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Slichter modes of the Earth revisited

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Abstract

Using the simple model of a spherical solid inner core oscillating in a rotating liquid outer core, we compute the frequencies of the Slichter modes of the earth. The fluid is assumed neutrally stratified (but with a radially varying density) and viscous, viscosity being taken into account non-perturbatively. The parameters are those given by earth models like PREM, 1066A or CORE11. The computed resonant frequencies are compared to those observed by Courtier et al. (2000) and claimed to be the Slichter frequencies. We show that our model cannot reproduce these frequencies and that previous models which did reproduce them are not consistent with the observed quality factors of the resonances.

Keywords: Slichter modes; Earth; Core

1. Introduction

The detection and measurement of the motion of matter inside the liquid core of the Earth is an important issue for the determination of the physical properties of this object. One possibility is the detection and measurement of the small oscillations of the solid inner core inside the liquid core, the so-called Slichter modes (Slichter, 1961). The detection of these motions is possible thanks to very precise gravimetric data since any mass motion generates a perturbation in the gravitational field.

Using data obtained from superconducting gravimeters, Smylie (1992) claimed the identification of the Slichter triplet—recall that because of the Earth's rotation, the Slichter modes are three—at periods very close to those forecast with an inviscid CORE11 model. These periods are 4.0166, 3.7687 and 3.5813 h for the prograde, axial and retrograde modes, respectively. However, these predictions have been criticized by Crossley et al. (1992) on the ground of an improper use of the static Love numbers to describe the motion of the inner core.

More recently, Courtier et al. (2000) presented the analysis of almost 300 000 h of superconducting observations; from these data, the observed periods of the Slichter modes were refined to 4.0150 ± 0.0010, 3.7656 ± 0.0015 and 3.5822 ± 0.0012 h. As these values are slightly off those predicted by Smylie (1992) using the inviscid CORE11 model, Smylie (1999) and Smylie and McMillan (2000) used this discrepancy to evaluate the viscosity of the fluid outer core near the inner core boundary (ICB). Since the ICB plays an important part in the dynamics of the core (its growth through the crystallization process is believed to be the engine driving the geodynamo), the measurement of any physical quantity, like viscosity, is of great value in the elaboration of Earth models.

However, the kinematic viscosity of the liquid core ν is believed to be very small in the sense that the

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associated Ekman number $E = \nu/2\Omega R_i^2$, is very small compared to unity, (Ω is the rotation rate of the Earth and R_i is the radius of the inner core). Common values of E range between 10^{-8} and 10^{-16} (Lumb and Aldridge, 1991). Because of these small values, viscosity effects are concentrated in thin boundary layers (Ekman layers) or internal shear layers (Rieutord, 2000). This is why viscosity is often taken into account through an asymptotic theory which divides the fluid domain into an inviscid interior and a viscous boundary layer. Smylie and McMillan (1998, 2000) used such a technique to estimate the effects of viscosity on the Slichter modes. Applying these results to the observed periods, Smylie (1999) and Smylie and McMillan (2000) deduced a dynamical viscosity of 1.22×10^{11} Pa s which seems to perfectly fit the observed splitting.

However, if we now compute the associated Ekman number $E = 1.22 \times 10^{11}/(2\Omega R_i^2 \rho_0)$, we find that $E \sim 0.05$, a value which is very much larger than usual values for the liquid core. This surprisingly high value of the Ekman number raises new questions.

First of all, is the asymptotic theory used to derive it still appropriate? Indeed, boundary layers introduce corrections which are of $O(\sqrt{E})$ to inviscid solutions and eigenfrequencies are modified like

$$\omega = \omega_0 + \omega_1\sqrt{E} + \omega_2 E + \cdots$$

We, therefore, see that, unless ω_2 is very small, E being 0.05, second order corrections may strongly modify the 'observed' viscosity. Recall that the discrepancy between viscous and inviscid periods is ∼2%.

Next, if it turns out that the asymptotic approach cannot be used, are the observed frequencies still interpretable as those of the Slichter modes, using appropriate density jump and viscosity? If so, what are the values of these parameters?

The aim of this paper is to answer these questions and somehow revisit the theoretical aspects of the Slichter oscillations using new methods independent from the previous and conflicting studies. For this purpose, we shall derive the equations governing the Slichter modes from first principles, simplify their fluid part using the subseismic approximation and solve them using a spectral method. When desired, viscosity will be included in a non-perturbative way to allow for high values. Contrary to previous work, we shall not take into account the elasticity of the boundaries because of the smallness of the effect, nor shall we take into account a stable stratification in the liquid core as it is likely too weak and thus not relevant to our problem; according to Wu and Rochester (1994), this latter effect cannot influence the Slichter eigenperiods more than 0.5%.

