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On the internal dynamics of turbulent plumes in the context of stellar convection

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Abstract. We investigate in some details the modeling of turbulent plumes which have been proposed by Rieutord and Zahn to represent the downflows in a stellar convective zone. We show in particular the limits of Taylor's hypothesis about turbulent entrainment and emphasize its connection with the flow's self-similarity. The role of the dissipation is shown to be important in the choice of the final asymptotic regime. This is illustrated by a paradox which is solved when dissipation is correctly taken into account. It is concluded that in stellar conditions, the lack of self-similarity implies the replacement of Taylor's hypothesis by a proper closure of the mean-field equations in order to obtain a reliable prediction on the large-scale dynamics.

Key words: convection – hydrodynamics – turbulence – Sun: interior, rotation – stars: interiors

1. Introduction: astrophysical motivations and theoretical background

In a recent paper (hereafter referred to as RZ95) Rieutord and Zahn (1995) proposed that cold diving turbulent plumes may play an important role in the dynamics and energy transport of stellar convection zones. Their proposition has been motivated by the results of direct numerical simulations of turbulent compressible convection showing the important role played by strong downwards flows which exhibit laminar plume-like behaviour (Cattaneo et al. 1991; Stein and Nordlund 1989; Nordlund et al. 1994).

In RZ95, the dynamics of these turbulent plumes has been modeled using Taylor hypothesis for turbulent entrainment. We recall that this hypothesis assumes that the entrainment of fluid outside the plume is proportionnal to the mean vertical velocity on the axis of the plume. However, it was mentioned in that paper the fact that this hypothesis could be justified only for self-similar flows. It is this last point and some others on the internal dynamics of a turbulent plume that we wish to clarify in the present paper.

In Sect. 2 we derive, two equivalent forms of the local averaged energy equation: the first form was used in RZ95 and utilizes the flux of enthalpy; the second form is the one classically used in the literature on turbulent plumes (see Turner 1986) and rather uses the flux of buoyancy ($\delta\rho\,\mathbf{v}$). As it is always the case in deriving the horizontally averaged equations of a turbulent plume, viscous and thermal diffusion terms are discarded. The two forms of the energy equations are then perfectly equivalent. Quite suprisingly, these two equations, horizontally averaged and used with Taylor hypothesis, do not give the same self-similar solutions for diving plumes in an isentropic atmosphere.

To understand this paradox, we turn back (Sect. 3) to the boundary layer equations of free shear flows. We look for self-similar solutions and show in passing that self-similarity implies Taylor hypothesis. We then solve the paradox by analysing the role of viscous dissipation: it turns out that in an isentropic atmosphere, if the plume develops on a scale small compared to that of the atmosphere, dissipation may be neglected and the two forms of the energy equations yield the same solution. However, if the plume develops on the whole scale height of the atmosphere, viscous dissipation becomes important and another régime takes place. This régime is fortunately correctly captured by the enthalpy form of the energy equation and therefore results of RZ95 are confirmed. However, it is now shown that turbulent dissipation plays a major part in the internal dynamics of a cold turbulent plume and more generally in stellar turbulent convection.

In Sect. 4, we discuss a little more Taylor hypothesis and in particular we stress the point that when it is applied without self-similarity, it is no more than a crude closure of turbulence equations and that only orders of magnitude can be expected.

2. The energy flow

We first reconsider the derivation of the plume's equations in the adiabatic atmosphere and more particularly the energy equation ((7) in RZ95).

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2.1. Local mean flow equations

Let us recall the three basic conservation equations

– mass

$$\partial_t \rho + \nabla\cdot(\rho \mathbf{v}) = 0 \tag{1}$$

– momentum

$$\rho(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) = \rho\mathbf{g} - \nabla P + \nabla\cdot[\sigma] \tag{2}$$

– energy

$$\rho c_v(\partial_t T + \mathbf{v}\cdot\nabla T) = -P\nabla\cdot\mathbf{v} + \nabla\cdot(\chi\nabla T) + [\sigma]:[\nabla\mathbf{v}] \tag{3}$$

where χ is the thermal (microscopic) diffusivity, $[\sigma]$ and $[\nabla\mathbf{v}]$ respectively the viscous stress and the velocity gradient tensors. The symbol : denotes double contraction between two tensors. Now we must recall that

– the plume is axisymmetric
– we are dealing with a perfect gas and the medium surrounding the plume is adiabatically stratified, the hydrostatic equilibrium being given by

$$\partial_z P_0 = \rho_0 g \qquad \text{or} \qquad \partial_z T_0 = g/c_p \tag{4}$$

The z-axis is oriented towards the center of the star.
– the flow being turbulent, we use the Reynolds decomposition of a physical quantity X into respectively a time-averaged and a fluctuating part $X = \overline{X} + X'$
– the density contrast between the plume and the medium is small and is mainly due to temperature fluctuations, thus pressure gradients in the plume are neglected and

$$\frac{\delta\rho}{\rho_0} = -\frac{\delta T}{T_0} \tag{5}$$

where the subscript 0 denotes the unperturbed medium quantities.

