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Turbulent plumes in stellar convective envelopes

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Abstract. Recent numerical simulations of compressible convection in a stratified medium suggest that strong downwards directed flows may play an important role in stellar convective envelopes, both in the dynamics and in the energy transport. We transpose this idea to stellar convective envelopes by assuming that these plumes are turbulent plumes which may be described by Taylor's entrainment hypothesis, whose validity is well established in various geophysical conditions. We consider first the ideal case of turbulent plumes occurring in an isentropic atmosphere, and ignore all types of feedback. Thereafter we include the effect of the backflow generated by the plumes, and take into account the contribution of the radiative flux. The main result is that plumes originating from the upper layers of a star are able to reach the base of its convective envelope. Their number is necessarily limited because of their conical shape; the backflow further reduces their number to a maximum of about 1000. In these plumes the flux of kinetic energy is directed downwards, but it is less than the upwards directed enthalpy flux, so that the plumes always carry a net energy flux towards the surface. Our plume model is not applicable near the surface, where the departures from adiabaticity become important due to radiative leaking; therefore it cannot predict the depth of the convection zone, which is determined mainly by the transition from the radiative regime above to the nearly adiabatic conditions below. Neither does it permit to evaluate the extent of penetration, which strongly depends on the (unknown) number of plumes. We conclude that, to be complete, a phenomenological model of stellar convection must have a dual character: it should include both the advective transport through diving plumes, which is outlined in this paper, and the turbulent diffusion achieved by the interstitial medium. Only the latter process is apprehended by the familiar mixing-length treatment.

Key words: convection – hydrodynamics – turbulence – Sun: interior, rotation – stars: interiors

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1. Introduction

One of the main weaknesses of stellar physics remains our poor description of thermal convection. It is true that the widely used mixing-length approach permits to construct models which represent fairly well the gross properties of stars, but it fails when one attempts to apply it to more subtle problems, such as convective penetration, the differential rotation of the Sun, or its magnetic activity. Over the two past decades, significant progress has been achieved through the numerical simulation of increasingly "turbulent" convection in a stratified medium, from the pioneering, admittedly crude modal expansions, to the recent fully compressible three-dimensional computations. But the results are still difficult to apply as such to realistic situations, because of the huge difference in the relevant control parameters, the Reynolds and Prandtl numbers.

In this paper we shall describe a new approach to these problems, which is inspired by the latest numerical experiments. Those have revealed that fully compressible convection is highly intermittent, and shows strong, long-lived, downward flows, which contrast with the slower, random upward motions (Cattaneo et al. 1991; Stein & Nordlund 1989; Norlund et al. 1994). These coherent structures, or plumes as they are called, originate in the upper turbulent boundary layer, where they are initiated by the strong temperature and density fluctuations, which arise there in the steep superadiabatic gradient. Similar plumes have also been observed in laboratory experiments of nearly incompressible convection (Castaing et al. 1989); there they move in both directions, up and down, as expected from the symmetry of this problem.

Again, it is impossible to directly apply the results of these numerical experiments to the convection zone of a star. But they suggest that coherent structures may play an important role in the dynamics of a turbulent convective layer, and thus open the possibility of a different description of stellar convection.

In a stellar convection zone, viscous coherent structures are extremely small (less than a millimeter!) and they do not play any role in the large scale dynamics. But the thermal structures are of much larger size: at the top of the solar convection zone they span about 1 Mm, as demonstrated by the granulation. A flow on this scale has a very large Reynolds number ($\sim 10^9$) and is therefore strongly turbulent. Due to that turbulence, the downflows originating in the uppermost layers entrain the surrounding fluid and carry with them an increasing amount of matter. The principal aim of this paper is to model these downflows, and to examine whether they can reach the base of the convection zone.

