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Inertial modes in the liquid core of the Earth

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Abstract

Using the simple model of an incompressible fluid, we have computed the eigenfrequencies of the lowest-order inertial modes (azimuthal wavenumber $m = 0,1,2$) in a spherical shell with the same aspect ratio as the liquid core of the Earth. The computed eigenfunctions show that all inertial modes have strong oscillating shear layers. For the very low Ekman number appropriate to the core, these layers might be the origin of some small-scale turbulence through shear instabilities. We have also studied the effect of a thin stable layer lying just below the core–mantle boundary, with the remainder of the core being neutrally stratified, as suggested by recent work. For plausible Nusselt numbers (0.8–0.9), the frequencies of the large-scale modes are only slightly increased (at best by 10^{-4}).

1. Introduction

Understanding the oscillating properties of the liquid core is a cornerstone for building models of global oscillations of the Earth. Although acoustic modes have been studied for a long time in connection with the interpretation of seismic data, core modes, such as inertial modes, have only recently attracted attention, thanks to the development of superconducting gravimeters (Aldridge and Lumb, 1987; Aldridge et al., 1989; Rieutord, 1991).

The eventual detection of these modes would have very important consequences for our understanding of the dynamical properties of the core. Indeed, these modes would give access to two important parameters: the viscosity (or more generally the dissipation) and the stratification of the fluid core. However, as such a detection will by no means be an easy matter, one needs to know the frequencies which should first be looked for.

The aim of this paper is to present our first results of computations of inertial modes in spherical shells with the same aspect ratio as the Earth's core. In contrast to Zhang's work (see, e.g. Zhang, 1993), we do not consider the coupling of inertial modes with convection. Indeed, as Zhang has shown, inertial modes excited by convection have large azimuthal wavenumbers and very low frequencies, and these features make their detection unlikely. Therefore, the fluid is assumed neutral with respect to convective instability. In addition, and for the sake of simplicity, we do not take into account the variations of density owing to adiabatic compression and we thus assume incompressiblity. However, we shall also consider the case where a stably stratified layer is lying just below the core–mantle boundary (CMB) as proposed by Lister and Buffett

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(1995). To treat this latter case and keep incompressibility, we use the Boussinesq approximation.

The paper is organized as follows: in Section 2 we specify all the details of the model, and the numerical technicalities are briefly given in Section 3. Results on inertial modes in a spherical shell like the Earth's core are discussed in Section 4, and the role played by a stable layer lying below the CMB is investigated in Section 5.

2. The model

We assume that the liquid core is contained in a rotating spherical shell of aspect ratio $\eta = 1221.5\ \text{km}/3480\ \text{km} = 0.351 \simeq 0.35$. In this first approach, we do not include the density variations owing to adiabatic compression across the shell, and thus we shall use a constant density fluid when the core is meant to be adiabatic (Section 4) and the Boussinesq approximation when a subadiabatic layer is included just below the CMB (Section 5). For the sake of generality, we give the equations of the flow in the latter case, setting the Brunt–Väisälä frequency to zero when necessary.

We use R, the radius of the CMB, as the length scale, $1/2\Omega$ as the time scale (where Ω is the angular velocity of the Earth) and $4\Omega^2 R/\alpha g$ as the temperature scale (where α is the volume coefficient of thermal expansion and g is the gravity at the CMB).

As we are interested in the modal properties of the system, we discard nonlinear terms and write the non-dimensional equations of the perturbations in the following way:

$$\begin{aligned}
&\lambda \mathbf{u} + \mathbf{e}_z \times \mathbf{u} = -\nabla p + T\mathbf{r} + E\Delta\mathbf{u} \\
&\lambda T + N^2(r) u_r = \frac{\mathrm{E}}{\mathrm{Pr}}\Delta T \\
&\nabla \cdot \mathbf{u} = 0 \\
&\mathbf{u} = \mathbf{0} \quad \text{on } r = \eta \text{ and } r = 1
\end{aligned} \tag{1}$$