We now express the energy equation (3) in terms of the buoyancy $\delta\rho$. We will show in the next subsection that the obtained expression yields a paradox if dissipation is neglected. Let us first develop Eq. (3) and neglecting second order terms in $\delta\rho$ and δT we get

$$\partial_t\delta T + \mathbf{v}\cdot\nabla T_0 + \frac{\delta\rho}{\rho_0}\mathbf{v}\cdot\nabla T_0 + \mathbf{v}\cdot\nabla\delta T =$$
$$-(\gamma-1)T_0\nabla\cdot\mathbf{v} + \frac{1}{\rho_0 c_v}\left(\nabla\cdot(\chi\nabla T) + [\sigma]:[\nabla\mathbf{v}]\right)$$

where γ is the adiabatic exponent. Developing the right hand side by using Eq. (1) and remembering that because of adiabaticity

$$\mathbf{v}\cdot\nabla T_0 = (\gamma-1)\frac{T_0}{\rho_0}\mathbf{v}\cdot\nabla\rho_0 \tag{6}$$

we get

$$\partial_t\delta T + \frac{\delta\rho}{\rho_0}\mathbf{v}\cdot\nabla T_0 + \mathbf{v}\cdot\nabla\delta T =$$
$$(\gamma-1)\frac{T_0}{\rho_0}\left(\nabla\cdot(\delta\rho\mathbf{v}) + \partial_t\delta\rho\right)$$
$$+\frac{1}{\rho_0 c_v}\left(\nabla\cdot(\chi\nabla T) + [\sigma]:[\nabla\mathbf{v}]\right)$$

Using expression (5) the former equation becomes for the instantaneous buoyancy transfer

$$-T_0\partial_t\delta\rho - T_0\mathbf{v}\cdot\nabla\delta\rho + \frac{\delta\rho}{\rho_0}T_0\mathbf{v}\cdot\nabla\rho_0 =$$

$$(\gamma-1)T_0[\nabla\cdot(\delta\rho\mathbf{v}) + \partial_t\delta\rho] + \frac{1}{c_v}\left(\nabla\cdot(\chi\nabla T) + [\sigma]:[\nabla\mathbf{v}]\right)$$

The left hand side can be transformed into

$$-T_0[\partial_t\delta\rho + \nabla\cdot(\delta\rho\mathbf{v})] + \frac{\delta\rho}{\rho_0}T_0\nabla\cdot(\rho_0\mathbf{v}) \tag{7}$$

but according to Eq. (1) the last term is of second order in $\delta\rho$ so neglecting this term and time-averaging we finally get

$$\overline{\nabla\cdot(\delta\rho\mathbf{v})} = -\frac{1}{c_p T_0}\overline{\left(\nabla\cdot(\chi\nabla\delta T) + [\sigma]:[\nabla\mathbf{v}]\right)} \tag{8}$$

which is known as the buoyancy equation. We took into account the fact that in the static atmosphere

$$\nabla\cdot(\chi\nabla T_0) = 0 \tag{9}$$

Because of the high Péclet and Reynolds numbers of the flow, we now neglect the microscopic heat diffusion and the viscous dissipation due to the mean flow. However, we keep the dissipation due to turbulence, that is

$$\overline{D} = \overline{[\sigma']:[\nabla\mathbf{v}']} \tag{10}$$

The buoyancy equation now reads

$$\overline{\nabla\cdot(\delta\rho\mathbf{v})} = -\frac{\overline{D}}{c_p T_0} \tag{11}$$

Now let's show that the buoyancy equation is equivalent to the enthalpy conservation equation given in RZ95.