In Section 2, we thus, present our model and formulate the non-dimensional equations to be solved. The numerical method is then presented along with a comparison of the results with previous work. In Section 3, we discuss the possibility of an identification of observed frequencies using our model and give our conclusions in Section 4.

2. The model

We consider the inner core (hereafter referred to as IC) as a spherical ball of surface density ρ_1 and mean density $\bar{\rho}_1$. It exists in a rotating fluid filling a spherical shell, the outer core, whose density varies according to a law $\rho(r)$, r being the radial spherical coordinate. The core-mantle boundary (CMB), which serves as the outer boundary of the fluid, is assumed to be a rigid sphere of radius R. The radius of the inner core is ηR; for later use, we introduce the density of the fluid at the ICB, ρ_2, thus, $\rho(\eta R) = \rho_2$. We also assume the fluid outer core of Newtonian type with a constant shear viscosity μ; in order to further simplify the equations of motion, we shall also assume that the variations of the kinematic viscosity $\nu = \mu/\rho(r)$ are unimportant and that it takes a constant value ν. As mentioned in Section 1, we assume the liquid core to be well mixed and therefore isentropic; thus, we set the Brunt-Väisälä frequency to zero.

Let us first write, in a rotating frame of angular velocity $\vec{\Omega}$, the equation of motion of the inner core of mass M_1. Since we are only interested in translational modes, the inner core is only characterized by the displacement of its mass center $\vec{r}$; Newton's law gives

$$\frac{d^2\vec{r}}{dt^2} + 2\Omega\vec{e}_z \times \frac{d\vec{r}}{dt} = -\left(1 - \frac{M_2'}{M_1}\right)\left(\frac{4\pi}{3}G\rho_2\vec{r} + \Omega^2\vec{e}_z \times (\vec{e}_z \times \vec{r})\right) - \frac{1}{M_1}\left\{\int p\, d\vec{S}_1 - \rho_2\nu\int [s]\, d\vec{S}_1\right\} \quad (1)$$

In this expression, we recognize Coriolis acceleration in the LHS, the integral of body forces (gravitational and centrifugal) in the first terms of the RHS (see Busse, 1974) and the integral of fluid stresses associated with the flow (pressure and viscous) in the last term of the RHS. M_2' is the mass of the inner core if it were of density ρ_2 and G is the constant of gravitation. Finally, note that $\mathrm{d}\vec{S}_1$ is oriented along the outward normal of the ICB (i.e. into the fluid) and that $[s]$ is the rate-of-strain tensor; $\vec{e}_z$ is the unit vector along the z-axis which is also the rotation axis.

These equations of motion need now be completed by those of the surrounding fluid. Taking into account the assumption we have made on the viscosity and that

$$\frac{\vec{\nabla}P}{\rho} = \vec{\nabla}h$$

for an isentropic fluid, h being the enthalpy, we have

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + 2\vec{\Omega}\times\vec{v} = -\vec{\nabla}(h+\phi_{\mathrm{g}}+\phi_{\mathrm{c}}) + \nu\,\Delta\vec{v} \quad (2)$$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{v}) = 0 \quad (3)$$

for momentum and mass conservation, ϕ_{g} and ϕ_{c} are the gravitational and centrifugal potentials, $\vec{v}$ is the fluid velocity. Considering only small amplitude motions, we linearize this system; moreover, since we are only interested in low-frequency motions, we can further, simplify the system by using the so-called subseismic approximation which filters out acoustic waves (Smylie and Rochester, 1981; Dintrans and Rieutord, 2001; Rieutord and Dintrans, 2002); hence, we get

$$\frac{\partial\vec{v}}{\partial t} + 2\vec{\Omega}\times\vec{v} = -\vec{\nabla}h' + \nu\,\Delta\vec{v} \quad (4)$$

$$\mathrm{div}(\rho\vec{v}) = 0 \quad (5)$$

where h' is the deviation of the enthalpy from equilibrium. These equations need to be completed by the following boundary conditions

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \quad \text{on} \quad S_1, \qquad \vec{v} = 0 \quad \text{on} \quad S_2 \quad (6)$$

where S_1 and S_2 designate the ICB and CMB surfaces, respectively. These boundary conditions just express that the velocity of the fluid on the boundaries is the velocity of the boundaries.

