2	0.142	-0.284	-0.29	
2	0.177	-0.355	-0.36	0.275
2	0.233	-0.466	-0.50	0.236
2	0.333	-0.667	-0.70	0.107
3	0.152	-0.203	-0.20	
3	0.189	-0.252	-0.25	
3	0.250	-0.333	-0.33	

Equation (7) shows that for a given (retrograde) nonaxisymmetric inertial modes, there is a critical negative Rossby number below which the critical layer exists and might excite the mode. This critical Rossby number can be derived from the resonance frequencies showing up in Figs. 8 and 9 and compared to the maximum Rossby number beyond which the mode is no longer excited. This Rossby number is derived from the spectrograms in Figs. 2 and 3. The results of this exercise are shown in Table II and illustrated with Fig. 17. The matching is remarkable, especially for the $m = 2$ and $m = 3$ modes.

When the Rossby number is decreased from zero, large-scale resonances appear in the order given by Figs. 8 and 9, until the purely toroidal mode of the associated m is excited. The $m = 1$ purely toroidal spin-over mode is an exception as it consists of a global solid body rotation around an equatorial axis, oscillating at the rotation frequency of the outer shell. Therefore, it can be mechanically excited by the boundary.

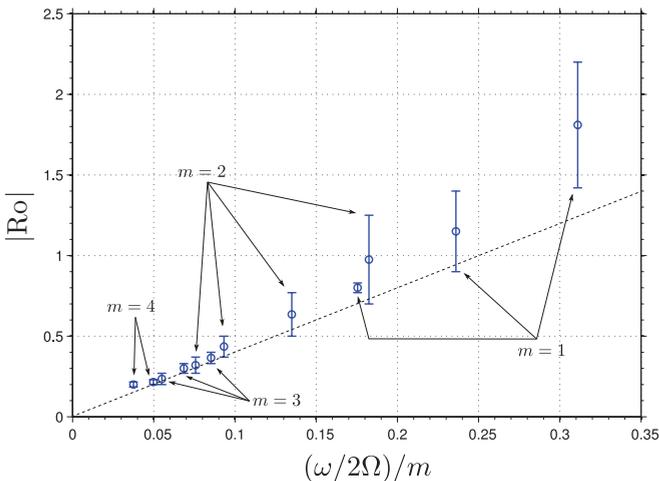


FIG. 17. (Color online) Graphic illustration of Table II: For each observed inertial mode, we display the Rossby number range of existence of the mode. The dashed line shows the (absolute value) of the critical Rossby number $4\hat{\omega}/m$ below which the mode is no longer excited. Note the good matching with the experimental value.

The spectrograms also show that when a new mode is excited the “old” one with the same m disappears. We understand this process as follows: The new mode is always of larger scale than the old one (Figs. 8 and 9 show that the strength of the resonances increases as the frequency increases), thus when the Rossby number decreases a less damped mode takes over a more damped one. In the differentially rotating fluid where these modes are unstable because of the critical layer, a more unstable mode overwhelms a less unstable one. This behavior is typical of instabilities where many modes are destabilized simultaneously and the most unstable dominates the bifurcated state (e.g., Rayleigh-Bénard instability or Taylor-Couette instability). The change of excited mode stops when the least damped of the series, the purely toroidal one, is excited.

This scenario thus explains why only nonaxisymmetric retrograde modes are excited and the Rossby number below which this is indeed the case. However, it does not explain why only antisymmetric modes are excited. We conjecture that the coupling with the spin-over mode may be the key of this selection. Indeed, from the work of Hollerbach [32] we know that the Stewartson layer generated by the spherical Couette flow can become unstable when the Rossby number is increased (in absolute value). Modes with the same symmetry as the original flow, namely symmetric with respect to equator, are destabilized. Their wavelength should scale with the width of the layer, namely $E^{1/4}$. From the numerical results of [32], we infer that the most unstable mode has an azimuthal wave number $m \sim 0.465E^{-1/4}$. In the context of the experiment described above, such modes have $m \sim 16$ ($E = 7.5 \times 10^{-7}$) or $m \sim 37$ ($E = 2.5 \times 10^{-8}$). Hence, the linear instability injects energy at rather high wave numbers, implying, thanks to nonlinear interactions and an inverse cascade of energy, that a large range of scales (with azimuthal wave numbers $0 < m \lesssim 40$) is excited, thus generating some turbulence. The equatorial symmetry could be broken by the spin-over mode [namely the $m = 1$ mode described by Eq. (5)], whose presence has been detected on the velocity signal. This mode is antisymmetric. The nonlinear interaction of this mode with disturbances growing over the turbulent Stewartson layer could thus select the observed antisymmetric inertial modes. However, more detailed numerical investigations are definitely necessary to answer this question.

