Wave Attractors in Rotating Fluids: A Paradigm for Ill-Posed Cauchy Problems

M. Rieutord,[1,2] B. Georgeot,[3] and L. Valdettaro[4]
[1]*Observatoire Midi-Pyrénées, 14 avenue Edouard Belin, F-31400 Toulouse, France*
[2]*Institut Universitaire de France*
[3]*Laboratoire de Physique Quantique, UMR 5626 du CNRS, Université Paul Sabatier, F-31062 Toulouse Cedex 4, France*
[4]*Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32, 20133 Milano, Italy*
(Received 1 February 2000; revised manuscript received 6 July 2000)

In the limit of low viscosity, we show that the amplitude of the modes of oscillation of a rotating fluid, namely inertial modes, concentrate along an attractor formed by a periodic orbit of characteristics of the underlying hyperbolic Poincaré equation. The dynamics of characteristics is used to elaborate a scenario for the asymptotic behavior of the eigenmodes and eigenspectrum in the physically relevant régime of very low viscosities which are out of reach numerically. This problem offers a canonical ill-posed Cauchy problem which has applications in other fields.

PACS numbers: 47.32.–y, 02.60.Lj, 04.20.Gz, 05.45.Ac

Rotating fluids encompass all fluids whose motions are dominated by the Coriolis force. These flows play an important role in astrophysics or geophysics where the large size of the bodies makes the Coriolis force a prominent force. Some engineering problems like the stability of artificial satellites also require the study of rotating fluids because of their liquid-filled tanks [1]. This latter problem is related to the existence of waves specific to rotating fluids, namely inertial waves, which easily resonate. These waves play also an important part in the oscillation properties of large bodies like the atmosphere, the oceans, the liquid core of the Earth [2], rapidly rotating stars [3], or neutron stars [4]. As such, they have been considered since the work of Poincaré on the stability of figures of equilibrium of rotating masses [5]. Pressure perturbations of inertial modes for inviscid fluids obey the Poincaré equation (PE) (christened by Cartan [6]) which reads $\Delta P - (2\Omega/\omega)^{-2}\partial^2 P/\partial z^2 = 0$ where $\Omega \vec{e}_z$ is the angular velocity of the fluid and ω is the frequency of the oscillation. Since $\omega < 2\Omega$ [7], the PE is hyperbolic (energy propagates along characteristics) and since its solutions must meet boundary conditions, the problem is ill posed mathematically. Although some smooth solutions exist (for instance, for a fluid contained in a full sphere or a cylinder), one should expect singular solutions in the general case. These latter solutions have been made explicit only recently thanks to numerical simulations which include viscosity to regularize the singularities and let this parameter be very small as in real systems [8,9].

In this Letter we wish to present a scenario, based on analytical and numerical results, for the asymptotic behavior of inertial modes at small viscosities. We use the case of a spherical shell as a container, which is relevant for astrophysical or geophysical problems, but it will be clear that this case is general. We will sketch only the main results; more details can be found in [9]. While the fluid mechanical problem is of much interest by itself, it opens new perspectives in the theory of partial differential equations (PDE) and also offers a toy model for some (very involved) problems of general relativity which we shall present briefly.

The model we use is a spherical shell whose inner radius is ηR and outer radius R ($\eta < 1$). The fluid is assumed incompressible with a kinematic viscosity ν. We write the linearized equations of motion for small amplitude perturbations for the velocity $\vec{u}$ in a frame corotating with the fluid; momentum and mass conservation imply

$$\frac{\partial \vec{u}}{\partial t} + \vec{e}_z \times \vec{u} = -\vec{\nabla} p + E\Delta\vec{u}, \qquad \vec{\nabla}\cdot\vec{u} = 0 \qquad (1)$$

when dimensionless variables are used; $(2\Omega)^{-1}$ is the time scale and $E = \nu/2\Omega R^2$ the Ekman number. When E is set to zero and $\vec{u}$ is eliminated, one obtains the Poincaré equation. In nature $E \ll 1$ and one is tempted to use boundary layer theory and singular perturbations to solve (1). However, this is feasible only when regular solutions exist for $E = 0$; this is the case when the container is a full sphere [7] but not when the container is a spherical shell. Indeed, numerical solutions of the eigenvalue problem issued from (1), where solutions of the form $\vec{u}(\vec{r})e^{\lambda t}$ are searched for [with $-1 \le \omega = \mathrm{Im}(\lambda) \le 1$], yield eigenmodes of the kind shown in Fig. 1. In this figure we see that the amplitude of the mode is all concentrated along a periodic orbit of characteristics of the PE; we found this property to be quite general, after extensive numerical exploration of least-damped modes of (1) [8,9], and will now explain its origin and consequences on the asymptotic spectrum of inertial modes. For this purpose we will use axisymmetric modes since the azimuthal dependence of solutions can always be separated out because of the axial symmetry of the problem.