Such turbulent plumes are observed in the Earth atmosphere, where they rise above concentrated heat sources (chimneys, volcanos, etc.). Actually, they may also have been seen in Castaing et al. convection experiment where they have been identified as jets (see Kadanoff 1990). Their theoretical interpretation is based on the entrainment hypothesis, which was first proposed by G.I. Taylor (cf. Morton et al. 1956); we shall recall it below. In astrophysics, they have been invoked by Schmitt et al. (1984) as responsible for the penetration into the stable domain underneath the solar convection zone. More recently, an estimate of that penetration has been made by Zahn (1991), which is also based on a crude plume model.

In the present paper we move a step forwards, by letting the plumes develop through the whole convection zone. We adapt to a medium which is highly stratified, in density and temperature, the treatment which is used in geophysical fluid dynamics. Furthermore, we take into account the backflow which results here from the confinement of the plumes in a limited volume. Our ambition is not to give just another phenomenology of thermal convection in stars, but to describe its dynamical properties in a more consistent way. We hope that this will lead eventually to a more realistic picture of stellar convection, including penetration and overshoot. Moreover, we anticipate that it may enable us to interpret the observed differential rotation, and perhaps to contribute to the understanding of the magnetic cycles.

We begin by describing the basic properties of plumes occurring in an isentropic envelope. After this ideal case, we consider the plumes as being the main actors of stellar convection, and we present a model of the solar convection zone which is based on this assumption. In particular, we examine the role of the backflow, which plays a severe role in limiting the depth the plumes may reach.

2. Turbulent plumes in an isentropic atmosphere

In this section we shall first examine the basic properties of turbulent plumes which rise or dive in an isentropic, plane-parallel atmosphere. We shall in both cases assume that the plume is fed steadily by a cold (or hot) layer of fluid lying on the top (or the bottom) of the atmosphere. A schematic view of the flow for a diving plume is sketched out in Fig. 1.

2.1. Equations of motions

The equations governing the structure of turbulent plumes have been established in the original work of Morton et al. (1956), but we refer also the reader to the book of Fischer et al. (1979) and to the excellent review by Turner (1986). For sake of clarity, we shall rederive these equations.

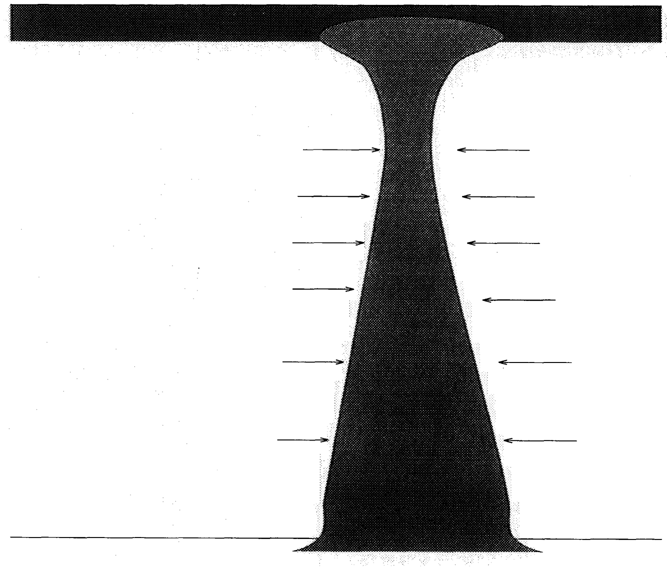

Fig. 1. The plume flow

One assumes that the flow is stationary, that the plumes are axisymmetric, and that all horizontal variations, namely those of the vertical velocity v_z, of the density excess $\delta\rho$ and of the excess of specific enthalpy δh, have the same gaussian profile:

$$\begin{cases} v_z(r,z) = V(z)\exp(-r^2/b^2) \\ \delta\rho(r,z) = \Delta\rho(z)\exp(-r^2/b^2) \\ \delta h(r,z) = \Delta h(z)\exp(-r^2/b^2) \end{cases} \qquad (1)$$

where $b \equiv b(z)$ is the effective radius of the plume. For convenience, we take the vertical coordinate z pointing downward; the density and the specific enthalpy of the isentropic atmosphere will be designated by $\rho_0(z)$ and $h_0(z)$.