Here $\boldsymbol{u}$ is the velocity field of the perturbations measured in a frame rotating with the angular velocity of the Earth and T is the temperature perturbation. p is the reduced pressure perturbation. We assume all the perturbations to be proportional to $\exp\lambda t$. We introduced the Ekman number E, Prandtl number Pr and the local Brunt–Väisälä frequency N, which are defined by

$$\mathrm{E} = \frac{\nu}{2\Omega R^2}, \quad \mathrm{Pr} = \frac{\nu}{\kappa}, \quad \mathrm{N}^2(r) = \frac{\alpha g}{4\Omega^2}\beta(r) \tag{2}$$

where ν and κ are respectively the kinematic viscosity and thermal diffusivity and $\beta(r)$ is the temperature gradient at r. It should be noted that we have assumed gravity proportional to radial distance as implied by a constant density mass distribution (see Chandrasekhar, 1961), and that the spherical coordinate system (r,θ,ϕ) is used.

The values of E, Pr and N are barely known for the Earth's core. We shall assume E is very small compared with unity and use $\mathrm{E} = 10^{-5}$ in numerical solutions (however, see Lumb and Aldridge (1991) for a review); The Prandtl number is probably less than unity, as for liquid metals; however, uncertainties are such that we set this number to unity for the computations of Section 5.

3. Numerics

To solve this eigenvalue problem, we follow the method presented by Rieutord (1991). We first expand the velocity on vectorial spherical harmonics as

$$\mathbf{u} = \sum_{l=m}^{+\infty} u_m^l(r)\mathbf{R}_l^m + v_m^l(r)\mathbf{S}_l^m + w_m^l(r)\mathbf{T}_l^m$$

where

$$\mathbf{R}_l^m = Y_l^m \mathbf{e}_r, \quad \mathbf{S}_l^m = \nabla Y_l^m, \quad \mathbf{T}_l^m = \nabla \times \mathbf{R}_l^m$$

As may be checked easily, each solution of (1) may be characterized by its azimuthal wavenumber m and its symmetry with respect to the equator. Symmetric solutions will be denoted by m^+ and antisymmetric ones by m^-.

We then expand the radial functions on Chebyshev polynomials. The eigenvalue problem is then simply a complex generalized eigenvalue problem of the form

$$[A]\mathbf{X} = \lambda[B]\mathbf{X} \tag{3}$$

where $[A]$ and $[B]$ are matrices whose dimension is given by the resolution. If L_{max} is the highest degree of the spherical harmonics expansion and N_T the highest degree of the Chebyshev polynomials' expansion, then the dimension of $[A]$ and $[B]$ is of order $L_{max} N_T$.

4. Inertial modes in the liquid core

We first set the Brunt–Väisälä frequency to zero and compute the eigenfrequencies (Table 1) and associated eigenfunctions (Fig. 1) of pure inertial modes. Hence, in these computations we assumed the simplest model for the liquid core of the Earth: an incompressible viscous fluid contained in a spherical shell of aspect ratio $\eta = 0.35$. Such a model will give, however, the gross features of the spectrum of long-period oscillations. We indeed expect that the inclusion of density variations owing to adiabatic compression will shift the frequencies by a small amount (10% say) but will not change the shape of the spectrum dramatically, as far as large-scale modes are concerned.

We selected large-scale inertial modes from the resolution of (3) by choosing the eigenvalues which are the closest to the real axis and thus correspond to the least damped modes. It turned out that the order in which the eigenvalues appear is sensitive to parameters such as the aspect ratio η, the Ekman number or the boundary conditions. For instance, a mode such as (6,1,0) (using Greenspan notations (Greenspan, 1969)), which is the second least damped in the 0^+ symmetry when the sphere is full, is shifted to the twelfth position when a core with $\eta = 0.35$ is there.

To present as clearly as possible this complex situation, we have selected in Table 1 and Fig. 1 all the modes which systematically appeared in the least damped ones, regardless of the boundary conditions and the Ekman number. These modes might thus be considered as the best candidates for detection and identification.