For understanding the concentration of kinetic energy along a periodic orbit of characteristics, it is necessary to consider in some detail the dynamics of these lines. Characteristics of PE are, in a meridional plane, straight lines making the angle arcsinω with the rotation axis. A numerical calculation of their trajectories shows that they

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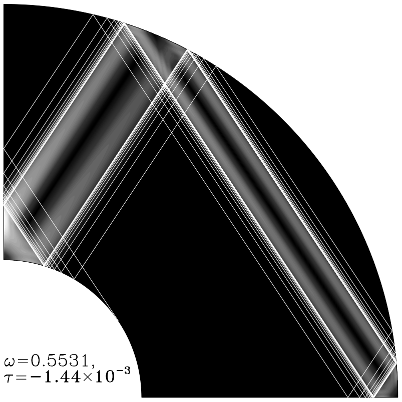

FIG. 1. Kinetic energy in a meridional section of a spherical shell of an inertial mode in a viscous fluid. For this numerical solution, $E = 10^{-8}$, 570 spherical harmonics and 250 Chebyshev polynomials have been used (the numerical method is described in [8]). The mode is axisymmetric and symmetric with respect to equator. $\eta = 0.35$ as in the Earth's core. ω is the frequency of this mode and τ its damping rate. Stress-free boundary conditions are used. The convergence of characteristics towards the attractor is also shown (white lines).

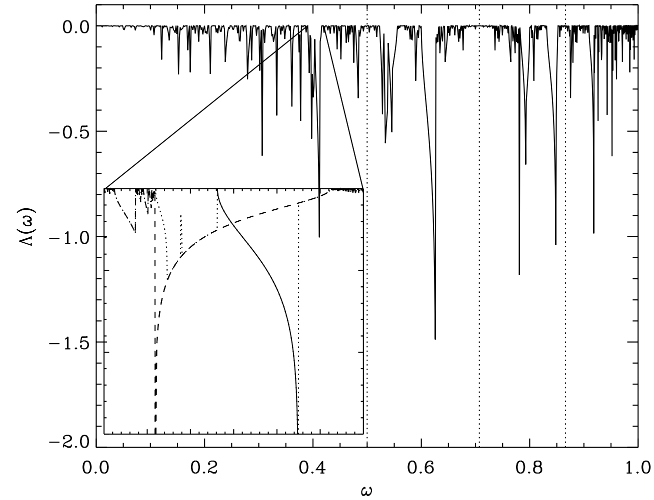

FIG. 2. LE $\Lambda(\omega)$ of the orbits as a function of ω for $\eta = 0.35$. Inset: Blowup showing the LE of two coexisting attractors (full and dashed thick lines).

generally converge towards a periodic orbit which we call, after [10], an *attractor*. The periodic orbit of Fig. 1 is one example of such an attractor.