Starting from (11) and using (5) again we have

$$\overline{\nabla\cdot(\rho\delta T\mathbf{v}) - \frac{\rho_0}{T_0}\delta T\mathbf{v}\cdot\nabla T_0} = \frac{\overline{D}}{c_p} \tag{12}$$

For the kinetic energy equation we have, multiplying (2) by $\mathbf{v}$ and averaging

$$\overline{\nabla\cdot(\frac{v^2}{2}\rho\mathbf{v})} = \overline{\delta\rho\mathbf{g}\cdot\mathbf{v}} + \overline{\nabla\cdot(\mathbf{v}'\cdot[\sigma'])} - \overline{D} \tag{13}$$

The second term on the right hand side will be neglected, owing to the high Reynolds numbers of the fluctuations, because it expresses energy transfer due to viscous forces, which is unimportant (but see Monin & Yaglom 1975 for details). However, the dissipation term is kept. Multiplying (12) by c_p and adding it to (13) we get, with the use of (4),

$$\overline{\nabla\cdot[(\delta h + \frac{v^2}{2})\rho\mathbf{v}]} = 0 \tag{14}$$

which is another form of the energy equation, precisely that used in RZ95 (see their Eq. (6)).

The buoyancy equation (11) can also be understood as an entropy equation expressing the fact that the entropy default induced by the presence of the cold plume in the atmosphere is reduced by dissipation. Actually, using (5), one can put it under the form

$$\overline{\nabla\cdot(\rho_0\delta s\mathbf{v})} = \frac{\overline{D}}{T_0} \qquad (15)$$

2.2. The energy paradox

We have now at our disposal three different forms of the energy equation; these are (11), (14) and (15). In RZ95 (14) was used and using Taylor's hypothesis with gaussian profiles for the velocity and the excess of density $\delta\rho$ (assumed small compared to ρ), the following equations were derived:

$$\begin{cases} \frac{d}{d\zeta}\left[\zeta^q\beta^2 u\right] = 2\alpha\zeta^q\beta|u| \\ \frac{d}{d\zeta}\left[\zeta^q\beta^2 u^2\right] = \Gamma\beta^2\zeta^q(\eta-1) \\ \frac{1}{2}(\eta-1)\zeta^{q+1}\beta^2 u - \frac{1}{3\Gamma}\zeta^q\beta^2 u^3 = F \end{cases} \qquad (16)$$

where we recall that ζ is the dimensionless depth, β the dimensionless radius of the plume, u the vertical velocity, α the entrainment constant and η the ratio of the density inside the plume to the density of the background. Γ and F are constants (but see RZ95 for all scalings). The system (16) admits self-similar solutions of the form

$$b = b_0\zeta, \quad u = u_0\zeta^{-\frac{q+2}{3}}, \quad \eta - 1 = R_0\zeta^{-\frac{2q+7}{3}} \qquad (17)$$

Let us now use (11) without taking into account the dissipation term. After integration over r and use of dimensionless quantities, it reads

$$\beta^2(\eta-1)u = C \qquad (18)$$

where C is another constant. Replacing the third equation of (16) by (18), we can also look for self-similar solutions which are

$$\beta = \beta_0\zeta, \quad u = u_0\zeta^{-\frac{q+1}{3}}, \quad \eta - 1 = R_0\zeta^{\frac{q-5}{3}} \qquad (19)$$

These solutions are clearly different from the preceding ones: the velocity now decreases as $\zeta^{-5/6}$ instead of $\zeta^{-7/6}$ when q=3/2 as for the monatomic gas.

Therefore, even if the local equations (11) and (14) are strictly equivalent, the use of Taylor's hypothesis leads to different solutions !

The origin of this paradox is of course to be found in the rough treatment of the original equations (11) and (14). The enthalpy equation (14) is true no matter whether dissipation is neglected or not. However, self-similar asymptotic solutions might not be the same with and without significant dissipation. Actually, we shall see in next section that this is indeed the case. The question arises then as to which asymptotic regime should be considered as relevant under given conditions. In other terms, under which conditions does dissipation become relevant ? We shall deal with this question in the next section by resorting to the boundary layer theory. However, it should be emphasized by now that the use of the entrainment equations without control cannot account for the existence of different asymptotic regimes. In order to shed some more light on Taylor's entrainment hypothesis, we now focus on the local equations and their resolution in the framework of the boundary layer theory and self-similarity. For this purpose, we use the buoyancy equation without dissipation and discuss later the use of the enthalpy equation.