The equations describing the vertical profile of the plume are derived from the three basic equations of fluid mechanics expressing the conservation of mass, momentum and energy.

2.1.1. Conservation of mass

In the steady flow of the plume, the mass conservation implies

$$\operatorname{div}(\rho \boldsymbol{v}) = \frac{1}{r}\frac{\partial r\rho v_r}{\partial r} + \frac{\partial \rho v_z}{\partial z} = 0\,;$$

when integrated over r, this equation becomes

$$\frac{d}{dz}\int_0^{+\infty} \rho v_z r dr \; + \; [r\rho v_r]_0^{+\infty} = 0. \qquad (2)$$

The entrainment hypothesis made by G.I. Taylor postulates that the radial inflow of matter is proportional to the central vertical velocity:

$$\lim_{r\to+\infty} r v_r = -\alpha b(z)|v_z(0,z)| \qquad (3)$$

where α is the entrainment constant. The absolute value guarantees that the plume is always accreting matter, whether it is directed upwards or downwards. Using the gaussian profile of v_z and $\delta\rho$, one casts the mass equation in its final form

$$\frac{d}{dz}\left[\left(\frac{\xi+1}{2}\right)\rho_0 b^2 V\right] = 2\alpha\rho_0 bV \qquad (4)$$

where we have introduced the density contrast $\xi = 1 + \Delta\rho/\rho_0$.

2.1.2. Conservation of momentum

Ignoring the viscous stresses, the flow obeys the steady Euler equation:

$$\partial_j(\rho v_j v_i + \delta p\, \delta_{ij}) = \delta\rho\, g_i\,,$$

where the static equilibrium has been subtracted. We assume that the plume is in pressure balance with the surrounding medium, although this is only true to first approximation (see Massaguer & Zahn 1980). We thus neglect δp and write the z-component as:

$$\frac{1}{r}\frac{\partial r\rho v_r v_z}{\partial r} + \frac{\partial \rho v_z^2}{\partial z} = \delta\rho\, g$$

which we integrate in the same way as the mass equation, to reach

$$\frac{d}{dz}\left[\left(\frac{2\xi+1}{3}\right)\rho_0 b^2 V^2\right] = 2gb^2\Delta\rho\,. \qquad (5)$$

2.1.3. Conservation of energy

We start from the steady energy equation

$$\mathrm{div}(\rho e \boldsymbol{v}) = \mathrm{div}(\chi \boldsymbol{\nabla} T) - p\,\mathrm{div}\,\boldsymbol{v}\,,$$

where e is the specific internal energy and χ the radiative conductivity, and rewrite it in a more convenient form by using the momentum equation and the definition of the specific enthalpy $dh \equiv de + d(p/\rho)$:

$$\mathrm{div}\left[(h + \frac{1}{2}v^2)\rho\boldsymbol{v} - \chi\boldsymbol{\nabla}T\right] = \rho\boldsymbol{v}\cdot\boldsymbol{g}.$$

This equation may be further simplified by remembering that

$$\boldsymbol{g} = \boldsymbol{\nabla} h_0$$

in an isentropic atmosphere (see below). Finally, the conservation of the energy flux is expressed by:

$$\mathrm{div}\left[\left(\delta h + \frac{1}{2}v^2\right)\rho\boldsymbol{v} - \chi\boldsymbol{\nabla}T\right] = 0, \qquad (6)$$

where we recognize respectively the enthalpy, kinetic and radiative fluxes.

To make contact with previous work, we first neglect the radiative flux. Proceeding as before, we integrate in the radial direction, and obtain

$$\frac{1}{2}\left(\frac{2\xi+1}{3}\right)\rho_0 b^2 V\Delta h + \frac{1}{6}\left(\frac{3\xi+1}{4}\right)\rho_0 b^2 V^3 = -\mathscr{F}/\pi\,, \qquad (7)$$

where $\mathscr{F}$ designates the total flux (enthalpy plus kinetic) carried by one plume.