The Lyapunov exponent (LE) of a trajectory, defined by $\Lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \ln|\frac{d\phi_{n+1}}{d\phi_n}|$ (ϕ_n is the latitude of the nth reflection point), describes how fast characteristics are attracted or repelled. Its computation as a function of frequency shows that attractors ($\Lambda < 0$) are ubiquitous in frequency space (see Fig. 2). Their existence shows that the dynamical system described by the characteristics is not Hamiltonian; the "dissipation" is purely geometrical and is due to the fact that, unlike billiards, the reflection on boundaries is not specular, but conserves the angle with the rotation axis. In fact, the dynamics of rays is a one-to-one one-dimensional map (from the outer boundary to itself), piecewise smooth, but with a finite number (twelve) of discontinuities. This kind of map has not been studied in the literature of dynamical systems, perhaps because it does not produce chaos because of its invertibility. Iterations of such a map generate fixed points which either correspond to attractors or to some neutral periodic orbits. Indeed, if $\eta = 0$ (i.e., the sphere is full), all orbits such that $\omega = \sin(p\pi/2q)$ with $(p, q) \in \mathbb{N}^2$, are neutral and periodic while those such that $\omega = \sin(r\pi)$, r being irrational, are neutral ergodic (quasiperiodic). When η is nonzero only a finite number of such neutral periodic orbits subsist; for instance, if $\eta = 0.35$ which is the aspect ratio of the Earth's liquid core, $q = 1, 2, 3, 4$ are the only possibilities. Interestingly, we face here a situation which is just the opposite of the one described by the KAM theorem in Hamiltonian systems: when the full sphere is perturbed by the introduction of an inner sphere, all ergodic orbits are instantaneously destroyed while the longer periodic orbits survive the smaller the denominator q is.

Apart from these isolated frequencies which become rarer and rarer as η increases, generic trajectories are in the basin of attraction of attractors. We were able to show [9] that the number of attractors at a given frequency is finite. The inset of Fig. 2 shows the typical case where an attractor exists in a frequency band $[\omega_1, \omega_2]$ with $\Lambda(\omega_1) = 0$, $\Lambda(\omega_2) = -\infty$ and $\omega_2 - \omega_1 \sim 1/N^2$ where N is the length of the attractor defined as its number of reflection points. Near ω_1, $\Lambda \sim \sqrt{\omega - \omega_1}$ and near ω_2, $\Lambda(\omega) \sim \frac{1}{N} \ln N(\omega - \omega_2)$. The latter implies that long attractors have small LE in a large fraction of $[\omega_1, \omega_2]$ (all these results are shown in [9]).

The existence of attractors for characteristics implies that solutions of the inviscid problem (i.e., of PE) are singular. This property can be made explicit in the simplified case of a 2D problem. Indeed, in this case the PE may be written $\partial^2 P/\partial u_+ \partial u_- = 0$ using characteristic coordinates; solutions may be constructed explicitly from an arbitrary function but, as shown in [11], regular eigenmodes exist only when neutral periodic orbits exist and eigenvalues are infinitely degenerate. When attractors are present, the scale of variations of the pressure vanishes on the attractors while its amplitude remains constant. As velocity depends on the pressure gradient, it diverges on the attractor; this divergence is like the inverse of the distance to the attractor which makes the velocity field not square integrable. This result seems to be valid also in 3D [9].

We therefore understand why solutions of (1) look like Fig. 1: the inviscid part of the operator focuses energy of the modes thanks to the action of the mapping made by characteristics while viscosity opposes this action via diffusion. The resulting picture of Fig. 1 therefore comes from a balance between inviscid terms and viscous ones; let us make this more quantitative.

For this purpose we first observe that the patterns drawn by the kinetic energy of the mode in Fig. 1 is in fact a shear layer whose width scales with E^σ and $\sigma \simeq 1/4$. Such a scaling is observed numerically and seems generic [8,9]; it implies that the damping rate of such modes scales like $E^{1/2}$ as clearly shown in Fig. 3. Now we may consider a wave packet traveling around an attractor in a slightly viscous fluid. The above mentioned balance, when applied to both the width and the amplitude of the packet, leads to a relation between the LE and the Ekman number such as $\Lambda \sim E^{1-3\sigma}$ with $\sigma < 1/3$ for an eigenmode of the viscous problem. We see that the constraint $\sigma < 1/3$ is met by actual shear layers. It therefore turns out that frequencies of eigenmodes of the viscous problem are such that $\Lambda \to 0$ when $E \to 0$ which means that they will gather around the roots of the equation $\Lambda(\omega) = 0$.

The above result shows the importance of the scaling verified by shear layers. A boundary layer analysis reveals that these shear layers are in fact nested layers which consist of an inner $\sigma = 1/3$ layer surrounded by a thicker layer. The inner $1/3$ layer can be fully explicited. Using coordinates along the shear layers (x) and perpendicular to it (y), we find that the φ component of the velocity verifies $\frac{\partial^3 u_\varphi}{\partial Y^3} = -i\frac{\partial u_\varphi}{\partial q}$, with $Y = y/E^{1/3}$ and $q = x/\sqrt{1-\omega^2}$ which is also the equation verified by the stream function in a steady shear layer of a rotating fluid [12]. Solutions which vanish in $Y = \pm\infty$ are self-similar and of the form $u_\varphi = q^\alpha H_\alpha(Y/q^{1/3})$ with $H_\alpha(t) = \int_0^\infty e^{-ipt}e^{-p^3}p^{-3\alpha-1}\,dp$. Besides, $\alpha = -\frac{1}{3}$ is the only admissible value to ensure a coherent evolution of the width and amplitudes after reflection on a boundary.