3. Entrainment and self-similarity

3.1. Boundary layer equations

Turbulent plumes as well as turbulent jets belong to the wide class of free shear flows. Because of their low spreading plumes and jets are usually treated using the boundary layer theory as originally proposed by Prandtl. Hence it is assumed that all the quantities of the system (1), (2), (11) can be evaluated in terms of a small parameter $\epsilon = b/L$, where b is the lateral length scale of the plume (ie its width) and L the longitudinal length scale of the flow in its main direction. Prandtl's idea relies on the fact that, due to shear, gradients are much stronger in the lateral than in the main direction. Besides, turbulent quantities such as the Reynolds stress are linked to the influence of shear. Thus the order of magnitudes of the terms appearing in (1), (2) and (11) are

$$\overline{u} = O(1), \qquad \overline{v} = O(\epsilon), \qquad \overline{u'v'} = O(\epsilon),$$

$$\overline{\delta\rho} = \mathcal{O}(1), \qquad \overline{\delta\rho'v'} = \mathcal{O}(\epsilon), \qquad \overline{\delta\rho'u'} = \mathcal{O}(\epsilon),$$

$$\partial_z = O(1), \qquad \partial_r = O(1/\epsilon)$$

where u is the vertical and v the horizontal component of velocity, z the vertical coordinate, r the radius, ∂_z and ∂_r the derivatives with respect to the corresponding coordinates.

Retaining only the O(1) terms, system (1), (2), (11) simplifies into

$$\frac{\rho_0}{r}\partial_r(r\overline{v}) + \partial_z(\rho_0\overline{u}) = 0 \qquad (20)$$

$$\frac{1}{r}\partial_r(r\rho_0\overline{uv}) + \partial_z(\rho_0\overline{u}^2) = \overline{\delta\rho}g \qquad (21)$$

$$\frac{1}{r}\partial_r(r\overline{\delta\rho v}) + \partial_z(\overline{\delta\rho u}) = 0 \qquad (22)$$

which we may solve with additional assumptions. It should be quoted that the turbulent fluxes of momentum $\overline{u'^2}$ and $\overline{v'^2}$ are also neglected compared to those due to the mean flow in the main direction. This is an additional empirical assumption rather than a boundary layer evaluation, although it is sometimes presented as such (see for example Schlichting 1961). It means that the kinetic energy of the mean flow is much larger than the turbulent kinetic energy due to one component of the turbulent velocity field. It turns out to be true within the precision of the boundary layer approximations but it should be kept in mind that the total turbulent kinetic energy, once summed over all the components of the fluctuating velocity field, is no longer negligible in all the plume flows available from laboratory experiments (see for example Shabbir and George 1994).

3.2. The self-similar solutions without dissipation

We now search the behaviour of the axial velocity and density contrast from the set (20), (21), (22). We assume that there is an asymptotic self-similar regime under which we can separate the variables in the following way

$$\overline{u} = U_m(z)f(\xi), \qquad \overline{\delta\rho} = \delta\rho_m(z)k(\xi)$$

$$\overline{u'v'} = U_m^2(z)L(\xi), \qquad \overline{\delta\rho' v'} = \delta\rho_m U_m N(\xi)$$

where $\xi = r/b$. However, we make no assumption about the shape of the radial profiles. RZ95 assumed a gaussian profile for the mean velocity and density contrast since, in laboratory experiments on plumes, gaussian curves usually fit experimental data. Here we shall not specify the profile functions f, k, L and N. This makes impossible an exact calculation of the searched quantities but still gives their variations with depth, because of the similarity hypothesis.

Now let's multiply equation (22) by r and integrate from zero to infinity with respect to this variable. We get

$$\delta\rho_m U_m b^2 = B \tag{23}$$

where B is a constant. This equation is the same as (18).

Now, doing the same with Eq. (20) but integrating only from zero to r we get

$$\overline{v} = -\frac{1}{\rho_0 b\xi}\partial_z[\rho_0 b^2 U_m A(\xi)] \tag{24}$$

with $A(\xi) = \int_0^\xi y f(y)dy$.

Using (23) and (24) in Eq. (21) we finally get

$$\begin{aligned} k(\xi) = &-\left(\frac{U_m d_z(\rho_0 b^2 U_m)}{b^2\delta\rho_m g}\right)\xi^{-1}d_\xi[A(\xi)f(\xi)] \\ &+\left(\frac{\rho_0 U_m^2 d_z b}{b\delta\rho_m g}\right)\xi^{-1}d_\xi[A'(\xi)\xi f(\xi)] \\ &+\left(\frac{d_z(\rho_0 U_m^2)}{\delta\rho_m g}\right)f^2(\xi) - \left(\frac{2\rho_0 U_m^2 d_z b}{\delta\rho_m b g}\right)f^2(\xi)f'(\xi)\xi \\ &+\left(\frac{\rho_0 U_m^2}{b\delta\rho_m g}\right)\xi^{-1}d_\xi[\xi L(\xi)] \end{aligned} \tag{25}$$

