Turbulent jets: Reichardt's inductive theory and intermittency corrections

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Abstract We investigate the so-called inductive closure relation given by Reichardt for mean turbulent momentum transport in jets (1941). It is shown to be in good agreement with measurements in the axisymmetric jet by Panchapakesan and Lumley(1993). Despite these results, it is conjectured that there is a lack of account for intermittency effects which is hidden by the experimental procedure of Panchapakesan and Lumley. Intermittency corrections are then proposed, based on the classical concept of intermittency factor and the corrected model is compared with the results of Bradbury (1965) for the plane jet. The agreement with experimental measurements appears to be very good. The underlying picture of the Reichardt 1941 model is discussed, mainly the fact that it expresses the self-injection of kinetic energy by the jet and the conversion of longitudinal into lateral momentum thus providing entrainment of the surrounding fluid.

1 Introduction

Momentum transport in a turbulent jet is determined by the physics of inhomogeneous turbulent shear flows, which is far from being understood, although it has motivated a great deal of experimental and numerical studies. In various contexts, among which geophysical and astrophysical applications, turbulent free shear flows occur as a part of larger scale dynamics. It then seems desirable to get a description of turbulent transfer processes in those flows by means of simple analytical relations, keeping heavy calculations for the larger scales under immediate interest. Being faced with a lack of deep physical understanding, analytical models are necessarily semi-empirical. Several models have been developed in the thirties and forties, among which the well-known mixing length and eddy viscosity representations, due to Prandtl. Although these approaches partly succeed in simulating reality, they are known to have shortcomings linked to the uncompleteness of the transport processes reproduction, or to a wrong underlying physical picture. Reichardt (1941) developed an empirical model based on a "momentum transfer length". It is known to give good results in simulating the mean velocity profiles in plane turbulent mixing layers and jets. It has also been proved to be valuable for calculations of these profiles in the round jet (Abramovicz 1963). We compare calculations based on Reichardt's model to experimental results due to Panchapakesan and Lumley (1993) for the round jet and to Bradbury (1965) for the plane jet. We also propose corrections to the model in order to take into account external intermittency effects.

2 The Model

We consider an incompressible fluid and an isothermal jet with surrounding medium at rest. The Reichardt model applies to the similarity zone of the jet and is used in the framework of Prandtl's boundary layer theory (Schlichting 1961), with the Reynolds decomposition applied to all quantities under interest. The small parameter is $\epsilon = b/L$, the ratio between the lateral and longitudinal length scales. One can then give an evaluation of the terms appearing in the dynamical equations, namely

$$\overline{u} = O(1), \overline{v} = O(\epsilon), \overline{u'v'} = O(\epsilon), \overline{u'^2} = O(\epsilon), \overline{v'^2} = O(\epsilon), \partial_z = O(1), \partial_y = O(1/\epsilon).$$

Keeping only the leading terms in the dynamical equations, one gets the simplified mass and momentum budgets:

$$\frac{1}{\chi}\partial_y(\chi\overline{v}) + \partial_z\overline{u} = 0 \tag{1}$$

$$\partial_z\overline{u}^2 = -\frac{1}{\chi}\partial_y(\chi\overline{uv}) \tag{2}$$

where $\chi = 1$ for a plane jet, $\chi = y$ for a round jet, u and v are the longitudinal and lateral component of the velocity respectively.

Reichardt's model introduces the closure relation

$$\overline{uv} = -\Lambda(z)\partial_y\overline{u}^2 \tag{3}$$

We assume self-similarity of the profiles $\overline{u} = U_m(z)g(\xi)$, where $\xi = y/b$. Let's first solve the problem for the round jet. The integration of (2) from $y = 0$ to infinity brings the integral conservation of momentum

$$U_m^2 b^2 = cst \tag{4}$$

Using (3) in (2) and using (4) we get the ordinary differential equation on $f = g^2$

$$\frac{-U_m^2}{b}\, d_z b\,[2f(\xi) + \xi f'(\xi)] = \frac{\Lambda}{b^2}\, U_m^2\, [f''(\xi) + \frac{1}{\xi} f''(\xi)] \tag{5}$$

whose boundary conditions are $f(0) = 1$ and $f(\infty) = 0$.

