Physics of the Earth and Planetary Interiors 117 (2000) 63–70

PHYSICS OF THE EARTH AND PLANETARY INTERIORS

www.elsevier.com/locate/pepi

A note on inertial modes in the core of the Earth

Michel Rieutord *

Observatoire Midi-Pyrénées, 14 avenue Edouard Belin, 31400 Toulouse, France

Received 3 November 1998; accepted 5 March 1999

Abstract

We analyse the consequences of the singular (at zero viscosity) nature of inertial modes in a spherical shell for the dynamics of the Earth's liquid core on the time scale of a day. We show that the singularities, essentially appearing as internal shear layers, although increasing the damping rates of the modes, cannot be invoked to rule out the possible role of the elliptical instability in the geodynamo. We also show that the search for core modes in the spectrum of oscillations of the Earth should be guided by the Lyapunov exponents associated with the maps built up by characteristics propagating in the shell.

Keywords: Geodynamo; Earth's core; Inertial modes

1. Introduction

It is well known that rapidly rotating fluids exhibit a special kind of oscillations, namely inertial waves, which are driven by the Coriolis force. As such, the liquid core of the Earth also exhibits these oscillations.

The role of inertial modes in the Earth's core has already been considered in the past (Aldridge et al., 1989 and references therein; Rieutord, 1995) since such modes may be the origin of (or contribute to) the geodynamo (Aldridge et al., 1997) or may open a new window onto the Earth's core if they are detected (by superconducting gravimeters for instance).

This note is intended to follow up our paper of 1995 and present some consequences, concerning the dynamics of the core, of the new results obtained recently on inertial modes in a spherical shell (Rieutord and Valdettaro, 1997; Rieutord et al., 1999; Dintrans et al., 1998; Fotheringham and Hollerbach, 1998). As in the preceding paper, we shall ignore magnetic fields in the dynamics of the modes; at a period of a day the coupling between inertial modes and magnetic modes is small indeed, as shown by Kerswell (1994), since Alfvén waves are much slower than inertial waves. In Section 2, we show why the incompressible fluid is a good model for the Earth's core; within this model, we then describe our current understanding of the asymptotic properties of inertial modes in a spherical shell at very low viscosities. We then draw some conclusions about the dynamics of the core.

* E-mail: rieutord@obs-mip.fr

PII: S0031-9201(99)00087-4

2. On the dynamics of the core for time scales of a day

Our current understanding of the structure of the core is that it is adiabatically stratified in its main part with possibly a stably stratified layer just below the core–mantle boundary (Lister and Buffett, 1995). This layer has, however, a marginal effect on core modes (see Rieutord, 1995) and we shall ignore it in the following.

Hence, we assume the core to be roughly isentropic and characterized by a density profile $\rho_0(r)$ where the density changes by 12% between the inner core boundary (ICB) and the core–mantle boundary (CMB). As we are interested by inertial modes whose typical period is larger than 12 h, we can filter out the sound waves and use the so-called subseismic or anelastic approximation. With such an approximation, mass conservation implies:

$$\mathrm{div}\,\rho_0\vec{v} = 0 \tag{1}$$

while the momentum equation yields:

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} + 2\vec{\Omega}\times\vec{v}\right) = -\vec{\nabla}P + \rho\vec{g} \tag{2}$$

and, since the fluid is assumed isentropic, we have from thermodynamics $\vec{\nabla}P = \rho\vec{\nabla}h$, where h is the specific enthalpy. Finally, the momentum equation simplifies into

$$\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v} + 2\vec{\Omega}\times\vec{v} = -\vec{\nabla}\chi \tag{3}$$

where χ is the sum of all potentials. Linearizing this equation and using non-dimensional quantities, we may rewrite Eqs. (1) and (3) as

$$\begin{cases} \mathrm{div}\,\vec{u} = -\vec{u}\cdot\vec{\nabla}\ln\rho_0 \\ i\omega\vec{u} + \vec{e}_z\times\vec{u} = -\vec{\nabla}\Pi \end{cases} \tag{4}$$

where the time scale is now $(2\Omega)^{-1}$, the rotation axis is parallel to the z-axis and Π is the generalized pressure. Perturbations have been assumed proportional to $\exp(i\omega t + im\varphi)$, where φ is the azimuthal angle of the cylindrical coordinates to be used below and m is an integer.