Finally, in light of the foregoing scenario, we can also interpret the linear drift of the frequencies with the Rossby number, as shown in Figs. 4 and 5. The linear dependence suggests that

$$\hat{\omega} = \hat{\omega}_{\text{crit}} - \alpha \frac{m}{4} (\text{Ro} - \text{Ro}_{\text{crit}}), \quad (8)$$

where $\text{Ro}_{\text{crit}} = -4\hat{\omega}_{\text{crit}}/m$ and α is a constant. With the expression of Ro_{crit} we may rewrite the previous equation as

$$\hat{\omega} = \hat{\omega}_{\text{crit}}(1 - \alpha) - \frac{\alpha m}{4} \text{Ro}, \quad (9)$$

which shows that α should adjust both the slope and the frequency at $\text{Ro} = 0$. Fitting experimental data, we derive the value of α for the modes used in Figs. 4 and 5. These are given in Table II. The physical meaning of α may be derived from our scenario. Indeed, if the modes are excited through their critical layer, their azimuthal phase velocity $-\hat{\omega}/m$ equals the

fluid angular velocity somewhere in the shear layer. We recall that the scaled angular velocity of the fluid viewed from the frame corotating with the outer shell varies from 0 down to $\text{Ro}/4 < 0$. Introducing the angular velocity profile $\tilde{\Omega}(s)$ such that $1 \geq \tilde{\Omega}(s) \geq 0$, we have

$$\hat{\Omega}(s, \text{Ro}) = \frac{\text{Ro}}{4} \tilde{\Omega}(s).$$

In our modeling of the spherical Couette flow, we assume that the rotation profile is independent of Ro .

If s_{cl} is the position of the critical layer, then $-\hat{\omega}/m = \frac{\text{Ro}}{4} \tilde{\Omega}(s_{\text{cl}})$. Using Eq. (9) leads to

$$\tilde{\Omega}(s_{\text{cl}}) = \alpha + (1 - \alpha) \frac{\text{Ro}_{\text{crit}}}{\text{Ro}}. \quad (10)$$

This expression shows that the constant α gives the asymptotic position of the critical layer when $\text{Ro} \rightarrow -\infty$. Since $\tilde{\Omega}(s)$ is a decreasing function of s , it also shows that, as the differential rotation increases (Ro more negative), the critical layer moves away from the rotation axis (s_{cl} increases). Furthermore, modes with a higher α have critical layers more inward to the tangent cylinder.

In Sec. II, we noted that the first $m = 1$ mode shows a different dependence with Ro when the Ekman number E is doubled (approximately). From Eq. (8), this translates into an increased α for a lowered E . Unfortunately, there is no univocal interpretation of this change: It may mean either that the critical layer moved inside the shear layer or that the rotation profile changed, or both. Either a more elaborated model or more precise measurements using other modes is necessary to disentangle the effects.

V. CONCLUSIONS

In this work we have been able to refine the scenario proposed by Kelley *et al.* for the selection of inertial modes in a spherical Couette flow [17]. We showed that the observed modes are most likely excited by a critical layer lying inside the shear layer which separates the fluid inside and outside the tangent cylinder. We demonstrated that this mechanism leads to a negative critical Rossby number, below which some nonaxisymmetric retrograde inertial modes can be excited. The predicted value of this critical Rossby number matches quite nicely the observed experimental values, thus giving support to the proposed mechanism. However, because of the strong differential rotation needed to excite some modes, the identification of a few oscillation frequencies with those of a

fluid inside a spherical shell in solid body rotation remains uncertain. Another pending question is that of the equatorial symmetry of the observed modes. Our scenario, along with the matching of frequencies of many modes, argues in favor of a selection of antisymmetric modes. However, this conclusion is not completely firm because some of the symmetric modes may match a few of the frequencies and also because the FWHM of two resonances show experimental values that are much less than those predicted by the model, which is *a priori* less dissipative.

These results underline the need of more detailed investigations of inertial oscillations within differentially rotating fluids. Preliminary numerical results [31] show that a global shear can indeed significantly modify the properties of inertial modes and that critical layers can destabilize some nonaxisymmetric modes. However, the very case of the mean flow associated with a quasiturbulent spherical Couette flow remains to be investigated.

Such a study may lead to the interesting perspective of reconstructing the interior differential rotation of the fluid by adjusting the prediction of the model to the observed value of the resonance frequencies. One could thus deduce details of the mean flow. Such a technique is similar to helioseismology techniques in solar research [33], but instead uses inertial modes to reconstruct the internal rotation of the fluid. Inertial modes are certainly the most appropriate modes to infer the rotational property of a star or a planet. Our comparison of the inertial frequencies of our model with the observed ones shows that a laboratory spherical Couette flow offers a unique playground to test this use of inertial waves.

Finally, let us point out that the ultimate challenge of modeling (this) experimental spherical Couette flows is to predict the amplitude of excited modes as a function of the Rossby number and to reproduce the sequence of their appearance.

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