According to the self-similarity condition, the coefficients in the above equation depend only on ξ and thus Λ is proportional to $b\, d_z b$. As b is a scaling parameter, we can set a strict equality without loss of generality. We also see that b spreads linearly with z. Once simplified, the above equation becomes

$$2f + (\xi + \frac{1}{\xi})f' + f'' = 0 \tag{6}$$

of which the solution is the gaussian profile $\overline{u} = U_m exp(-\xi^2/4)$. From (1) we then get

$$\overline{v} = U_m\, d_z b\, [(\frac{2}{\xi} + \xi)\exp(-\xi^2/4) - \frac{2}{\xi}] \tag{7}$$

It is then straightforward, from (3) to get the Reynolds stress

$$\overline{u'v'} = \overline{uv} - \overline{u}\,\overline{v} = \frac{2}{\xi}\, U_m^2\, d_z b\, \exp(-\xi^2/4)[1 - \exp(-\xi^2/4)] \tag{8}$$

This profile obtained for the Reynolds stress is shown in figure 1 and compared to the experimental data of Panchapakesan and Lumley with the measured quantity $d_z b = 0.0576$. We find good agreement between measurements and the model. The calculated curve seems too high near the maximum but there is a lot of scatter in the experimental data. It should be quoted that the curve calculated by Panchapakesan and Lumley from the measured mean velocities is even higher. So, the results given by the model seem quite acceptable. This is a good test because Reichardt had no Reynolds stress data at his disposal and based his model on the observation of the mean velocity profile only.

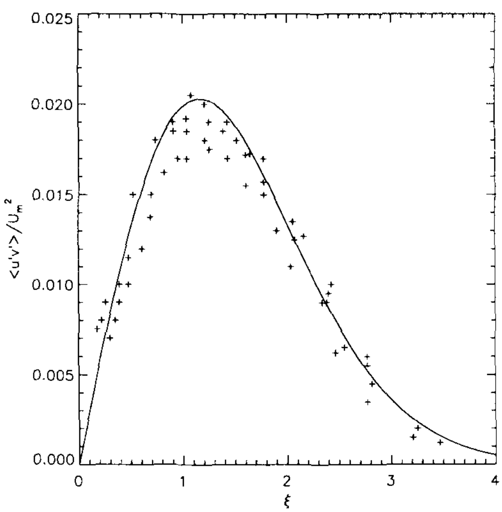

Fig. 1. The Reynolds stress, normalized to the square of the mean axial velocity. The experimental data of Panchapakesan and Lumley are compared to the calculated curve.

Now, it is well known that in the outer part of the jet, external intermittency effects lower the efficiency of the turbulent transport. The gaussian fits to the mean velocity profile become poor in that zone. Curiously, this does not appear in Panchapakesan and Lumley's measurements as shown by the quality of their gaussian fit to the data even in the external part of the jet. We believe that this is because they used a moving probe with a periodic motion of low frequency, and may have missed the appearance of laminar zones. In order to get intermittency corrections to Reichardt's model, we introduce the classical intermittency factor

$$\gamma(y,z) = \frac{1}{T}\int_0^T I(y,z,t)dt \tag{9}$$

where I(y,z,t) is the intermittency function. The intermittency factor at position ξ is also defined as the probability of the measurement location ξ to be within the turbulent domain The random position of the turbulent/non turbulent interface is not a single-valued function of z but for sufficiently long time-averaged values of the intermittency function, one may expect the laminar zones due to the folding of the interface to have a negligible life time compared to the total time spent out of the turbulent zone. Moreover, the random process describing the evolution of the interface location is expected to follow the central limit theorem if the dispersion is weak enough. Thus, we take the classical representation of a gaussian intermittency factor as a sufficiently good approximation, except near the axis, where the convergence to a gaussian process is not achieved for obvious reasons of symmetry. The intermittency factor thus reads

$$\gamma = \frac{1}{\sqrt{2\pi}\sigma}\int_\xi^\infty \Upsilon(\xi')\exp(\frac{(\xi'-\xi_0)^2}{2\sigma^2})d\xi' \tag{10}$$

where Υ is the Heaviside function. We then propose to rewrite Reichardt's model considering that it is valid only during the turbulent periods and neglecting the laminar contributions to momentum transfer. The latter point is justified by the experimental fact that laminar momentum fluxes are small within the precision of the boundary layer's approximations. We then have

$$\overline{uv} = -\gamma\Lambda\partial_y(\frac{\overline{u}^2}{\gamma}) \tag{11}$$

Resuming the calculations as before, with the only change $\chi = 1$ we get for the integral conservation of momentum: $U_m^2 b = cst$ and the solutions are the following ones:

$$\overline{u} = U_m\sqrt{f(\xi)} \qquad \text{with} \qquad f(\xi) = \frac{\int_\xi^\infty \Upsilon(\xi')\exp(-\frac{(\xi'-\xi_0)^2}{2\sigma^2})d\xi'}{\int_0^\infty \exp(-\frac{(\xi'-\xi_0)^2}{2\sigma^2})d\xi'} \tag{12}$$

and

$$\overline{v} = U_m d_z b[\xi\sqrt{f} - \frac{1}{2}\int_0 \xi\sqrt{f}d\xi\prime] \tag{13}$$

$$\overline{u'v'} = \frac{1}{4}U_m^2\, d_z b\, d_\xi[\int_0^\xi \sqrt{f}d\xi']^2 \tag{14}$$