Expressing the velocity field in terms of $\vec{\nabla}\Pi$, we can cast the two preceding equations into a generalized Poincaré equation which reads:

$$\Delta\Pi - \frac{1}{\omega^2}\frac{\partial^2\Pi}{\partial z^2} = -\frac{\rho_0'}{r\rho_0}\left(\frac{\omega}{1-\omega^2}s\frac{\partial\Pi}{\partial s} + \frac{m\Pi}{1-\omega^2} - \frac{z}{\omega}\frac{\partial\Pi}{\partial z}\right) \tag{5}$$

where r and s are respectively the radii of spherical and cylindrical coordinates. The equilibrium density profile has been assumed spherically symmetric.

This equation shows that terms with second order derivatives are not modified by the density profile. Hence the characteristics of this generalized Poincaré equation are the same (i.e., straight lines) as those of the original Poincaré equation (when $\rho_0 = \mathrm{const.}$). Therefore, the geometrical properties of the inviscid solutions to be described below, where we discard density variations, can be transferred easily to the more realistic situation described by Eq. (5).

3. Inertial modes in a spherical shell

We shall now present some recent results obtained for inertial modes of an incompressible fluid in a rotating spherical shell. The problem with such modes is that they obey the Poincaré equation, which is hyperbolic, while they meet some kind of Neuman condition (in fact an oblique derivatives condition) on the two bounding spheres; namely,

$$-\omega^2\vec{n}\cdot\vec{\nabla}\Pi + (\vec{n}\cdot\vec{e}_z)(\vec{e}_z\cdot\vec{\nabla}\Pi) + i\omega(\vec{e}_z\wedge\vec{n})\cdot\vec{\nabla}\Pi = 0$$

where $\vec{n}$ is the unit vector normal to the spheres. Such a problem is ill-posed mathematically and therefore of considerable difficulty.

3.1. The inviscid solutions

The difficulties come from the fact that solutions of the inviscid problem are in most cases singular. At the moment, the only known regular solutions of the Poincaré equation in a sphere or a spherical shell are:

- All inertial modes in the full sphere where pressure perturbations are polynomials (see Greenspan, 1969).
- Some purely toroidal modes in a spherical shell, non-axisymmetric, with frequencies in the set $\{(2\Omega)/(m+1), \quad m \in \mathbb{N}\}$ (see Rieutord and Valdettaro, 1997); the first of these modes ($m=1$) is the Poincaré (or spin-over) mode.

All other modes of the spherical shell are singular and we may divide the singularities into three categories:

- Divergence on the axis
- Divergence at the critical latitude
- Divergence along the limit cycle to which characteristics converge.

The first one, described in Rieutord and Valdettaro (1997), is a divergence in $1/\sqrt{s}$ on the axis. This is a ''mild'' singularity as it is integrable.

The second one, first studied by Stewartson and Rickard (1969) in the case of non-axisymmetric modes in a thin shell, appears on convex bodies when a grazing characteristic exists. It may be shown (Rieutord et al., 1999) that the velocity diverges as $1/\sqrt{d}$ on the grazing characteristic which is tangent to the inner sphere at the critical latitude; d is the distance to this characteristic. Like the first one, this singularity is integrable.