We shall now turn to dimensionless variables using the radius of the outer core R as the length scale and $(2\Omega)^{-1}$ as the time scale. The enthalpy perturbation h' is simply p'/ρ and will be transformed into a non-dimensional pressure p. The equations of motion now read

$$\lambda\vec{u} + \vec{e}_z\times\vec{u} = -\vec{\nabla}p + E\,\Delta\vec{u} \quad (7)$$

$$\vec{\nabla}\cdot\rho\vec{u} = 0 \quad (8)$$

$$\vec{u} = \lambda\vec{X} \quad \text{on} \quad r=\eta, \qquad \vec{u}=0 \quad \text{on} \quad r=1 \quad (9)$$

for the fluid, and

$$\lambda^2\vec{X} + \lambda\vec{e}_z\times\vec{X} = -\frac{(1-\alpha)}{4}\{\gamma\vec{X} + \vec{e}_z\times(\vec{e}_z\times\vec{X})\} - \frac{3\alpha}{4\pi\eta^3}\left\{\int p\,\mathrm{d}\vec{S}_1 - E\int[s]\,\mathrm{d}\vec{S}_1\right\} \quad (10)$$

for the solid. We introduced the dimensionless numbers

$$E = \frac{\nu}{2\Omega R^2}, \qquad \gamma = \frac{4}{3}\pi\frac{G\rho_2}{\Omega^2}, \qquad \alpha = \frac{\rho_2}{\bar{\rho}_1}$$

and the dimensionless complex eigenvalue $\lambda = \tau + i\omega$; τ is the damping rate and ω the frequency ($i^2 = -1$).

2.1. Method

We solve these equations by projecting the velocity and pressure fields on the normalized spherical harmonic base, namely

$$p = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{+\ell} p_m^\ell(r)Y_\ell^m \quad (11)$$

$$\vec{u} = \sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{+\ell} u_m^\ell(r)\vec{R}_\ell^m + v_m^\ell(r)\vec{S}_\ell^m + w_m^\ell(r)\vec{T}_\ell^m \quad (12)$$

where

$$\vec{R}_\ell^m = Y_\ell^m\vec{e}_r, \qquad \vec{S}_\ell^m = r\,\vec{\nabla}Y_\ell^m, \qquad \vec{T}_\ell^m = r\,\vec{\nabla}\times\vec{R}_\ell^m \quad (13)$$

Using some simple rules (Rieutord, 1987), we find the following set of differential equations for the radial variables

$$\lambda u^\ell = -(\ell-1)a_\ell w^{\ell-1} + (\ell+2)a_{\ell+1}w^{\ell+1} + imv^\ell - \frac{\partial p^\ell}{\partial r} + E\,\Delta_\ell(ru^\ell) \quad (14)$$

$$\lambda\ell(\ell+1)v^\ell = B_\ell w^{\ell-1} - B_{\ell+1}w^{\ell+1} + im(u^\ell - v^\ell) - \ell(\ell+1)\frac{p^\ell}{r} + \frac{E}{r}\frac{\partial}{\partial r}(r\,\Delta_\ell ru^\ell) \quad (15)$$

$$\lambda w^\ell = \frac{im}{l(l+1)}w^\ell + A_\ell r^{\ell-1}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u^{\ell-1}}{r^{\ell-2}}\right) + A_{\ell+1}r^{-\ell-2}\frac{\mathrm{d}}{\mathrm{d}r}(r^{\ell+3}u^{\ell+1}) + E\,\Delta_\ell w^\ell \quad (16)$$

where we introduced

$$\Delta_\ell = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\ell(\ell+1)}{r^2}$$

and

$$a_\ell = \sqrt{\frac{\ell^2-m^2}{(2\ell-1)(2\ell+1)}},$$

$$A_\ell = \frac{a_\ell}{\ell^2}, \qquad B_\ell = (\ell^2-1)a_\ell$$

Mass conservation now reads

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2\rho u^\ell) - \ell(\ell+1)\rho v^\ell = 0 \quad (17)$$

Note that we dropped everywhere the index m which is the same for all the dependent variables.

We need now to express the boundary conditions met by the radial variables. On the outer boundary, we simply have

$$u^\ell = v^\ell = w^\ell = 0 \quad \text{at} \quad r=1 \quad (18)$$

for all ℓ's; on the inner boundary we have

$$u^\ell = v^\ell = 0 \quad \text{for } \ell \neq 1, \qquad w^\ell = 0 \quad \forall\ell$$

$$u_1^1 = v_1^1 = \frac{-\lambda(X-iY)}{2N_1^1} \quad (19)$$

$$u_{-1}^1 = v_{-1}^1 = \frac{-\lambda(X+iY)}{2N_1^1} \quad (20)$$

$$u_0^1 = v_0^1 = \frac{\lambda Z}{N_1^0} \quad (21)$$

we introduced the components of the displacement of the inner core $\vec{X} = X\vec{e}_x + Y\vec{e}_y + Z\vec{e}_z$. The coefficients N_1^0 and N_1^1 are the norms of associated spherical harmonics such as