From now on, we shall drop the subscript $_0$, since there will be no ambiguity anymore between the local values of density and temperature, and their counterparts in the isentropic atmosphere.

2.1.4. Thermodynamics

For simplicity, we shall assume throughout this paper that the fluid is a perfect monatomic gas, with $\gamma = 5/3$ being the adiabatic exponent. We thus complete the system of governing equations with the two thermodynamic relations:

$$\delta h = c_p \delta T$$

$$(\rho + \delta\rho)(T + \delta T) = \rho T\,,$$

where the latter expresses the pressure balance between the plume and the surrounding medium.

2.1.5. The isentropic atmosphere

As is well known, the isentropic atmosphere is a polytrope of index $q = 1/(\gamma - 1)$. In plane-parallel geometry, the density and the temperature vary with depth z as:

$$\begin{cases} \rho(z) = \rho_0(z/z_0)^q \\ T(z) = T_0(z/z_0). \end{cases} \qquad (8)$$

The origin of the vertical coordinate z is taken at the "surface", where pressure, density and temperature vanish altogether, and the subscript $_0$ now designates the reference level, for which we choose here the base of the atmosphere. The reference depth z_0 is related to the reference temperature by

$$z_0 = \frac{c_p T_0}{g}\,.$$

2.1.6. Non-dimensionalization

It is convenient to put the governing equations in non-dimensional form. We take z_0 as the length scale, V_1, the initial velocity (just below the cold layer for a diving plume), as the velocity scale and we normalize all the thermodynamic quantities by their value at the reference level, i.e. the base of the atmosphere. With these new variables, the isentropic atmosphere is described by

$$\begin{cases} \rho(\zeta) = \zeta^q \\ T(\zeta) = \zeta \end{cases} \qquad (9)$$

while the plume equations now read:

$$\begin{cases} \dfrac{d}{d\zeta}\left[\left(\dfrac{\xi+1}{2}\right)\zeta^q\beta^2 u\right] = 2\alpha\zeta^q\beta|u| \\ \dfrac{d}{d\zeta}\left[\left(\dfrac{2\xi+1}{3}\right)\zeta^q\beta^2 u^2\right] = \Gamma\zeta^q\beta^2(\xi-1) \\ \dfrac{1}{2}\left(\dfrac{2\xi+1}{3}\right)(\Theta-1)\zeta^{q+1}\beta^2 u + \\ \qquad \dfrac{1}{3\Gamma}\left(\dfrac{3\xi+1}{4}\right)\zeta^q\beta^2 u^3 = -F \end{cases} \tag{10}$$

where we have set

$$\begin{cases} \zeta = z/z_0 \\ \beta = b/z_0 \\ u = V/V_1 \\ \Theta\,\xi = 1 \\ \Gamma = \dfrac{2gz_0}{V_1^2} \\ F = \dfrac{\mathcal{F}}{\pi\rho_0\, g z_0^3 V_1}. \end{cases} \tag{11}$$

As it will be useful later, we also give the expression for the Mach number

$$\mathrm{Ma} = \sqrt{\frac{2q}{\Gamma}}\frac{u}{\zeta^{1/2}} \quad . \tag{12}$$

2.1.7. Boundary conditions

The set of differential Eqs. (10) needs now to be completed by suitable boundary conditions. Since these are all applied at the start of the plume, we could as well call them initial conditions, although the problem is independent of time.