The third one is the strongest and the most interesting as it implies non-square integrable solutions. An intuitive idea of the phenomenon may be obtained by considering a purely two-dimensional case, such as the one considered by Maas and Lam (1995), which derived from the analysis of internal gravity waves in a basin. In such a case, the Poincaré equation reduces to

$$\frac{\partial^2 \psi}{\partial \xi_+ \partial \xi_-} = 0 \qquad (6)$$

when characteristics coordinates $(\xi_+, \ \xi_-)$ are used. In this case, the stream function ψ in the domain can be constructed through a mapping of an arbitrary function defined on some fundamental interval (see Maas and Lam, 1995 or Maas et al., 1997). When the characteristics converge towards a limit cycle, the scale of the mapping tends to zero while the amplitude of the mapped function remains constant. It may be shown that gradients diverge as the inverse distance to the limit cycle which implies that the velocity is not square integrable. Two examples of limit cycles are shown in Fig. 1.

Another difficulty with these solutions is that they may be infinitely degenerate. In the 2D-case mentioned above, this is proved by Maas and Lam (1995) as the solution appears to be defined from an arbitrary function. This degeneracy might not persist,

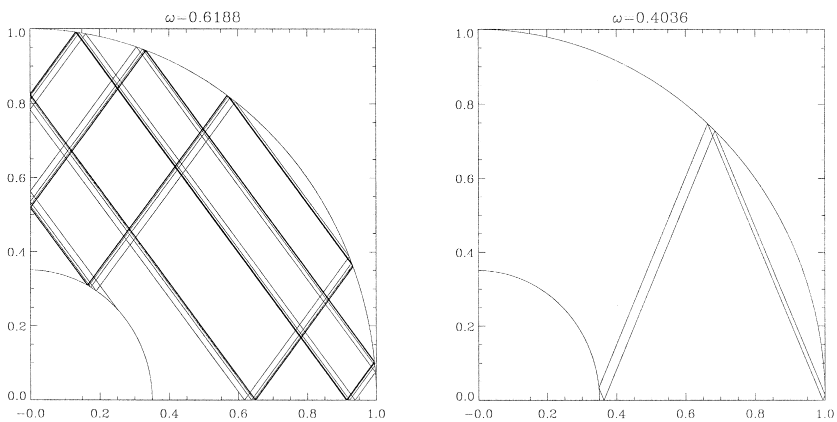

Fig. 1. Two periodic orbits of characteristics for a spherical shell similar to the Earth's liquid core. The ratio of the inner shell radius to the outer one is $\eta = 0.35$. In the left-hand plot we have drawn characteristics starting from an arbitrary point on the inner shell to show the rapid convergence of the path towards the limit cycle. In the right-hand plot, we have drawn the asymptotic periodic orbit only. The viscous solutions corresponding to these orbits are shown in Fig. 2.

however, in a spherical shell because of curvature terms, but this is not yet proved.

Finally, let us note that all these results which may be described in the meridional plane of the shell also apply to non-axisymmetric modes. Indeed, the symmetry of the equations and the container with respect to the rotation axis makes the φ variable separable from the two others: thus once the solutions are projected on the Fourier basis $e^{im\varphi}$ the Fourier components m are solutions of independent equations whose characteristics are independent of m and therefore identical to those of axisymmetric modes. An exception, however, is the set of purely toroidal modes, which is a degenerate case (Rieutord et al., 1999).

3.2. Viscous solutions

As may be foreseen, viscosity will regularize the above solutions by smoothing out all the singularities and also removing an eventual degeneracy.

As shown by Fig. 2, the amplitude of a mode in the asymptotic regime is concentrated along the limit cycle of the characteristics. These ''rays'' were first