We are now in a position to propose a scenario for the asymptotic behavior of inertial modes when the viscosity vanishes. Eigenfunctions reduce to nested shear layers concentrated along attractors while the associated eigenvalues converge toward the frequency ω_i such that $\Lambda(\omega_i) = 0$ for the associated attractor. Furthermore, we can constrain this convergence of eigenfrequencies; indeed, since $\Lambda \sim \sqrt{\omega - \omega_i}$, one finds that $\omega = \omega_i + aE^{2-6\sigma} + \ldots$ and $\tau = \mathrm{Re}(\lambda) = -bE^{1-2\sigma}$ when $E \to 0$; Fig. 3 shows that this law agrees well with the numerical results, in the case shown, with $\sigma = 1/4$.

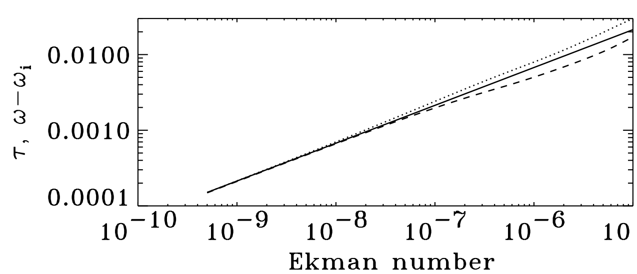

FIG. 3. Asymptotic behavior of an eigenvalue. The dashed line is $\omega - \omega_i$ as a function of E, while the dotted line is for the damping rate τ. The solid line represents the "theoretical" law $E^{1/2}$. $\omega_i = 0.403\,112\,887$ is a root of $\Lambda(\omega) = 0$ when $\eta = 0.35$.

In addition, we noticed earlier that for a finite number of ω such that $\omega = \sin(p\pi/2q)$ all orbits of characteristics are periodic; this implies that in the vicinity of these frequencies very long attractors with very small average LE accumulate as shown by Fig. 4; therefore, these frequencies will be accumulation points of the asymptotic spectrum. Moreover, around these frequencies eigenmodes are weakly damped. On the contrary, modes whose frequency is in the frequency band of short attractors (like the one of Fig. 1) are more strongly damped. It therefore turns out that the LE curve in Fig. 2 will strongly constrain the distribution of least-damped modes in the complex plane at finite viscosities: such modes will avoid the large frequency bands of short-period attractors and concentrate around frequencies where $\Lambda(\omega) = 0$ especially those with $\omega = \sin(p\pi/2q)$.

This general evolution of the spectrum is well illustrated in Fig. 5. Here, the least-damped eigenvalues have been computed for $E = 10^{-8}$. We clearly see frequency bands of attractors avoided by weakly damped eigenvalues but see the gathering of these eigenvalues around $\sin(\pi/4)$ and, but less conspicuously, around $\sin(\pi/6)$.

To complete the picture, we need now mention that a few regular modes survive among all these singularities; such modes are purely toroidal modes or r modes [13] which are nonaxisymmetric. They avoid the constraint of characteristics, for their velocity field has no radial component; this property makes their characteristics independent of frequency (they are circles and vertical straight lines) and authorizes smooth solution at zero viscosity. The associated eigenvalues $\omega = 1/(m+1)$, $m \in \mathbb{N}^*$ seem to be the only eigenvalues of the Poincaré operator in a spherical shell.

