

More concerning the anelastic and subseismic approximations for low-frequency modes in stars

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ABSTRACT

Two approximations, namely the subseismic approximation and the anelastic approximation, are used to filter out the acoustic modes when computing low-frequency modes of a star (gravity modes or inertial modes). In a previous paper, we observed that the anelastic approximation gave eigenfrequencies much closer to the exact ones than the subseismic approximation. Here, we try to clarify this behaviour and show that it is a result of the different physical approach taken by each approximation. On the one hand, the subseismic approximation considers the low-frequency part of the spectrum of (say) gravity modes and turns out to be valid only in the central region of a star; on the other hand, the anelastic approximation considers the Brunt–Väisälä frequency to be asymptotically small and makes no assumption concerning the order of the modes. Both approximations fail to describe the modes in the surface layers but eigenmodes issued from the anelastic approximation are closer to those including acoustic effects than their subseismic equivalent.

We conclude that, as far as stellar eigenvalue problems are concerned, the anelastic approximation is better suited for simplifying the eigenvalue problem when low-frequency modes of a star are considered, while the subseismic approximation is a useful concept when analytic solutions of high-order low-frequency modes are needed in the central region of a star.

Key words: stars: oscillations.

1 INTRODUCTION

When considering the low-frequency modes of a star, namely gravity modes or inertial modes, the compressibility of the fluid is often a side effect in the determination of eigenfrequencies and eigenmodes; in other words, the dynamics of these modes may be simplified by neglecting the elasticity of the fluid or, equivalently, by filtering out acoustic modes. This is the aim of the subseismic and anelastic approximations; the resulting equations for eigenmodes are much simpler than the original ones and are very useful when dealing with the low-frequency oscillations of rotating stars (e.g. Dintrans & Rieutord 2000).

Recently, we compared these two approximations (Dintrans & Rieutord 2001 referred to as Paper I hereafter). We found that in the two cases that we analysed, namely two polytropes, the anelastic approximation performed much better than the subseismic approximation. We attributed this behaviour to an inconsistency of the subseismic approximation, but our argument turns out to be not general and Smeyers (2001) showed that for low-frequency high-order modes, the subseismic approximation gives the first-order equations

in regions not close to the surface of the star. These results prompted us to re-examine this question in order to clarify the origin of the different behaviour of the two approximations. For this purpose we will focus, in Section 2, on two asymptotic developments: first, where we use, as in Smeyers (2001), the frequency as a small parameter and secondly, where we use the Brunt–Väisälä frequency as the small parameter. These asymptotic developments will prove to be at the origin of each of these approximations and will allow us to clarify the physics attached to each of them. In Section 3, using the same examples as in Paper I, we will compare the approximate eigenfunctions with their exact counterparts and show the better behaviour of the anelastic approximation. Our conclusions are drawn in Section 4.

2 THE ASYMPTOTIC EQUATIONS

As was shown in Paper I, both approximations imply Cowling's approximation; we shall therefore neglect the perturbation of the gravitational potential and will start from the following equations:

$$\rho' + \text{div}(\rho\xi) = 0, \quad (1)$$

$$\omega^2\xi = \nabla \left(\frac{P'}{\rho} \right) - \frac{N^2}{\rho g} \delta P e_r, \quad (2)$$

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$$\delta P = c^2 \delta \rho, \quad (3)$$

where we have assumed a time-dependence of the form $\exp(i\omega t)$ and considered adiabatic oscillations. ξ is the displacement; P' and ρ' , respectively, denote the Eulerian fluctuations of pressure and density, whereas δP , $\delta \rho$ are their Lagrangian counterparts. Thus, we have

$$\delta P = P' + \frac{dP}{dr} \xi_r, \quad \delta \rho = \rho' + \frac{d\rho}{dr} \xi_r,$$

with a pressure gradient satisfying the hydrostatic equilibrium $dP/dr = -\rho g$. Also, ρ is the equilibrium density, $\mathbf{g} = -g\mathbf{e}_r$ the gravitational acceleration and $\gamma = (\partial \ln P / \partial \ln \rho)_S$ the first adiabatic exponent. Finally, c^2 and N^2 , respectively, denote the squares of speed of sound and the Brunt–Väisälä frequency such as

$$c^2 = \gamma \frac{P}{\rho}, \quad N^2 = g \left(\frac{1}{\gamma} \frac{d \ln P}{dr} - \frac{d \ln \rho}{dr} \right). \quad (4)$$

2.1 The subseismic view

As a first exercise we derive the equations verified by the low-frequency gravity modes.