$$Y_1^0 = N_1^0\cos\theta, \qquad Y_1^1 = N_1^1\sin\theta e^{i\varphi},$$

$$N_1^0 = \sqrt{\frac{3}{4\pi}}, \qquad N_1^1 = \sqrt{\frac{3}{8\pi}}$$

We can now rewrite (10) as three equations

$$\lambda^2\tilde{Z} + i\lambda\tilde{Z} = -\omega_{\mathrm{g}}^2\tilde{Z} + \frac{3\alpha}{4\pi\eta N_1^1}\left(p_{-1}^1 - 2E\frac{\mathrm{d}v_{-1}^1}{\mathrm{d}r}\right)_\eta \quad (22)$$

$$\lambda^2\bar{Z} + i\lambda\bar{Z} = -\omega_{\mathrm{g}}^2\bar{Z} + \frac{3\alpha}{4\pi\eta N_1^1}\left(p_1^1 - 2E\frac{\mathrm{d}v_1^1}{\mathrm{d}r}\right)_\eta \quad (23)$$

$$\lambda^2 Z = -\omega_{\mathrm{g}z}^2 Z + \frac{\alpha}{\eta}\sqrt{\frac{3}{4\pi}}\left(p_0^1(\eta) - 2E\frac{\mathrm{d}v_0^1}{\mathrm{d}r}\right)_\eta \quad (24)$$

where we introduced $\tilde{Z} = X + iY$, $\bar{Z} = X - iY$, $\omega_{\mathrm{g}}^2 = (1-\alpha)(\gamma-1)/4$ and $\omega_{\mathrm{g}z}^2 = (1-\alpha)\gamma/4$. These three equations can be simplified using the boundary conditions (19–21) and introducing $x = \bar{Z}/(-2N_1^1)$, $y = \tilde{Z}/(-2N_1^1)$ and $z = Z/N_1^0$ to remove normalization constants. Thus, we find that

$$\lambda u_1^1 - iu_1^1 = -\omega_{\mathrm{g}}^2 x - \frac{\alpha}{\eta}\left(p_1^1 - 2E\frac{\mathrm{d}v_1^1}{\mathrm{d}r}\right)_\eta \quad (25)$$

$$\lambda u_{-1}^1 - iu_{-1}^1 = -\omega_{\mathrm{g}}^2 y - \frac{\alpha}{\eta}\left(p_{-1}^1 - 2E\frac{\mathrm{d}v_{-1}^1}{\mathrm{d}r}\right)_\eta \quad (26)$$

$$\lambda u_0^1 = -\omega_{\mathrm{g}z}^2 - \frac{\alpha}{\eta}\left(p_0^1 - 2E\frac{\mathrm{d}v_0^1}{\mathrm{d}r}\right)_\eta \quad (27)$$

which we use as boundary conditions for u_m^1, $m = +1$, 0, −1 and with the boundary conditions

$$v_1^1(\eta) = \lambda x, \qquad v_0^1(\eta) = \lambda z, \qquad v_{-1}^1(\eta) = \lambda y \quad (28)$$

Eqs. (14)–(17) are solved using a collocation method on the Gauss–Lobatto grid (Rieutord and