This equation implies that the coefficients on the right hand side are constants. It yields the following conditions:

$$d_z b = cst \tag{26}$$

$$\frac{\rho_0 U_m^2}{b\delta\rho_m g} = cst \tag{27}$$

$$\frac{d_z(\rho_0 U_m^2)}{\delta\rho_m g} = cst \tag{28}$$

$$\frac{d_z(\rho_0 U_m b^2)}{\rho_0 U_m b} = cst \tag{29}$$

The last condition is obviously Taylor's entrainment relation (compare to the first equation of system (16)). If now we separate the variables in the buoyancy equation (22) we obtain after straightforward simplifications:

$$\begin{aligned} &-\left(\frac{d_z(\rho_0 b^2 U_m)}{\rho_0 U_m b}\right)\xi^{-1}d_\xi[k(\xi)A(\xi)] \\ &+\xi^{-1}d_\xi[\xi N(\xi)] + d_z b\{\,\xi^{-1}d_\xi[k(\xi)A'(\xi)] \\ &\quad -2[f(\xi)k(\xi)] - \xi d_\xi[f(\xi)k(\xi)]\,\} = 0 \end{aligned} \tag{30}$$

which yields no additional condition.

The fact that condition (29) is exactly Taylor's assumption provides a direct proof that Taylor's relation is necessary in a self-similar regime.

In an isentropic atmosphere where

$$T_0 \propto z, \qquad \rho_0 \propto z^q, \qquad g = cst$$

we get from the preceding conditions, together with (23)

$$U_m \propto z^{-\frac{q+1}{3}}, \delta\rho_m \propto z^{\frac{q-5}{3}} \tag{31}$$

as previously obtained in (19). The preceding power-laws apply to the self-similar plume regime when dissipation can be neglected in the buoyancy equation.

We shall now control whether they are compatible with the enthalpy equation (14). Let us put the relevant quantities in this latter equation under the self-similar form

$$\overline{\delta T} = \delta T_m k(\xi), \quad \overline{\delta T' v'} = \delta T_m U_m N(\xi)$$

$$\overline{u'^2} = U_m^2 H(\xi), \quad \overline{v'^2} = U_m^2 Q(\xi)$$

where δT_m is immediately found from $\delta\rho_m$ using (5) and the other mean quantities have been previously given. We shall neglect turbulent correlations of order higher than two (see the discussion in Sect. 4). Reporting in Eq. (14) we get the additional self-similarity conditions

$$\frac{b\, d_z(\rho_0\delta T_m U_m)}{\rho_0\delta T_m U_m} = cst \tag{32}$$

$$\frac{b\, d_z(\rho_0 U_m^3)}{\rho_0 U_m^3} = cst \tag{33}$$

$$\frac{U_m^2}{\delta T_m} = cst \tag{34}$$

It may be checked that the power-laws (31) found previously obey these conditions. Thus, they are perfectly compatible with the enthalpy equation.

The self-similar solution (17) used in RZ95 verifies the similarity conditions (26), (27), (28), (29), (32), (33), (34), but (23) is replaced by the additional requirement

$$\rho_0 U_m^3 b^2 = cst \quad (35)$$

thus leading to the power-laws given by (17). As these latters are incompatible with (23) that is, with the buoyancy equation without dissipation, one must conclude that to get the solution of RZ95, dissipation must play an important part. The previous solution (31) and that in RZ95 might then correspond to different asymptotic regimes, the physical relevance of which we discuss now.

3.3. The two asymptotic regimes: the role of dissipation

The question to examine now is the importance of dissipation, and the conditions under which each asymptotic regime is relevant. Eqs. (11) and (14) will not prove to be sufficient for a full discussion. We need complementary forms of the energy equation. A kinetic energy budget on the mean flow can only be obtained by averaging Eq. (2) and multiplying by $\overline{\mathbf{v}}$. After some rearrangements and neglecting pressure and mean flow viscous terms (see Sect. 2.1), we get

$$\nabla\cdot[\frac{\rho_0}{2}\overline{\mathbf{v}}^2\overline{\mathbf{v}} + \rho_0\overline{\mathbf{v}}\cdot\overline{[\mathbf{v}'\otimes\mathbf{v}']}] = \rho_0\overline{[\mathbf{v}'\otimes\mathbf{v}']}: [\nabla\overline{\mathbf{v}}] + \overline{\delta\rho}\mathbf{g}.\overline{\mathbf{v}} \quad (36)$$

It can easily be checked that the second order correlations in this equation contain terms which are $O(1)$ from boundary layer evaluations and thus cannot be neglected. The first term on the right hand side is known as the gradient production of turbulent kinetic energy: due to shear of the mean flow, kinetic energy is transferred to the turbulent fluctuations, and this amounts to an energy loss for the mean velocity field. The second term on the right hand side is obviously the production term from potential energy due to the density contrast.