We therefore assume that the frequency reads as $\omega = \varepsilon \omega_1$ and that d/dr scales as ε^{-1} , with $\varepsilon \ll 1$, since we focus on high radial order modes. Developing the dependent variables generically as

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots, \quad (5)$$

and using the classical expansion of the variables on the spherical harmonics,

$$\xi(r, \theta, \phi) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} \xi_m^\ell(r) Y_\ell^m(\theta, \phi) \mathbf{e}_r + \chi_m^\ell(r) \nabla Y_\ell^m$$

and dropping (ℓ, m) indices, we find that $P'_0 = P'_1 = 0$, $\rho'_0 = 0$, $\xi_0 = 0$ and that

$$\begin{aligned} \frac{d}{dr}(r\chi_0) &= -\frac{N^2}{\omega_1^2} \xi_1, \\ \frac{d}{dr}(r^2 \xi_1) &= \ell(\ell+1)r\chi_0. \end{aligned} \quad (6)$$

A system which is slightly different from that obtained when setting $P' = 0$, which yields the subseismic equations:

$$\frac{d}{dr}(r\chi) = \left(1 - \frac{N^2}{\omega^2}\right) \xi, \quad (7)$$

$$\frac{d}{dr}(r^2 \xi) = \ell(\ell+1)r\chi + \frac{g}{c^2} r^2 \xi. \quad (8)$$

However, if we substitute the expansion (5) into (7) and (8), we recover (6); therefore, equations (7) and (8) contain terms of higher order than (6).

In fact, our expansion (5) breaks down near the origin $r = 0$ where the regularity of the solutions (i.e. that $\xi \propto r^{\ell-1}$) is not ensured. This comes from the fact that terms such as f/r are no longer negligible compared with derivatives df/dr .

This difficulty is avoided by Smeyers (2001) with the use of the variable

$$\tau = \frac{\sqrt{\ell(\ell+1)}}{\varepsilon} \int_0^r \frac{N(r')}{r'} dr'$$

instead of the radial variable r . τ serves as a fast variable while r , the slow variable, is assumed to be small compared with the scale of variation of the Brunt–Väisälä frequency or the background density.

Using this transformation, Smeyers (2001) shows that (8) is verified by the solution at leading order while (7) is approximately verified.

Near the surface layers, Smeyers (2001) has shown that within this development, the Eulerian pressure perturbation is no longer negligible and that the subseismic approximation does not apply.

Thus the subseismic equations govern the oscillations of high radial order gravity modes in the central parts of the star. No constraint is imposed on the Brunt–Väisälä frequency and the equations to be solved are

$$\text{div } \xi = \frac{g}{c^2} \xi_r, \quad \omega^2 \xi = \nabla \left(\frac{P'}{\rho} \right) + N^2 \xi_r \mathbf{e}_r.$$

2.2 The anelastic view

Let us now turn to the anelastic approximation. In this case, it is more convenient to write the equations of motion for the velocity field $\mathbf{v}$ rather than the displacement ξ , that is,

$$\begin{aligned} i\omega \rho' + \frac{1}{r^2} \frac{d}{dr}(r^2 \rho u) - \ell(\ell+1) \frac{\rho v}{r} &= 0, \\ i\omega u &= -\frac{d}{dr} \left(\frac{P'}{\rho} \right) + \frac{N^2}{i\omega \rho g} (i\omega P' - \rho g u), \\ i\omega r v &= -\frac{P'}{\rho}, \end{aligned} \quad (9)$$

$$i\omega(P' - c^2 \rho') = -\frac{\rho c^2 N^2}{g} u,$$

where we used the following spherical harmonics decomposition for $\mathbf{v}$ (for clarity, we still dropped in the previous system the ℓ, m indices):

$$\mathbf{v}(r, \theta, \phi) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} u_m^\ell(r) Y_\ell^m(\theta, \phi) \mathbf{e}_r + v_m^\ell(r) \nabla Y_\ell^m.$$

We now assume that the Brunt–Väisälä frequency is vanishingly small; note that as this quantity often diverges at the surface of the star models, it is more appropriate to assume that ω_N , the frequency of the lowest-order gravity mode, is vanishingly small. Thus we write

$$N(r) = \varepsilon n(r), \quad \omega = \varepsilon \omega_1, \quad P' = P'_0 + \varepsilon P'_1 + \dots$$