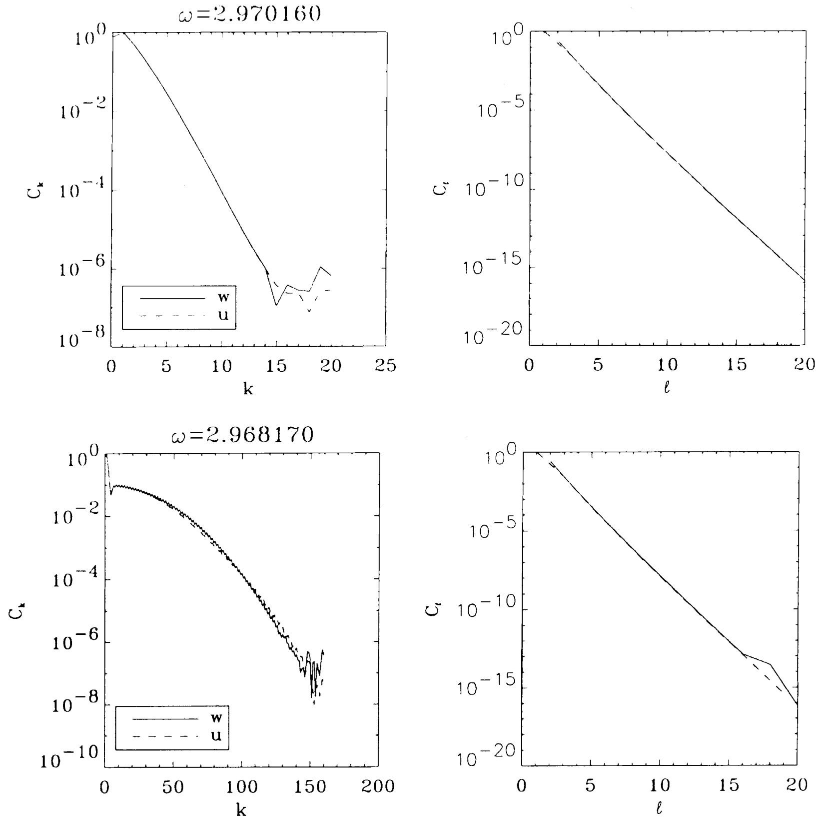

Fig. 1. Top: convergence of the Chebyshev coefficients, which are associated with a Gauss–Lobatto discretization (left) and of the spherical harmonics coefficients (right) for and inviscid solution of the axial Slichter mode using Busse's model. Bottom: same as above but for a viscous fluid with $E = 10^{-6}$; we see there that radial resolution needs to be strongly increased to resolve the Ekman layers. For all figures we used $\rho_1 = 13\,\text{g/cm}^3$, $\rho_2 = 12\,\text{g/cm}^3$ and $\eta = 0.349$.

Valdettaro, 1997). The advantages of this method over finite-difference methods come from its spectral nature which ensures fast convergence. Moreover, high density of grid points near the boundaries is appreciable to deal with Ekman layers (Fornberg, 1998).

The convergence of the solutions with the radial resolution and the number of spherical harmonics is shown in Fig. 1 for both an inviscid and viscous solution. There, we clearly see that inviscid solutions require only a modest resolution; a precision of 10^{-6} is achieved with eight spherical harmonics[1] and 20 radial grid points. However, if viscosity is included Ekman layers need to be resolved; for $E = 10^{-6}$, we need 160 radial grid points to achieve the same precision (the number of spherical harmonics is unchanged).

[1] This fast convergence comes from the fact that the Slichter modes are outside the inertial frequency band [0, 2Ω] and do not suffer from a strong coupling by the Coriolis force.

2.2. Comparison with existing work

A first test of our method can be made by comparing our results with those of Busse (1974) after setting $\rho = Cte$ and $E = 0$ in Eqs. (14)–(17) and (27). We only use the axial mode since Busse (1974) obtained quantitative results for that mode only; our results are in very good agreement with his results. To be complete, we give in Fig. 2, the variations of the periods of the triplet for this model when the inner core density is varied. Detailed numerical values are given in Table 1.

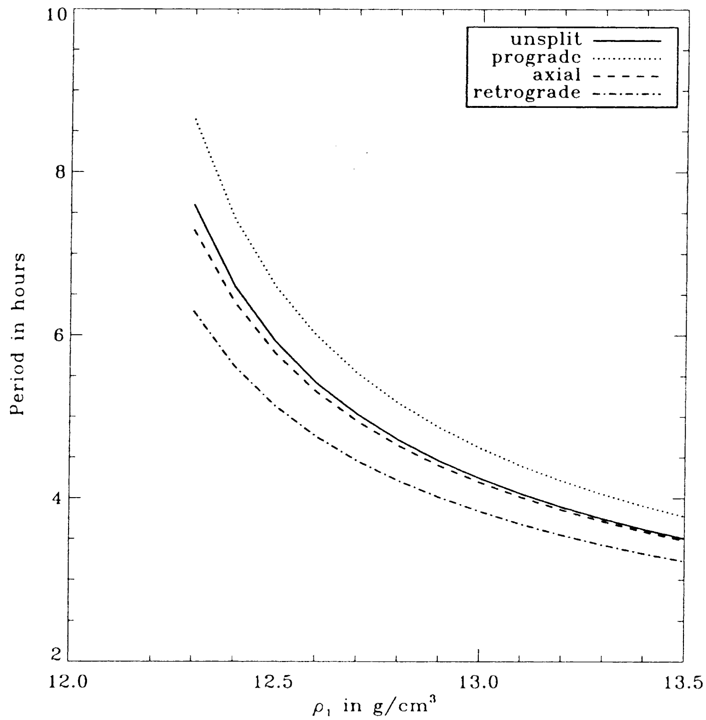

Fig. 2. Inviscid split periods vs. the density of the inner core using Busse's model; note that for this model $\bar{\rho}_1 = \rho_1$. The density of the fluid is kept constant at $\rho_2 = 12\,\mathrm{g/cm^3}$. The solid curve represents the classical result for inviscid unsplit modes. The curves are very insensitive to changes in the outer core density. The effect of rotation is a change on the order of −10/−1/+10% in the period of the retrograde/axial/prograde mode.