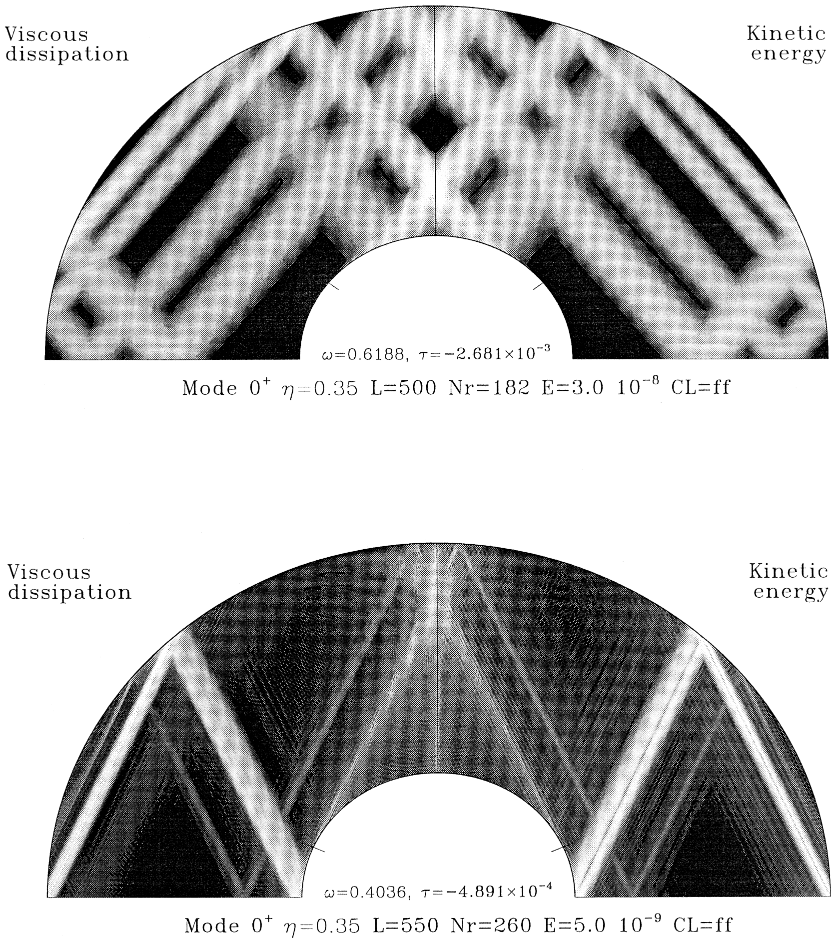

Fig. 2. Plots of the viscous dissipation (left) and kinetic energy (right) in the meridional plane for two axisymmetric inertial modes associated with a periodic orbit. Note the amplitude increase near the axis and the appearance, in the mode with $\omega = 0.404$, of some energy along the characteristic grazing the inner sphere at the critical latitude. L is the number of spherical harmonics used to compute such solutions, N_r is the number of points in the radial grid, and τ is the damping rate. Stress-free boundary conditions are used in both cases. The tickmarks on the inner shell shows the critical latitude.

observed in a spherical shell by Hollerbach and Kerswell (1995) in the case of the spin-over mode, and Tilgner (1999) generalized these computations to the spheroidal shell; rays are in fact detached shear layers which are very similar to the stationary Stewartson layers described in Moore and Saffman (1969). Previous attempts to describe these layers may be found in Stewartson (1972), Walton (1975) or more recently Kerswell (1995); unfortunately, these works overlook the nested structure of these shear layers and focus mainly on the $E^{1/3}$ scaling ($E = \upsilon/2\Omega R^2$ is the Ekman number with υ as the kinematic viscosity and R the radius of the CMB). However, as noted by Rieutord and Valdettaro (1997) and Dintrans et al. (1998), velocity gradients in these layers rather scale with $E^{1/4}$ and therefore imply a damping rate of the modes scaling as $E^{1/2}$ whatever the boundary conditions (no-slip or stress-free). The detailed analysis of these nested layers, which contains both scalings $E^{1/3}$ and $E^{1/4}$, will be presented in a forthcoming paper (Rieutord et al., 1999).

Also shown in Fig. 2 is the axial singularity emphasizing the ray near the axis. The singular grazing characteristic also shows up in the background as another, but much less intense, shear layer.

4. Implications for the dynamics of the core of the Earth

We shall now discuss two aspects of the dynamics of the core which may be affected by the special nature of inertial modes in a spherical shell. These are the role of the elliptical instability in the geodynamo and the detection of inertial modes of the core in the oscillation spectrum of the Earth.