Because of numerical constraints, we studied in detail the situation only for $E = 10^{-5}$. This is a rather large value for the Earth's core, and thus to have a better idea of the situation at very low values of E, we computed the modes using stress-free boundary conditions. In such a case, viscous dissipation occurs mainly in the internal shear layers. As it is weaker than in the Ekman boundary layers, the eigenfrequencies are closer to their asymptotic limit $E = 0$. The comparison of these frequencies with those computed with rigid boundary conditions gives an idea of the effect of Ekman layers: it is always less than 1% for $E = 10^{-5}$.

Finally, one may note that most of the modes that we selected have a period in the 15–17 h band. This is a rather favourable situation as far

Table 1
Eigenvalues of some large-scale inertial modes in a spherical shell with $\eta = 0.35$

λ_{rr}	λ_{ff}	Period (h)	Symmetry	Identification	
$-5.977 + 0.74383i$	$-1.621 + 0.74354i$	16.14	1^+	(3,1,1)	0.7550
$-6.095 + 0.75200i$	$-1.540 + 0.75131i$	15.96	0^-	(5,2,0)	0.76506
$-6.197 + 0.50079i$	$+0.50000i$	24.0	1^-	(2,1,1)	0.50000
$-6.744 - 0.69936i$	$-1.825 - 0.69901i$	17.17	1^-	(6,1,1)	-0.7021
$-7.190 + 0.66216i$	$-2.484 + 0.66109i$	18.12	0^+	(4,1,0)	0.65465
$-7.331 + 0.74731i$	$-1.385 + 0.74730i$	16.06	2^-	(5,3,2)	0.7482
$-7.463 + 0.81736i$	$-3.117 + 0.81736i$	14.68	0^+	(6,2,0)	0.83022
$-7.593 + 0.81308i$	$-2.312 + 0.81278i$	14.76	2^+	(6,4,2)	0.8217
$-8.224 + 0.33355i$	$-0.058 + 0.33333i$	36.0	2^-	(3,1,2)	0.33333
$-8.714 + 0.68559i$	$-4.460 + 0.68428i$	17.50	0^+	(8,2,0)	0.67719

λ_{rr} and λ_{ff} refer to the eigenvalues computed respectively with rigid or stress-free boundary conditions on both shells. i is such that $i^2 = -1$ and the real part has been multiplied by 10^3. The third column gives the corresponding period for the Earth. The corresponding mode for the full sphere using Greenspan's notations (Greenspan, 1969) and its frequency are given in the last two columns. All computations were done for $E = 10^{-5}$ and $N = 0$ using a resolution of $L_{max} = 32$ and $N_T = 26$.

as gravimetry is concerned. Indeed, because noise increases as a power of the period, this band has the minimum noise for inertial modes; in addition, the noise produced by tides is rather weak around 16 h.

The shape of the modes is represented through various quantities in Fig. 1. In each picture, lines for the characteristics and critical latitudes are given by a tick mark on the outer and inner shell, respectively. We recall that the critical latitude is