In the traditional approach, a plume is described by its mass flux M, momentum flux P and buoyancy flux B (which is proportional to the enthalpy flux). With our dimensionless variables, these quantities would be expressed as:

$$\begin{cases} M = \left(\dfrac{\xi+1}{2}\right)\zeta^q\beta^2 u \\ P = \left(\dfrac{2\xi+1}{3}\right)\zeta^q\beta^2 u^2 \\ B = \dfrac{1}{2}\left(\dfrac{2\xi+1}{3}\right)(\Theta-1)\zeta^{q+1}\beta^2 u\,. \end{cases} \tag{13}$$

However, we find it more convenient here to specify instead the density contrast ξ, the size β and the velocity u at the starting level of the plume. At this point one has to distinguish between rising and diving plumes. For the rising plume we set

$$\begin{cases} \beta = b \\ u = -1 \\ \xi = 1/\nu \end{cases} \quad \text{at} \quad \zeta = 1\,, \tag{14}$$

while for the diving plume, because of the vanishing density at $\zeta = 0$, the initial conditions must be imposed at a small, but non-zero depth $\zeta = \varepsilon$:

$$\begin{cases} \beta = b \\ u = 1 \\ \xi = \nu \end{cases} \quad \text{at} \quad \zeta = \varepsilon\,. \tag{15}$$

The initial density contrast is characterized by ν, a real number larger than unity.

For the purpose of numerical integration, the governing equations (10) have been rewritten using the momentum flux P and a modified mass flux defined as

$$M' = \left(\frac{2\xi+1}{3}\right)\zeta^q\beta^2 u \tag{16}$$

We give in the Appendix the differential system which results from these transformations.

2.2. *Asymptotic solutions*

The equations governing the plume flow are nonlinear and in most cases it is necessary to integrate them numerically. However, the system has asymptotic solutions which can easily be obtained in analytic form.

2.2.1. Diving plumes

The first and somewhat extreme case is that of a diving plume, during its initial phase, when its density is much larger than that of the surrounding medium, i.e. when $\xi \gg 1$. We shall call this regime the "free-fall" phase, in which the flow velocity is small and entrainment still negligible. The mass and momentum equations then become

$$\begin{cases} \dfrac{d}{d\zeta}(\xi\zeta^q\beta^2 u) = 0 \\ \dfrac{d}{d\zeta}(\xi\zeta^q\beta^2 u^2) = \dfrac{3}{2}\Gamma\,\xi\,\zeta^q\beta^2 \end{cases} \tag{17}$$

which reduce to the free-fall equation

$$\frac{du^2}{d\zeta} = 3\Gamma$$

with the following solution:

$$\begin{cases} u^2 = 1 + 3\Gamma(\zeta-\varepsilon) \\ \xi\beta^2 = \dfrac{b^2\,(\varepsilon/\zeta)^q}{[1+3\Gamma(\zeta-\varepsilon)]^{1/2}} \end{cases} \tag{18}$$