4.1. Elliptical instability

The elliptical instability may be understood as a parametric instability of a rotating fluid which destabilizes some non-axisymmetric inertial modes (Lumb et al., 1993; Kerswell, 1993; Kerswell and Malkus, 1998). In the case of the core, the parametric forcing comes from the deformation of the CMB and ICB by the tidal potential. The growth rate of the instability is of the order of $\varepsilon\Omega/2$, where ε is the tidal elongation of around 5×10^{-8} at the CMB (Aldridge et al., 1997). This growth rate must be compared to the damping rate of inertial modes which comes from both the Ekman layers and the internal shear layers. As mentioned above, these layers imply a damping rate scaling with $E^{1/2}$, which is of order 10^{-8} when molecular viscosity is taken. This is of the same order as the growth rate and therefore we cannot conclude. However, as mentioned by Rieutord (1995), the damping rate may be strongly increased if the shear layers develop some turbulence. To evaluate this possibility, we now compute the Reynolds number associated with the $E^{1/4}$-layers. We have

$$Re = RoE^{-3/4}$$

where Ro is the Rossby number of the flow. If we estimate the amplitude of the velocity field from the amplitude of the tidal deformation at CMB, $V = \varepsilon R/(6\ \mathrm{h})$, we find $V \sim 8 \times 10^{-6}\, m/s$ and thus $Re \sim 1.5 \times 10^4$, which is large enough to bear some turbulence. However, such turbulence cannot appear since the time scale of the shear instability $T \sim (RE^{1/4})/V \sim 530$ days is much larger than the period of forcing.

From this discussion it is clear that the elliptical instability cannot be dismissed because of the singular nature of inertial modes. Other mechanisms of dissipation like Ohmic diffusion also operate, but it was shown by Kerswell (1994) that this latter mechanism is no more efficient than the viscous one. Therefore, we still cannot exclude that this instability may play some part in the geodynamo by, for instance, triggering the inversions.

4.2. Core oscillations

As the detection of core oscillations is a very good way of getting information on the core, it is important to consider the first implications of the peculiar nature of inertial modes. The foregoing discussion has shown that the damping rate is a priori not much changed by the presence of shear layers: it is likely enhanced, but the scaling is still the one given by Ekman layers.

The new point raised by the shape of the modes concerns their excitability. At time scales of the order of a day, the exciting mechanisms are basically

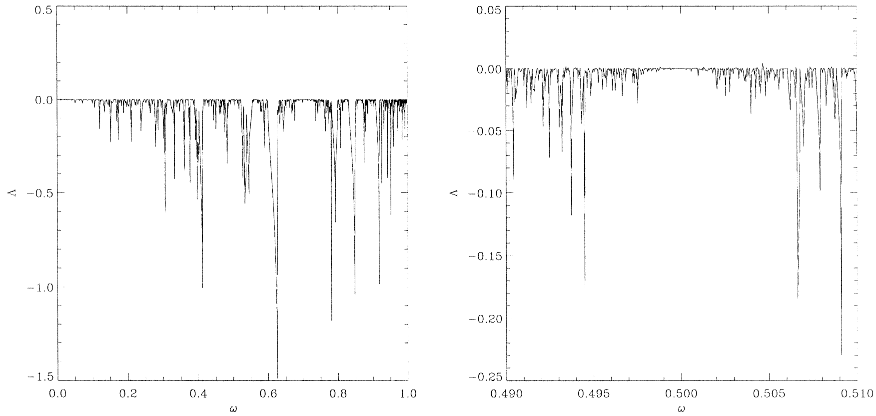

Fig. 3. Lyapunov exponents as a function of the frequency for trajectories of characteristics in shell with aspect ratio $\eta = 0.35$. The right-hand plot is a zoom on the frequencies close to the period of a day. On such diagrams the spikes of the curves in fact go to minus infinity: they correspond to orbits which touch the inner sphere near the critical latitude.

the tidal potential or the precession of the core, both of which are large scale forcing. Because of the shape of inertial modes, featuring thin shear layers, the response is likely to be weak.