Before testing with more realistic models, it is worth noticing that the inviscid frequencies are governed by three non-dimensional parameters: α, γ, η. However, only α has a strong influence on frequencies through the ω_g^2 and ω_{gz}^2 constants. Indeed, α measure the effective density jump $\bar{\rho}_1 - \rho_2$ which is not to be confused with the actual density jump $\rho_1 - \rho_2$ which does not influence the period of the Slichter modes. As $\bar{\rho}_1$ is somehow better evaluated from global Earth models, the uncertainty on the effective density jump mainly comes from ρ_2. To appreciate the influence of this uncertainty on frequencies, we can evaluate its influence on ω_g^2 (since $\gamma \simeq 640$ for the PREM model, the difference between ω_g^2 and ω_{gz}^2 is small. We find that

$$\frac{\delta\omega_g^2}{\omega_g^2} = \frac{\delta\gamma}{\gamma} + \frac{\delta\alpha}{1-\alpha} = \delta\rho_2[\rho_2^{-1} + (\bar{\rho}_1 - \rho_2)^{-1}]$$

Because $\rho_2 \gg \bar{\rho}_1 - \rho_2$, we see that variations of α give the most important contributions to variations of frequencies.

A further test can now be made with a more realistic model as proposed by Smylie et al. (1992). However, as this model was contested by Crossley et al. (1992), we shall first compare our equations to the ones used by Smylie.

In Smylie's formulation, the motion of the inner core is taken into account in the boundary conditions met by the fluid. Rewriting Eq. (15) of Smylie et al.

Table 1
Frequencies and periods of the Slichter modes for Busse's model

	Unsplit	Prograde	Axial	Retrograde
ω	2.82212	2.59355	2.85638	3.12166
Period (h)	4.24051	4.61423	4.18965	3.83361

The unsplit frequency refers to the case without rotation. These values have been obtained with $\rho_1 = 13\,\mathrm{g/cm^3}$, $\rho_2 = 12\,\mathrm{g/cm^3}$ and $\eta = 0.349$.

(1992) and using our notations, we find

$$u_\ell(\eta) = \lambda\varepsilon h_\ell^{\mathrm{I}} p^l(\eta) \quad \text{with} \quad \varepsilon = \frac{4\Omega^2 R_{\mathrm{IC}}}{g_{\mathrm{ICB}}}$$

the h_ℓ^{I} are dimensionless Love numbers which Smylie et al. (1992) tabulate. We note in passing that except for $\ell = 1$, all these numbers are less than 0.1; since $\varepsilon \sim 6\times10^{-3}$, we cannot expect that elasticity modifies the eigenfrequencies more than a percent.

As is shown by our Eqs. (25)–(27), the IC motion is contained in the $\ell = 1$ equation. We therefore need to concentrate on the h_1-Love number which we shall now express using the equation of motion along the z-axis.

Neglecting viscosity we rewrite (21) and (27) as

$$u_0^1 = \lambda z, \qquad \lambda^2 z = -\omega_{gz}^2 z - \frac{\alpha}{\eta} p_0^1(\eta)$$

Thus, we may write the boundary condition

$$u_0^1 = -i\omega\frac{\alpha}{\eta}\frac{p_0^1(\eta)}{\omega_{gz}^2 - \omega^2}$$

where, we set $\lambda = i\omega$. If we compare this equation to the one proposed by Smylie et al. (1992), we find the expression for the Love number

$$h_1^{\mathrm{I}} = -\frac{\alpha}{\eta\varepsilon}\frac{1}{\omega_{gz}^2 - \omega^2} \tag{29}$$

This expression reveals that this Love number depends on the frequency of the mode. This dependency is obviously neglected by Smylie et al. (1992). This is clearly not possible since ω^2 necessarily close to ω_{gz}^2; indeed, ω_{gz}^2 measures the restoring gravitational force which pulls back the inner core to its equilibrium position; if no other mechanism were intervening, then we would have $\omega = \omega_{gz}$, but because of fluid dragging the ω is slightly off ω_{gz}.

Our conclusion, therefore, clearly supports that of Crossley et al. (1992) who pointed out that dynamical Love numbers needed to be used for computing the frequency of the Slichter modes. We shall therefore further test our model by comparing it to the results of Crossley et al. (1992).