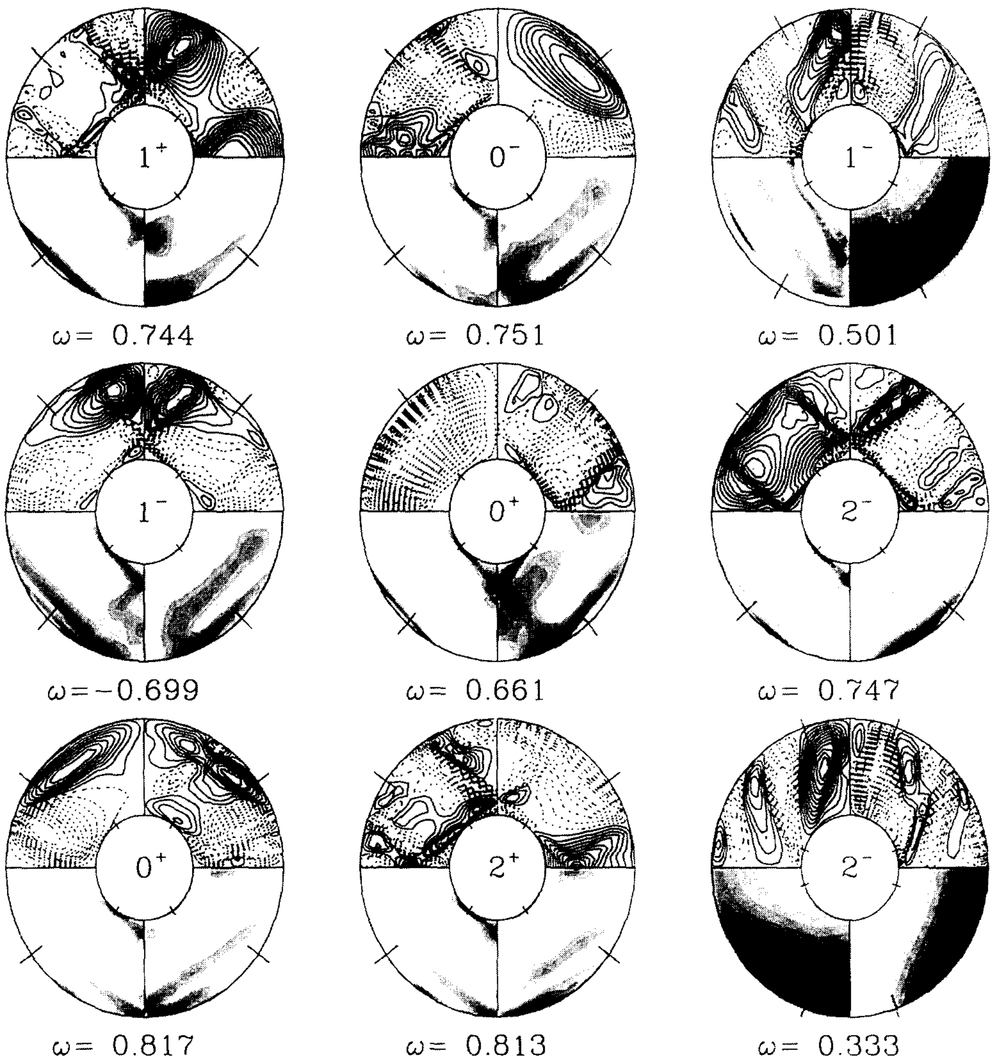

Fig. 1. A plot of nine large-scale inertial modes in a spherical shell like the Earth's core. In the upper part of each picture we have drawn isocontours of the radial velocity component (imaginary part on left, real part on right) for non-axisymmetric modes, and for axisymmetric modes meridional streamlines are plotted. In the lower part of each picture are drawn the longitude- and time-averaged kinetic energy (right) and viscous dissipation (left); dark areas represent high values. To enhance internal shear layers, stress-free boundary conditions have been used for all modes except for the spin-over mode ($\omega = 0.5$), for which rigid boundary conditions were used (for this plot Ekman layers are removed to make internal layers visible). Tick marks on the outer shell indicate the direction of lines of characteristics lines, and those on the inner shell are at critical latitudes. The symmetry of the mode is indicated at the centre of each plot.

$\theta_c = \arcsin(\omega)$ (ω is the imaginary part of λ) and that the direction of characteristics makes an angle θ_c with the rotation axis.

A major feature of the inertial modes which appears in Fig. 1 is the internal shear layers. These are particularly conspicuous on the plots of the mean dissipation or kinetic energy. There we may see that these layers are often between layers with high kinetic energy. This reveals that internal shear layers are in most cases the dominant feature of the interior flow. Another point, actually noticed by Hollerbach and Kerswell (1995) with the spin-over mode (the inertial mode excited by precession), is that shear layers often emerge at a critical latitude on the inner boundary and propagate around the characteristics cone. We show here that this is a general feature of inertial modes when an inner core is present.

A strong difference from the full-sphere case is the oscillatory nature of the velocity field lines for axisymmetric modes. Indeed, when the fluid completely fills the sphere, the flow pattern is, up to viscous damping, steady. When the inner core is present the internal viscous shear layers are shifted in phase (by $\pi/2$) with respect to the original flow pattern; this is visible in Fig. 1 for the first 0^+ mode ($\omega = 0.661$): the imaginary part of the meridional flow is very similar to the flow pattern without the core, whereas the real part shows rather clearly the presence of shear layers. The complete discussion of the properties of these modes will be presented elsewhere (Rieutord and Valdettaro, 1995, in preparation).