Ending this Letter, it is worth emphasizing the role of the geometrical approach allowed by the dynamics of

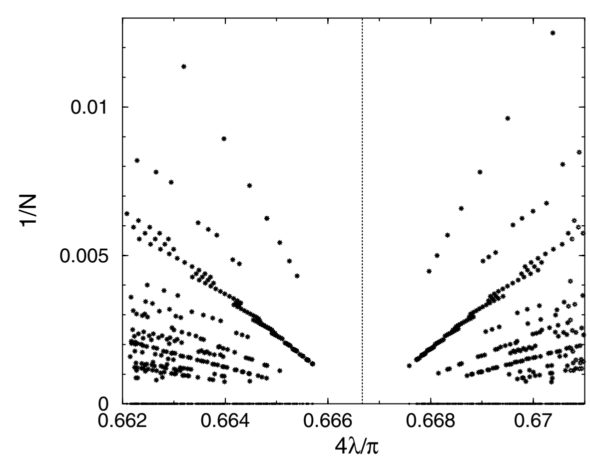

FIG. 4. Inverse of the length N of attractors with $N < 100$ for $\eta = 0.35$, near the accumulation point $\pi/6$; each point corresponds to an attractor with $\Lambda = 0$ and therefore to a point in the asymptotic spectrum. Note the lengthening of the attractors as $\pi/6$ is approached. Here $\eta = 0.35$.

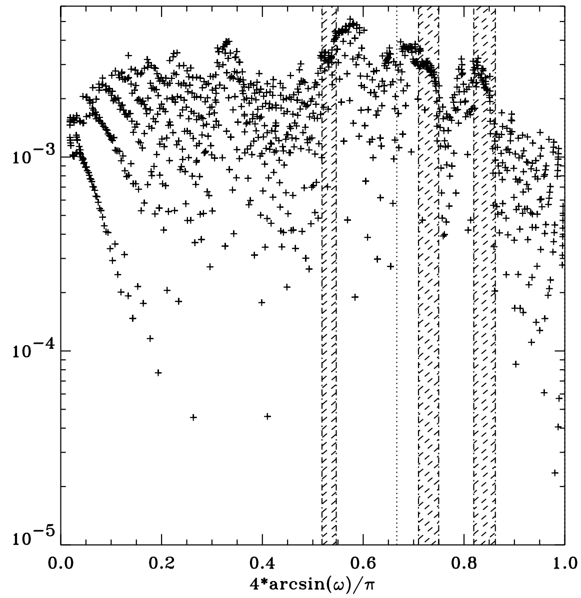

FIG. 5. Distribution of the eigenvalues associated with least-damped axisymmetric modes in the complex plane. Hatched frequency bands denoted bands occupied by simple attractors; the dotted line is for $\sin(\pi/6)$. The Ekman number is 10^{-8} and $\eta = 0.35$. We used a resolution of 700 spherical harmonics and 270 radial grid points.

characteristics, for describing the asymptotic properties of inertial modes; in the domain of very low Ekman numbers ($10^{-10} \rightarrow 10^{-20}$), typical of astrophysical or geophysical fluids, these modes are out of reach numerically.

The foregoing presentation shows that inertial modes display a very rich dynamical behavior which comes from the ill-posedness of the underlying inviscid problem. Here we discussed the case of the spherical shell, but our results are general and can be extended to any container; this is important since natural containers are usually not perfect geometrical objects. Hence, fortunately, a curve like Fig. 2 is structurally stable (see our discussion relative to the core of the Earth in [2]).

We note that the relevance of attractors has also been shown experimentally in stratified fluids [14]. Some configurations of conducting fluids bathed by a magnetic field, obeying the PE, will also display attractors [15]. These properties are in fact very general and extend to mixed-type PDE as illustrated by the case of gravito-inertial modes [16]. We think that similar results should hold for systems which are solutions of PDE of hyperbolic or mixed type meeting boundary conditions. As an example, our results may have applications in general relativity and the problem of "closed timelike curves" (CTC), that is the problem of the existence of physical systems which permit travels backward in time. Such systems like wormholes have been studied by various authors [17]; they set many problems among which is that of causality. Such a problem is also at the origin of the ill-posedness of the Poincaré problem, and we showed that it leads to many kinds of singularities.

We therefore see that inertial oscillations of a fluid inside a container offer a paradigm which may guide our intuition for problems in other fields of physics which are also ill-posed Cauchy problems.

We would like to thank Boris Dintrans and Leo Maas for very helpful discussions. Part of the calculations have been carried out on the Cray C98 of IDRIS at Orsay and on the CalMip machine of CICT in Toulouse which are gratefully acknowledged.

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