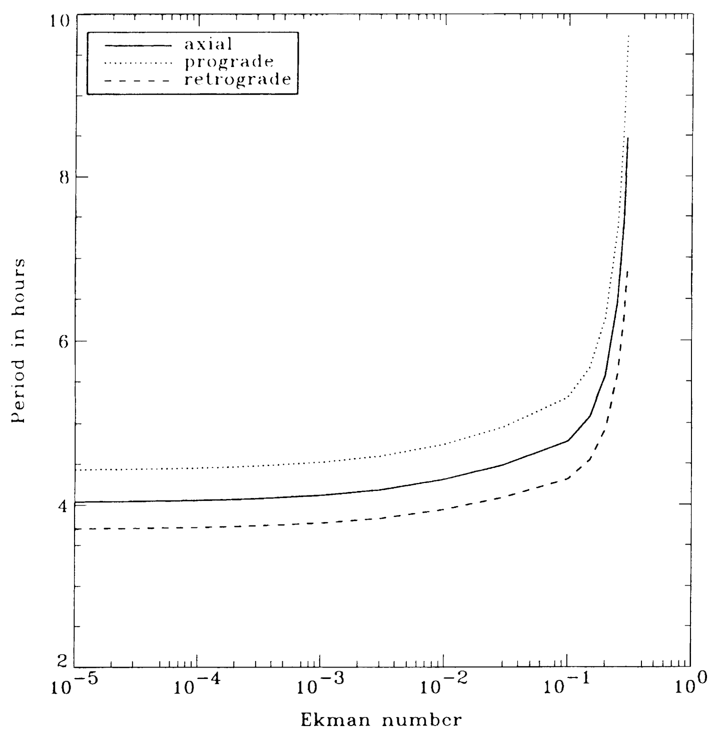

Fig. 3. Periods vs. Ekman number using Busse's model with $\bar{\rho}_1 = 13\,\mathrm{g/cm^3}$ and $\rho_2 = 12\,\mathrm{g/cm^3}$. The periods of all modes increase with the Ekman number. The effect of viscosity for $E < 10^{-5}$ is not significant for our problem.

For this purpose, we consider first a non-rotating PREM model (Dziewonski and Anderson, 1981) neglecting viscosity and Brunt-Väisälä frequency. Equations to be solved are very simple and we write directly their $\ell = 1$ spherical harmonic component

$$u = i\omega z, \qquad i\omega u = -\frac{\partial p}{\partial r},$$
$$i\omega v = -\frac{p}{r}, \qquad \frac{\partial u}{\partial r} + \left(\frac{2}{r} + \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) u - \frac{2v}{r} = 0$$

which meet boundary conditions (18) and (27), i.e.

$$i\omega u(\eta) = -\omega_{gz}^2 z(\eta) - \frac{\alpha}{\eta} p(\eta), \qquad u(1) = 0$$

The parameters of PREM, useful for the calculations are

$$\rho_2 = 12.16634\,\mathrm{g/cm^3}, \qquad \bar{\rho}_1 = 12.894\,\mathrm{g/cm^3}$$

Note that $\bar{\rho}_1 > \rho_1 = 12.7636\,\mathrm{g/cm^3}$. For the fluid, we use the dimensionless fit of the density profile

$$\rho(r) = 1 + 0.25355(1 - r^{2.15}), \qquad 0.351 \le r \le 1$$

Thus, doing, we find a period $P_0 = 4.966$ h shorter than the value (5.42 h) proposed by Crossley et al. (1992). We surmise that instead of using $\bar{\rho}_1$, Crossley et al. (1992) used ρ_1, i.e. the surface density of the inner core; indeed, in such a case, we find a period of 5.46 h, quite close to their value and this better matching repeats for all the models. Our values are $P_0 = 4.309$ h for the 1066A model and $P_0 = 5.426$ h for CORE11.

We now include rotation but still no viscosity and find, for the 1066A model, the following triplet periods:

- retrograde mode: $P = 3.894$ h (4.95);
- axial mode: $P = 4.255$ h (4.438);
- prograde mode: $P = 4.687$ h (4.896).

In parenthesis, we give the values found by Crossley et al. (1992); our periods are always 4% shorter than theirs. As before, the choice of the surface density of the inner core instead of its mean density explains the longer periods found by Crossley et al. (1992); numerics may also play a small part.

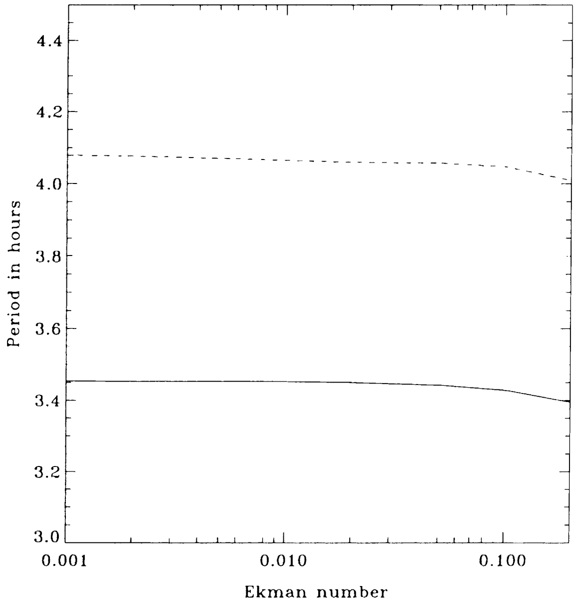

Fig. 4. Periods of the prograde (dashed line) and retrograde (solid line) modes as a function of the Ekman number keeping the frequency of the axial mode constant by adjusting the fluid density ρ_2; other parameters are those given by PREM. The effective density jump varies between $1.33\,\mathrm{g/cm^3}$ at $E = 10^{-3}$ and $1.91\,\mathrm{g/cm^3}$ at $E = 0.2$; because of this, the periods slightly decrease when viscosity increases unlike in Fig. 3 where $\bar{\rho}_1 - \rho_2$ is constant.