Note that we make no assumption concerning the scale of the perturbations, which may be of the order of unity. First orders yield the equations

$$P'_0 = \rho'_0 = 0, \quad (10)$$

$$\nabla \cdot \rho \mathbf{u}_0 = 0, \quad (11)$$

$$i\omega_1 u_0 = -\frac{d(P'_1/\rho)}{dr} - \frac{n^2}{i\omega_1} u_0, \quad (12)$$

$$i\omega_1 r v_0 = -P'_1/\rho, \quad (13)$$

$$i\omega_1 (P'_1 - c^2 \rho'_1) = -\frac{\rho n^2 c^2}{g} u_0 \quad (14)$$

from which we write the anelastic system

$$\text{div}(\rho \xi) = 0, \quad \omega^2 \xi = \nabla(P'/\rho) + N^2 \xi_r \mathbf{e}_r.$$

As for the subseismic approximation the perturbed equation of state is eliminated; but from (14), we note that the Eulerian pressure perturbation is of the same order as $c^2 \rho'$.

The subseismic equations can be obtained by just dropping out the Eulerian fluctuation P' in δP , in the original equations (1)–(3);

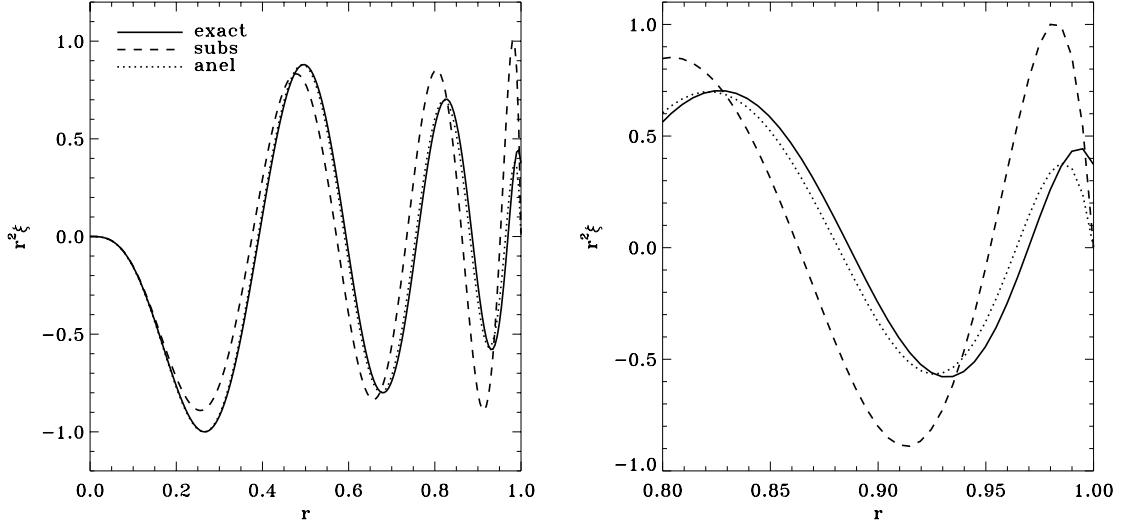


Figure 1. Normalized eigenfunctions ξ with $\ell = 2$ and $k = 5$ for the homogeneous polytrope. The solid line shows the exact solution ($\omega^2 \simeq -3.61 \times 10^{-2}$), while the dashed and dotted lines correspond to its subseismic and anelastic approximations, respectively (with $\omega_{\text{subs}}^2 \simeq -3.26 \times 10^{-2}$ and $\omega_{\text{anel}}^2 \simeq -3.53 \times 10^{-2}$). On the right-hand side, the surface layers have been magnified.

from (14), we see that this is not the case in the anelastic approximation. On the other hand, the fluctuation of density in the mass conservation equation can be neglected.

We therefore see that the anelastic approximation applies when the Brunt–Väisälä frequency is small compared with the acoustic frequencies but does not impose any constraint on the scale of the solutions. Near the surface the anelastic solution differs from the exact solution because of the different boundary condition: exact solutions verify $\delta P = 0$, a condition which transforms into $\xi = 0$ or $u = 0$ for the approximate solution.

2.3 Comments

The foregoing developments show that the anelastic approximation applies under rather more general conditions than the subseismic approximation; indeed, the only requirement is the smallness of the Brunt–Väisälä frequency or, in other words, a large separation between the acoustic spectrum and the gravity spectrum. As this latter condition is often met in stars we can expect that the anelastic approximation performs better when applied to star models.