Now, let's multiply (2) by $\mathbf{v}$ and get the total kinetic energy equation, as we did previously to get the enthalpy equation. We then subtract expression (36) and find the equation for the turbulent kinetic energy:

$$\nabla\cdot(\frac{\rho_0}{2}\overline{\mathbf{v}'^2\overline{\mathbf{v}}}) = -\overline{D} + \overline{\delta\rho'\mathbf{v}'}\cdot\mathbf{g} - \rho_0\overline{[\mathbf{v}'\otimes\mathbf{v}']} : [\nabla\overline{\mathbf{v}}] \quad (37)$$

where pressure and additional viscous terms have been dropped as before, as well as the third order correlations.

From this latter equation, it appears that the dissipation term is at most of the same order as the inertial terms (turbulent kinetic energy transfer and gradient production). With typical orders of magnitude, this means that $\overline{D} \lesssim \rho_0 V^3/z$.

Now, let's examine more carefully Eq. (11) in order to compare its left hand side to the dissipation term. Considering typical orders of magnitude we have :

$$c_p T_0 \nabla\cdot(\overline{\delta\rho\mathbf{v}}) \sim c_p T_0 \frac{\delta\rho V}{z} \sim \frac{\rho_0 V^3}{z}\frac{c_p T_0}{gz} \quad (38)$$

where it has been taken into account that $g\delta\rho V \sim \rho V^3/z$, that is, in a plume the amount of potential energy flux is of the same order, as a source term, as the kinetic energy flux. This latter condition differentiates the jet and the pure plume regime in a buoyant jet (see Appendix).

Let us consider the case $c_p T_0 \gg gz$ which means that the height z of the plume is much smaller than the atmospheric scale height. If we remember that dissipation $\overline{D}$ is at most of order $\rho_0 V^3/z$, then from (38), we immediately see that

$$c_p T_0 \nabla\cdot(\overline{\delta\rho\mathbf{v}}) \gg \overline{D}$$

so that to first order

$$\nabla\cdot(\overline{\delta\rho\mathbf{v}}) = 0 \quad (39)$$

which means that the flux of buoyancy is conserved and leads to the power-laws found using (23) together with the similarity conditions. In addition, if the height of the plume is small compared to the atmospheric scale height, the background density and temperature are nearly constant, ie we must consider that c_p tends to infinity, which brings $q = 0$ in the power-laws. We then recover the well-known Boussinesq regime (see for example Turner 1986).

The regime described by (39) may be interpreted as follows: the plume carries an enthalpy flux which is converted into kinetic energy fluxes via the work of the buoyancy force. The latter compensates dissipation and provides kinetic energy to the mean flow. As the enthalpy term is dominant, it is barely altered. Note that dissipation being of the same order as the gradient production term (see Eq. (37)) it evacuates a part of the turbulent kinetic energy created by the mean shear (see (36)). Thus the gradient production term acts more or less as a dissipation for the mean flow; it is the well-known "eddy viscosity" effect, although the analogy with viscous friction must be taken carefully.

Let us now turn to the case where the plume height is of the order of the atmospheric scale height. In that case, we get a constraint on the density contrast:

$$\delta\rho V g \sim \frac{\rho_0 V^3}{z} \Longleftrightarrow \frac{\delta\rho}{\rho_0} \sim \frac{V^2}{gz} \sim M^2\frac{z_0}{z}$$

where M is the Mach number and $z_0 = c_p T_{00}/g$ is the atmospheric scale height, T_{00} being the temperature at the bottom of the convective zone. As we assume $z/z_0 \sim 1$ we find that the density contrast is of order the square of the Mach number.

In this regime,

$$\frac{\rho_0\delta h V}{z} \sim \frac{c_p T_0\delta\rho V}{z} \sim g\delta\rho V \sim \rho_0\frac{V^3}{z}$$

which means that the enthalpy flux is now of the same order as the kinetic energy flux.