In the context of the core dynamics, the presence of thin shear layers may be the origin of small-scale turbulence. Indeed, using an Ekman number of 10^{-16}, and given the thickness of the layer is $O\{E^{1/3}\}$ (see Kerswell, 1995), the shear instability will grow on a time scale of order $T_s = E^{1/3}R/V$ (approximately 3 h) if we take $V = 1$ mms^{-1} as the amplitude. This is sufficiently short for the instability to grow during an oscillation, especially for long-period modes. With the same numbers, we computed the Reynolds number of such shear layers, and found a value around 10^5, which is sufficiently large to produce some turbulence. We may thus conjecture that this turbulence will damp the modes more efficiently than is predicted by linear theory.

5. The influence of a stable layer

In view of the uncertainties in the heat flux and the adiabatic gradient at the CMB, Lister and Buffet (1995) suggested that the heat flux at the CMB may well be lower than that which would be conducted up the adiabat, implying therefore the presence of a stable layer in the upper part of the liquid core. J. Lister (personal communication, 1994) proposed that the thickness of this layer is given by $d = 1000(1 - \mathrm{Nu})$ km and its stable stratification is characterized by the Brunt–Väisälä frequency at the CMB: $N_m = 0.0002(1 - \mathrm{Nu})$ s^{-1}. In these expressions Nu is a Nusselt number defined as the ratio of the actual heat flux to that which would be conducted up the adiabat. As the layer is subadiabatic this number is less than unity. The stability of this layer is also supposed to decrease linearly with depth and vanish at depth d where the adiabat takes over. We shall then use the following distribution for $N(r)$:

$$\begin{aligned} N &= 0 \qquad \eta \leqslant r \leqslant \eta_s \\ N &= N_m\left(\frac{r-\eta_s}{1-\eta_s}\right) \qquad \eta_s \leqslant r \leqslant 1 \end{aligned} \tag{4}$$

where $\eta_s = 0.713 + 0.287\mathrm{Nu}$ is the inner radius of the stable layer in units of the outer radius of the core.

Realistic values of the Nusselt number would be around 0.8 – 0.9 (J. Lister, personal communication, 1994). To have a general idea of the effect, we computed the frequencies for various Nusselt numbers between zero and unity. Results are summarized in Table 2. They show that for the realistic values mentioned above the frequencies increase only very slightly (less than 10^{-4}). In this case, the shape of the large-scale inertial modes is barely modified by this layer.

6. Conclusions

We have computed the largest-scale (and least damped) inertial modes in a spherical shell like the liquid core of the Earth. Our predictions show that most of these modes have periods around 16 h, which is rather favourable as far as

Table 2
Relative increase in frequencies of inertial modes when a stable layer is included just below the CMB

Nu	$\Delta\omega/\omega$
1	0.0
0.9	$\approx 10^{-6}$
0.8	$\approx 10^{-4}$
0.5	$\approx 10^{-2}$
0.2	$\approx 10^{-1}$
0.0	$> 10^{-1}$

gravimetric detection is concerned. These periods are outside the tidal bands and on the high-frequency side of the inertial modes' spectrum. The frequencies for the real core may, however, be slightly different from the one predicted, as our model uses an incompressible fluid. The inclusion of density variations along the adiabat will be taken into account in a forthcoming paper. However, as the core density variations are currently estimated to be around 10% (Smylie et al. 1984), we expect the frequency shift owing to these variations to be rather small.

We have also shown that other phenomena, such as viscous dissipation or the presence of a small stably stratified layer below the CMB, have a very small influence on the frequencies. Concerning the damping of the modes we have shown that, because of the very low value of molecular viscosity, one can expect the generation of some small-scale turbulence by internal shear layers of the modes in the finite-amplitude regime and therefore an enhancement of their decay rates.

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