Concerning the subseismic approximation, it is clear that it can be applied in the central region of a star but that surface layers should be avoided. Smeyers introduces the notion of a boundary layer to describe the regions where his asymptotic solutions are valid. However, these boundary layers are somehow special since their thickness can be comparable to the radius of the star (in polytropes for instance).¹

Broadly speaking, it turns out that the subseismic approximation has a rather local character, while the anelastic approximation has a global one. As eigenvalue problems are global problems in nature, the anelastic approximation should be better suited for these problems.

¹Classical boundary layers have a thickness that is very small compared with the size of the domain and which tends to zero as the small parameter is decreased.

3 EXAMPLES

As in Paper I we consider two polytropes: one of constant density and one of index $n = 3$.

3.1 The homogeneous star model

In this case analytic solutions exist either for the exact or the approximate equations (see Paper I).

In the asymptotic case of large wavenumbers ($k \rightarrow \infty$), one finds that

$$\omega^2 = \Delta - \sqrt{\Delta^2 + \ell(\ell + 1)} = -\frac{\ell(\ell + 1)}{2\Delta} + \mathcal{O}\left(\frac{\ell^2(\ell + 1)^2}{\Delta^3}\right),$$

with $\Delta = \gamma[k(\ell + k + \frac{5}{2}) + \ell + \frac{3}{2}] - 2$. Therefore,

$$\omega^2 \simeq -\frac{2\ell(\ell + 1)}{\gamma} \frac{1}{2k(2k + 2\ell + 5) + 4\ell + 6 - 4/\gamma}.$$

Now using (22) of Paper I we find that for the anelastic approximation

$$\omega_{\text{anel}}^2 = -\frac{2\ell(\ell + 1)}{\gamma} \frac{1}{2k(2k + 2\ell + 5) + 4\ell + 6},$$

while the subseismic expression (21) can be rewritten as

$$\omega_{\text{subs}}^2 = -\frac{2\ell(\ell + 1)}{\gamma} \frac{1}{2k(2k + 3\ell + 4 + 2/\gamma) + 6\ell + (2\ell + 4)/\gamma}.$$

It is clear from these three expressions that, for high-order modes, the anelastic approximation is very close to the exact expression. In fact, one finds that $(\omega^2 - \omega_{\text{anel}}^2)/\omega^2 \sim k^{-2}$ while $(\omega^2 - \omega_{\text{subs}}^2)/\omega^2 \sim k^{-1}$, i.e. the anelastic expression converges *quadratically* while the subseismic one converges only *linearly*.

This better behaviour of the anelastic approximation is confirmed by the shape of the eigenfunctions as shown in Fig. 1. There, we clearly see that the subseismic solution is good only in the central regions ($r < 0.2$), while the anelastic approximation remains close to the exact solution almost to the surface.