This new regime may be interpreted as follows: since the kinetic energy flux is bounded by the enthalpy flux, when these two fluxes are of the same order of magnitude, a steady state is reached when dissipation is just compensated by the work of buoyancy. As a consequence, the enthalpy flux is frozen, and the condition $\rho_0 U_m^3 b^2 = cst$ of RZ95 holds, bringing the power-laws given in the latter paper and recalled in Sect. 2.

The role of the gradient production term as an "eddy viscosity" effect in (36) remains the same.

Of the two asymptotic regimes that we have just described, the first one is never reached by cold plumes coming from the top of the convection zone since there $c_p T \sim gz$. Only the regime described in RZ95 is relevant. However, if the plume originates lower, it can undergo the two regimes, the second one corresponding to the final equilibrium state, which controls the energy transfer. The important point is the role of dissipation in the establishment of this equilibrium: as the energy available from buoyancy has to compensate dissipation, it does not provide any longer energy to the mean flow, which therefore carries frozen mean fluxes. Thus, dissipation is a very important ingredient of the plume dynamics, although it does not appear in the global enthalpy budget of Eq. (14).

3.4. The entrainment hypothesis

The entrainment hypothesis was introduced by Taylor in the forties and applied to the study of turbulent plumes and thermals. It postulates that the radial inflow of matter in the plume is proportional to the mean vertical velocity on the plume axis. Presented this way, it appears as a crude dimensional estimation, according to the fact that the vertical mean velocity on the axis is a natural velocity scale for the problem.

As we have shown above, Taylor's hypothesis is a natural consequence of self-similarity (see also Batchelor 1954). Morton et al. (1956) extended the use of the entrainment assumption to conditions where the flow obviously lacked self-similarity. They obtained good results in the estimate of a plume terminal height in the atmosphere. Much later, Turner (1986) argued that the entrainment assumption had been applied successfully in the study of plumes rising from volcanos on a height of 40 kms by Wilson et al. (1978). He suggested that perhaps, Taylor's entrainment hypothesis could have more physical meaning than initially thought and reflect some inner equilibrium between turbulence responsible for accretion of surrounding fluid and mean flow. This deserves some comments.

First of all, the experiment of Morton et al. (1956) was carried on a height of 3 kms, which is insufficient to make density variations clearly destroy the self-similar regime. On the contrary, these effects should be present in the study of Wilson et al. (1978). However, the formula borrowed from Morton et al. by these authors for the plume height can be found on purely dimensional grounds, as noticed by Turner himself. The striking fact is that the numerical coefficient in the formula, computed with Taylor's hypothesis, agrees with observations of the terminal plume height in several eruptions.

This success of the entrainment relation may be explained if we remember that in turbulent boundary layers the ratio between lateral and longitudinal velocities is always around 0.1. Therefore, we rather understand these good results as correct estimations of the main orders of magnitude.

These are relevant to predict geometrical quantities, such as the plume terminal height, but insufficient to simulate properly the actual evolution of the dynamical variables such as the velocity field or the density contrast. If one is mainly concerned with the latters, Taylor's hypothesis should not be used out of its range of validity, that is, out of the asymptotic self-similar regimes, where it appears as a consequence of self-similarity, as previously shown.

Note that if one compares the two possible asymptotic regimes, there is a priori no reason that α should remain the same, after the transition from a regime to another, as well as it actually changes from the jet to the plume regime in a buoyant jet (see Appendix).

In the stellar context, many processes are alike to break the self-similarity of the flow, making the use of Taylor's assumption rather uncertain. We now discuss this point in further details.

4. Discussion

The question is: does an asymptotic self-similar plume regime always exist? This amounts to asking if conditions (26)-(34) are always fulfilled. A simple way to see that this is not always the case is to consider gravity variations in a stellar convective envelope. If the self-gravity of the envelope is neglected then g varies as

$$g = \frac{GM_s}{R_s^2(1+\frac{z_0-z}{R_s})^2}$$

where R_s and M_s stand respectively for the radius and mass of the star and z_0 is the depth of the convective zone. Consequently, for the density we get

$$\rho_0 \propto [\frac{z/z_0}{(1+\frac{z_0-z}{R_s})}]^q$$

One can then verify that the similarity conditions (26)-(34) are never reached simultaneously.