3. The role of viscosity

The foregoing calculations have shown that if the observed periods of Courtier et al. (2000) are indeed those of the Slichter triplet, they are much shorter than what is predicted using PREM. In fact, all the models tested (PREM, 1066A, CORE11) have an effective density jump too small to fit the observed periods.

Still using an inviscid fluid, if we adjust the density jump $\bar{\rho}_1 - \rho_2$ so that the frequency of the axial mode meets the observed one,[2] we find as Crossley et al. (1992), that the predicted triplet is wider in frequency than the observed one, i.e.

$$\text{Prediction (using PREM)}: \quad \omega_{\text{pro}} = 2.915, \qquad \omega_{\text{retro}} = 3.443$$

$$\text{Observation}: \quad \omega_{\text{pro}} = 2.981, \qquad \omega_{\text{retro}} = 3.341$$

[2] Using PREM, we had to decrease ρ_2 to $11.6146\,\mathrm{g/cm^3}$.

The question is now: could viscosity reconcile theory and observations? As shown in Fig. 3, the main effect of introducing a viscosity is to increase the periods of the triplet. As we shall see, it is impossible that viscosity (of a Newtonian fluid) reconciles the theoretical prediction and the observations.

We first observe that the discrepancy is of the order of 2.5% which means that Ekman numbers need to be around 10^{-3} which are high values as found by Smylie (1999). Adjusting the density jump so that the axial frequency is kept to the observed one, we computed the two remaining frequencies and plotted them as function of viscosity starting from $E = 10^{-3}$ (Fig. 4). We observe that while the prograde mode can be adjusted to observations (using an Ekman number of 0.2!), the retrograde mode cannot be matched. An independent third parameter, yielding a frequency change of 5%, would be necessary.

4. Discussion

In this paper, we revisited the theoretical determination of the Slichter modes and applied a new numerical method (a spectral one) to solve the equations of the problem. In passing, we observe that our analysis disagrees with that of Smylie et al. (1992) but confirm the conclusions of Crossley et al. (1992).

Using inviscid models, we could test the results of our code and find a good agreement with the previous work of Busse (1974) and Crossley et al. (1992).

As the periods given by the inviscid models do not match the observations of Courtier et al. (2000), we tried to use viscosity to remedy to the mismatch as Smylie (1999) did with his theory. Unfortunately, we could not find a density jump and viscosity which reconcile theory and observations on the three periods of the triplet. Moreover, the density jumps (recall that for this problem, it is the difference between the mean density of the inner core and the fluid density at the ICB) necessary to reproduce the axial mode and one of the prograde or retrograde mode, are far too large compared to those admissible by global Earth models according to the values given by Masters and Shearer (1990); they even exceed (see Fig. 4) the value found by Souriau and Souriau (1989) considered as likely an upper limit. We, therefore, do not confirm the results of Smylie (1999) who found agreement between observed and predicted periods; this agreement may be the consequence of the use by Smylie (1999) of a first order asymptotic theory which is not accurate enough with the resulting viscosities.

The disagreement between theoretical predictions and the observations of Courtier et al. (2000) may be explained by either an incorrect identification of the Slichter triplet in the data or an important physical effect missing in the model. We feel that the first possibility is the most likely. Indeed, Smylie (1992) measured a quality factor of the resonances of $100 \lesssim Q \lesssim 400$ and direct measurements on Fig. 5 of Courtier et al. (2000) shows even higher values, while Ekman numbers found by Smylie and McMillan (2000) imply quality factors less than 10, strictly. This contradiction between the data and the viscosity obtained with the data renders very likely that the peaks shown by Courtier et al. (2000) are not those of the Slichter modes. Moreover, since previous attempts by Hinderer et al. (1995) and Jensen et al. (1995) to confirm Smylie's results also failed, we feel that the Slichter modes are still not detected, unfortunately.

As stressed by previous authors, the detection of these modes is certainly a formidable problem and some progress on the theoretical models will be helpful. As was argued in Section 2.2, neither the elasticity of the boundaries of the outer core nor a hypothetical stably stratified region are likely to modify the frequencies significantly. On the other hand, the rheological properties of the fluid near the ICB may be important. For instance, the fluid may penetrate into the inner core and therefore oscillate in a porous medium or it may contain a floating solid phase and therefore be a two-phase fluid. These effects may affect both the frequencies and damping rates and it is clear that a strong damping rate will render the Slichter modes undetectable. Such issues need to be cleared out.

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