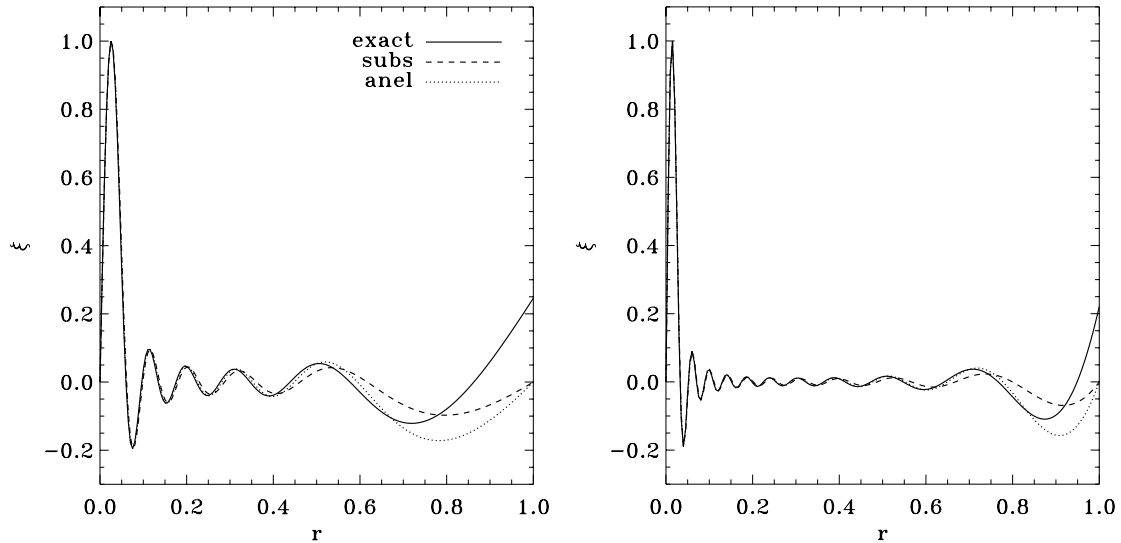


Figure 2. Normalized eigenfunctions ξ with $\ell=2$ and $k=10$ (left) and $k=20$ (right) for the polytrope $n=3$. As in Fig. 1, exact solutions are denoted by solid lines (with $\omega_{10}^2 \simeq 1.98 \times 10^{-3}$ and $\omega_{20}^2 \simeq 5.99 \times 10^{-4}$), while the dashed and dotted lines correspond to their subseismic and anelastic approximations, respectively (with $\omega_{\text{subs}}^2 \simeq 2.08 \times 10^{-3}$, $\omega_{\text{anel}}^2 \simeq 2.02 \times 10^{-3}$ for $k=10$ and $\omega_{\text{subs}}^2 \simeq 6.15 \times 10^{-4}$, $\omega_{\text{anel}}^2 \simeq 6.04 \times 10^{-4}$ for $k=20$).

3.2 The polytrope $n=3$

For a polytrope $n=3$ a similar behaviour exists, although the difference between the two approximations is less pronounced. In Paper I we observed that the eigenfrequencies converged at different rates, with the anelastic approximation converging faster. Here we plot two eigenmodes of high order ($k=10$ and 20) computed with the two approximations and with the complete equations (see Fig. 2). As expected, while both approximations describe the central regions very accurately, the anelastic one remains closer to the ‘exact’ solutions over a larger volume. For the $k=20$ mode, it departs noticeably from the exact solution close to the surface ($r \sim 0.83$).

4 CONCLUSION

In this paper we tried to clarify the differences between the subseismic and anelastic approximations that both aim to describe the low-frequency spectrum. The subseismic approximation appears when one concentrates on the low-frequency high-radial order modes in the central region of a star; no constraint is imposed on the Brunt–Väisälä frequency.

On the other hand, the anelastic approximation assumes a weak stratification but imposes no constraint on the degree of the mode.

Hence, while the anelastic approximation makes the Brunt–Väisälä frequency, and thus the frequency of all gravity modes, vanishingly small compared with a acoustic frequencies, the subseismic approximation focuses on gravity modes for which the radial

order is very large and hence have small frequencies compared with acoustic ones.

In other words, the anelastic approximation removes the elasticity of the fluid by rejecting acoustic frequencies to infinity and therefore allows for a description of the full spectrum of gravity modes, while the subseismic approximation, keeping ω_c and ω_N in a finite ratio, concentrates on one part of the spectrum, namely that containing high radial order modes, which are the least sensitive to the elasticity of the fluid. This situation is summarized in Fig. 3.

Since in stars the situation is often that $\omega_N \ll \omega_c$, the use of the anelastic approximation is recommended as it is probably closer to the solutions of the complete equations; on the other hand, the subseismic approximation may be useful when one needs an analytic expression of gravity modes in the central regions of a star.

Finally, it is worth mentioning the work of Durrant (1989) who discussed these two approximations in the context of the atmospheric sciences. In this field, where the subseismic approximation is called the ‘pseudo-incompressible approximation’ and the anelastic approximation the ‘modified anelastic approximation’, the subseismic approximation appears to be superior to the anelastic approximation as it conserves energy, a property that is important for non-linear problems. This result shows that the best choice for filtering out acoustic modes is dependent on the problem at hands. Therefore, our results which favour the anelastic approximation when searching for low-frequency modes of stars, may be specific to eigenvalue problems.

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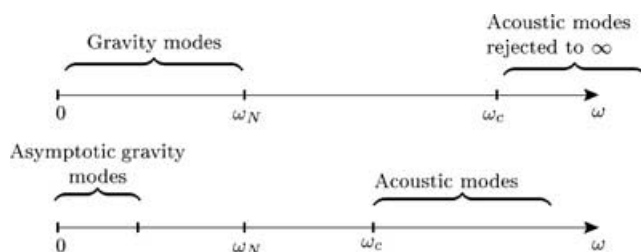


Figure 3. A schematic picture of the modes of a star viewed from the anelastic viewpoint (above) and the subseismic viewpoint (below). ω_N and ω_c are, respectively, the frequency of the lowest-order gravity and acoustic modes.

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