The best candidates are all the modes with the same symmetries as the exciting sources and with close frequencies so as to resonate. At first sight, these are the purely toroidal modes, which are large-scale modes. The first of them, the Poincaré mode ($m = 1$), is indeed already detected as the Free Core Nutation. The second one ($m = 2$) would have been a good candidate, since it has the same φ-dependence as the tidal potential and a period (36 h) close to a multiple of the tidal forcing; but it has the wrong equatorial symmetry, since the tidal force is a poloidal field.

To search for other candidates, one may look in the ''windows'' defined by the Lyapunov exponent associated with the limit cycle of the frequency as shown in Fig. 3. Indeed, at each frequency one may compute the trajectory of characteristics which generically converge towards a limit cycle as illustrated by Fig. 1a. The rate of convergence is measured by the Lyapunov exponent, which is defined by

$$\Lambda = \frac{1}{N} \sum_{n=1}^{N} \ln \left| \frac{\mathrm{d}\theta_{n+1}}{\mathrm{d}\theta_n} \right| \quad \text{with } N \to \infty$$

where θ_n is the colatitude of the nth reflection point along the characteristic path. In other words, Λ is just the mean (taken along the path of characteristics) of the logarithm of the contraction or dilation of the application which maps the nth reflection point into the $(n+1)$th. The convergence of characteristics towards a limit cycle implies that Λ is negative. [1] In the asymptotic regime of small Ekman numbers, least damped modes will be found in regions where the map is not converging too rapidly and therefore in regions of the spectrum where the Lyapunov exponent is close to zero. Indeed, it turns out from Rieutord and Valdettaro (1997) and Dintrans et al. (1998), that shear layers appears only if the Ekman number is below some critical value depending on the length of the limit cycle; short-period orbits show first their shear layers and correspond to more strongly damped modes; longer ones corresponding to low (absolute) value of the Lyapunov exponent seem to be associated with least-damped modes. Such a window with small Lyapunov exponent exists around the frequency of a day as shown by Fig. 3b. One may thus find some $m = 2$ modes excited by the tides but their detection will be difficult because of the oceanic and atmospheric tides. Another window around $\omega = \sqrt{2}\,\Omega$ (period around 17 h) may be more favourable as it is outside

[1] It may be zero for some exceptional values of the frequency for which the map is either neutral or algebraically converging.

the tidal bands, but as a consequence the excitation mechanism of such modes is much less clear.

One may wonder how small perturbations like the flattening of the CMB or ICB, or a difference between the true value of η and 0.35 may influence the shape of the Lyapunov exponent curve. Long-period orbits are indeed sensitive to slight changes in the boundaries shape but they transform in most cases into other long-period orbits which still have a small (in modulus) Lyapunov exponent. On the other hand, short period orbits (which are responsible for the spikes in the diagram) are not much affected by such slight changes. The main reason is that their position in the spectrum is controlled by the value of the critical latitude ϑ; i.e., the latitude at which characteristics graze the inner boundary. This angle changes only slightly with the flattening ε according to the law

$$\vartheta = \vartheta_0 - \varepsilon \sin 2\vartheta_0$$

when $\varepsilon \ll 1$. Changing the aspect ratio η by 1% implies no perceptible difference in Fig. 3. Therefore, we may say that the Lyapunov curve as shown by Fig. 3 is structurally stable.

5. Conclusions

To briefly conclude this note, we may stress the fact that order of magnitude arguments, taking into account the special shape of inertial modes in a spherical shell, cannot rule out the possibility that the elliptical instability might play some role in the geodynamo. More detailed calculations are needed to give a reliable answer to this interesting question. The same conclusion also applies for the observations of core modes: here too more detailed modelling including the forcing mechanisms, is needed to guide the search for such modes.