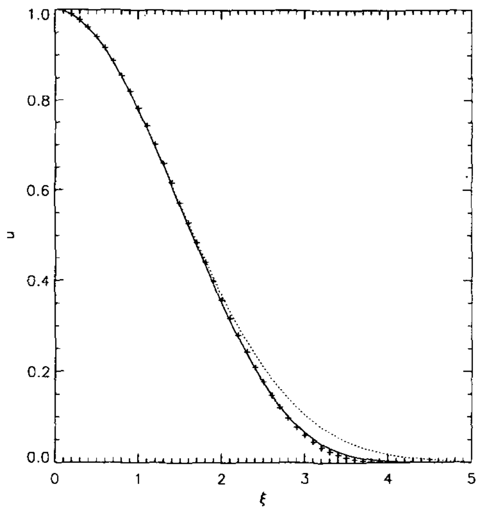

Fig. 2. The ratio $\overline{u}/U_m$: comparison between the data of Bradbury and the theoretical curve. The dotted curve is the gaussian obtained without intermittency corrections.

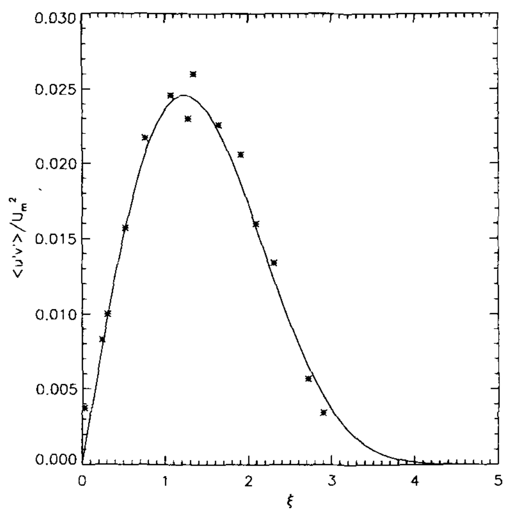

Fig. 3. The Reynolds stress: comparison of the calculated curve with the experimental data of Bradbury (recorded in the similarity zone).

We compare the calculated $\overline{u}/U_m$ to Bradbury's data in figure 2, with the experimentally determined values $d_z b = 0,066$, $\xi_0 = 2,83$ and $\sigma = 0,62$. The theoretical and experimental curves are very close to each other. Figure 3 shows the comparison between calculated and measured $\overline{u'v'}/U_m^2$. We only kept the data recorded far from the jet source, to be sure that they have been obtained in the region of approximate self-similarity. Experimental points show a great deal of uncertainty near the maximum but they are well represented by the calculated curve. It is then very likely that Reichardt's model with intermittency factor gives the quantities available by the boundary layer theory with good accuracy.

3 Conclusion

The success of this model is surprising, owing to its roughness. One may then wonder what underlying physical picture it could contain. Relation (3) is a way of expressing that the lateral flux of momentum is obtained through a mixing process whose result is to substract a part of the longitudinal momentum flux to the main flow and redistribute it. In other words, there is a kinetic energy self-injection of the jet to provide entrainment of the surrounding fluid via this mixing process, occuring on a length scale Λ much smaller than the boundary layer's width. In this picture, the Reichardt relation appears as a new version of the mixing length concept, but based on a different approach as Prandtl's. Its originality is to characterize turbulent mixing by the whole lateral momentum flux and not only by the Reynolds stress. Indeed, the mean lateral component of the velocity is also a product of turbulent transfer processes, as already inferred by Prandtl in his own mixing length model. But in contradiction to the latter, Reichardt's model deals with momentum fluxes rather than with velocities and is free of the microscopic analogy underlying Prandtl's representation. Now, the reason why Reichardt's model provides an adequate representation of the momentum transfer process is still open. Observations on turbulent jets show a small-scale nearly homogeneous turbulence to which Kolmogorov's ideas seem to apply well. They also show large vortical structures occuring on length scale b. It might be worth asking whether the very simple diffusion equation proposed by Reichardt's model could describe transport effects associated with the interaction of a "universal" turbulent velocity field characterized by the Kolmogorov spectrum and larger flow-dependent structures. We shall investigate this point later.

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