However, variations of g are not the only reason for not having self-similar solutions; the presence of rising counter flows, background rotation and magnetic field makes the possibility of self-similar solutions rather implausible. To make progress, it is then necessary to get solutions of the system (1), (2) and (3) without the assumption of self-similarity.

This implies to investigate the turbulent mechanisms involved in the plume dynamics: basically, this is a problem of coupled heat and momentum transfer in a free turbulent boundary layer, travelling in a specific medium. Additionally, there are several phenomena that interfere with the plume dynamics, some of which we just enumerated above, and which are basic ingredients of stellar environment.

Another fundamental question, dealing with convection, is the treatment of the energy transfer. Our rough treatment, neglecting the effect of turbulent pressure fluctuations and third order correlations in (37) is sufficient to point out the existence of two asymptotic regimes. However, pressure fluctuations are known to redistribute the energy towards more isotropy in the small scales of the energy cascade, and there are some cases where third order correlations do play a significant dynamical role (Monin & Yaglom, 1975). It is difficult to evaluate a priori the consequences of turbulent kinetic energy transfer processes on the mean dynamics but we cannot exclude its influence on the evolution of the plume after a sufficiently long time.

5. Conclusion

We have shown in this paper that viscous dissipation plays an important part in the final asymptotic régime of a turbulent plume in an isentropic atmosphere. To our knowledge, this is the first time that a dissipative process is shown to control the dynamics of a turbulent free shear flow.

We have also shown that this effect is totally non-Boussinesq. It suggests that once this régime is reached, all the work done by buoyancy is dissipated and that the kinetic energy flux saturates at a level of the order of the enthalpy flux. Because of turbulent dissipation, the kinetic energy flux remains smaller than the enthalpy flux and therefore the turbulent plumes carry some heat flux unlike their laminar sisters observed in numerical simulations which convert efficiently potential energy into kinetic energy thanks to their relatively high Reynolds number (a few tens say, but see Cattaneo et al. 1991).

We have also emphasized that Taylor hypothesis rigorously applies only to self-similar flows. The use of this model without this condition may still give correct orders of magnitude, as this is the case, for instance, when the terminal height of a plume is computed, provided one inputs the right physics in the equations.

However, a more detailed description of the turbulence is now needed to take into account the effects of magnetic fields or rotation. This requires a solution of the closure problem for the transport of momentum and temperature. This point will be presented in a subsequent paper.

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Appendix: buoyant jets

A well-known example of transition between two self-similar asymptotic regimes is that of buoyant jets, when the buoyant flow receives an important initial flux of momentum. It then starts with a jet rather than with a plume behaviour, the influence of buoyancy forces being negligible compared to the flux of momentum. Buoyancy is dominant only when the velocity is sufficiently lowered by entrainment. Let us come back to equations (20) and (21) with the self-similarity assumption but neglecting the buoyancy term. Eq. (24) is still valid but now, multiplying (21) by r and integrating from zero to infinity with respect to this variable we get

$$\rho_0 U_m^2 b^2 = M = cst \qquad \text{(A1)}$$

Reporting into (21) we get

$$-U_m d_z(\rho_0 b^2 U_m)\frac{1}{\xi}d_\xi[A(\xi)f(\xi)]$$
$$+(d_z b\rho_0 U_m^2)\frac{1}{\xi}d_\xi[A'(\xi)f(\xi)] + b(z)d_z(\rho_0 U_m^2)f(\xi)^2$$
$$-(2\rho_0 U_m^2 d_z b)\xi f(\xi)f'(\xi) + \rho_0 U_m^2\frac{1}{\xi}d_\xi(\xi L(\xi)) = 0 \qquad \text{(A2)}$$

from which we have the conditions

$$d_z b = cst \qquad \text{(A3)}$$

$$\frac{d_z(\rho_0 b^2 U_m)}{\rho_0 b U_m} = cst \qquad \text{(A4)}$$

$$b\frac{d_z(\rho_0 U_m^2)}{\rho_0 U_m^2} = cst \qquad \text{(A5)}$$

Condition (A4) is again Taylor's assumption but the entrainment constant has no reason to be the same as in the pure plume regime. Actually, it is known to be slightly less in an isothermal jet (0.05) than in a plume (0.083). The difference is small but the power-law solutions are completely different. For an isothermal medium

$$U_m \propto z^{-1}$$

and from (22) and (23) still valid we get

$$\delta\rho_m \propto z^{-1}$$

without additional similarity conditions. In this case, the density contrast, or equivalently heat, is transported passively, while its transfer is coupled to the momentum transfer in the pure plume regime.

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