Acknowledgements

I would like to thank Lorenzo Valdettaro a lot for letting me use his programs to compute ray-trajectories and associated Lyapunov exponents. I need also thank Keith Aldridge and Gary Henderson for their careful reading of the manuscript. The high resolution computation of the inertial modes in Fig. 2 was done with the Cray C98 of the Institut du Développement et des Ressources en Informatique Scientifique (IDRIS), which is gratefully acknowledged. Support from the ‘‘Dynamique et Bilan de la Terre’’ INSU-CNRS program is also gratefully acknowledged.

References

Aldridge, K., Lumb, I., Henderson, G., 1989. A Poincaré model for the Earth’s fluid core. Geophys. Astrophys. Fluid Dyn. 48, 5–23.

Aldridge, K., Seyed-Mahmoud, B., Henderson, G., van Wijngaarden, W., 1997. Elliptical instability of the Earth’s fluid core. Phys. Earth Plan. Int. 103, 365–374.

Dintrans, B., Rieutord, M., Valdettaro, L., 1998. Gravito-inertial waves in a rotating stratified sphere or spherical shell. J. Fluid Mech., in press.

Fotheringham, P., Hollerbach, R., 1998. Inertial oscillations in a spherical shell. Geophys. Astrophys. Fluid Dyn. 89, 23–43.

Greenspan, H.P., 1969. The Theory of Rotating Fluids. Cambridge Univ. Press.

Hollerbach, R., Kerswell, R., 1995. Oscillatory internal shear layers in rotating and precessing flows. J. Fluid Mech. 298, 327–339.

Kerswell, R., 1993. The instability of precessing flow. Geophys. Astrophys. Fluid Dyn. 72, 107–144.

Kerswell, R., 1994. Tidal excitation of hydromagnetic waves and their damping in the Earth. J. Fluid Mech. 274, 219–241.

Kerswell, R., 1995. On the internal shear layers spawned by the critical regions in oscillatory Ekman boundary layers. J. Fluid Mech. 298, 311–325.

Kerswell, R., Malkus, W., 1998. Tidal instability as the source for Io’s magnetic signature. Geophys. Res. Lett. 25, 603–606.

Lister, J., Buffett, B., 1995. How much of the Earth’s core would be stably stratified if Nu < 1?. Phys. Earth Planet. Inter. 91, 17–30.

Lumb, L., Aldridge, K.D., Henderson, G., 1993. A generalized ‘‘core resonance’’ phenomenon: Inferences from a Poincaré core model. In: Le Mouël, J.L., Smylie, D., Herring, T. (Eds.), Dynamics of Earth’s Deep Interior and Earth rotation. IUGG, p. 51.

Maas, L., Lam, F.-P., 1995. Geometric focusing of internal waves. J. Fluid Mech. 300, 1–41.

Maas, L., Benielli, D., Sommeria, J., Lam, F.-P., 1997. Observation of an internal wave attractor in a confined, stably stratified fluid. Nature 388, 557–561.

Moore, D., Saffman, P., 1969. The structure of free vertical shear layers in a rotating fluid and the motion produced by a slowly rising body. Phil. Trans. R. Soc. London 264, 597–634.

Rieutord, M., 1995. Inertial modes in the liquid core of the Earth. Phys. Earth Planet. Inter. 91, 41–46.

Rieutord, M., Valdettaro, L., 1997. Inertial waves in a rotating spherical shell. J. Fluid Mech. 341, 77–99.
Rieutord, M., Georgeot, B., Valdettaro, L., 1999. Inertial waves in a rotating spherical shell. Part II. To be submitted to J. Fluid Mech.
Stewartson, K., 1972. On trapped oscillations in a slightly viscous rotating fluid. J. Fluid Mech. 54, 749–761.
Stewartson, K., Rickard, J., 1969. Pathological oscillations of a rotating fluid. J. Fluid Mech. 35, 759–773.
Tilgner, A., 1999. Non-axisymmetric shear layers in precessing fluid ellipsoidal shells. Geophys. J. Int., to appear.
Walton, I., 1975. Viscous shear layers in an oscillating rotating fluid. Proc. R. Soc. London 344, 101–110.