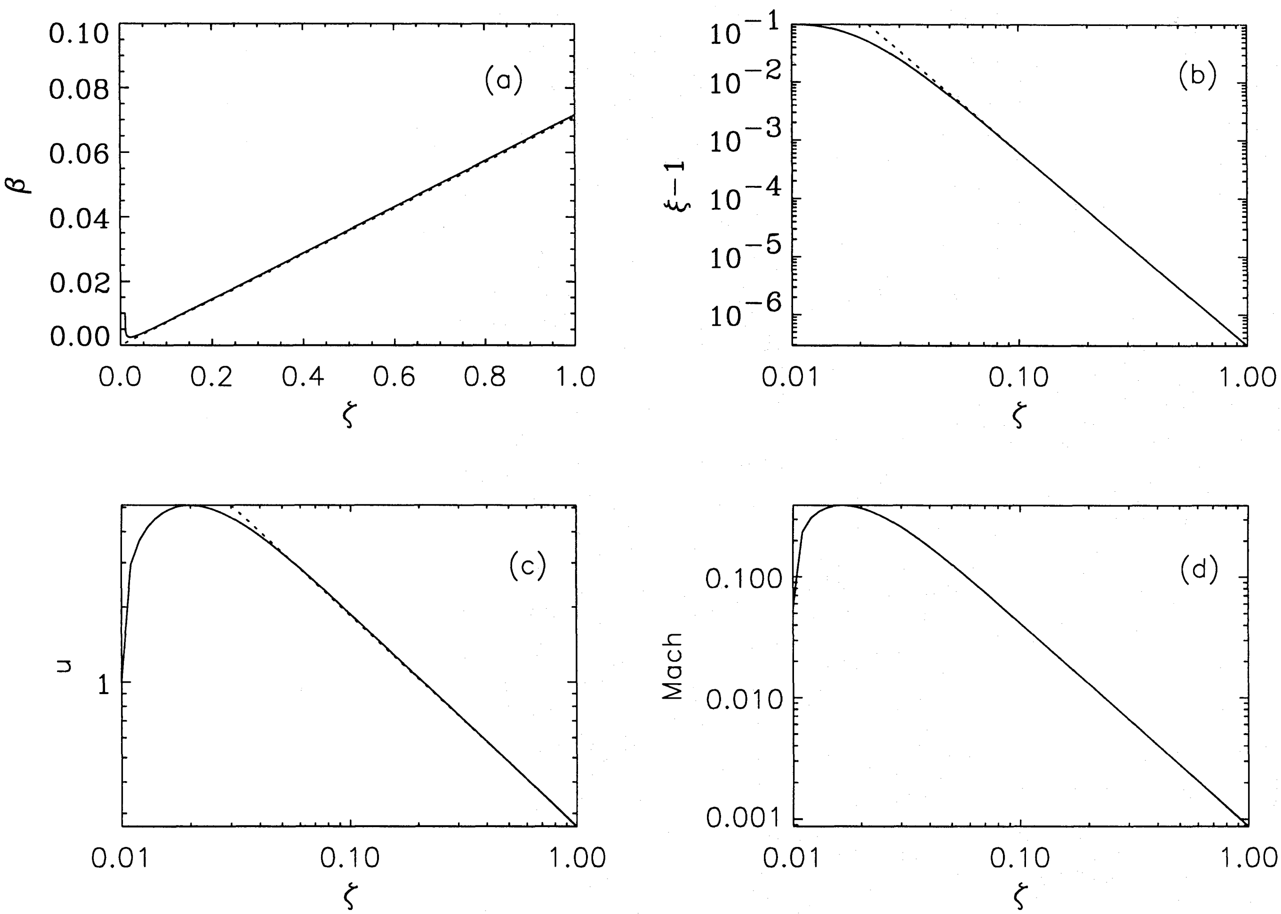

Fig. 2a–d. The radius (**a**), density contrast (**b**), velocity (**c**) and Mach number (**d**) for a diving plume. The dotted line represents the asymptotic solution reached when the plume fluid is well mixed. The parameters $\Gamma = 10^5$, $\nu = 1.1$, $\gamma = 5/3$, $b = 0.01$ and $\varepsilon = 10^{-2}$, are chosen to match approximately the solar conditions

This solution is however not consistent with the energy equation, because the self-similarity of the plume profile, which we have postulated by imposing a Gaussian, cannot be conserved in this free-fall solution.

Another asymptotic regime which has an analytical solution is the limiting case of very small density fluctuations, i.e. $\xi - 1 \ll 1$. It corresponds to the developed phase of the plume, when it is fully controlled by entrainment. The equations simplify into

$$\begin{cases} \frac{d}{d\zeta}\left[\zeta^q \beta^2 u\right] = 2\alpha\zeta^q\beta|u| \\ \frac{d}{d\zeta}\left[\zeta^q\beta^2 u^2\right] = \Gamma\beta^2\zeta^q(\xi-1) \\ \frac{1}{2}(\xi-1)\zeta^{q+1}\beta^2 u - \frac{1}{3\Gamma}\zeta^q\beta^2u^3 = F \end{cases} \tag{19}$$

The asymptotic solutions are of the form

$$\beta = \beta_0\zeta^p, \quad u = u_0\zeta^r, \quad \xi - 1 = R_0\zeta^m \tag{20}$$

Once these power laws are inserted in Eqs. (19), one easily determines the exponents:

$$\begin{cases} p = 1 \\ r = -\frac{q+2}{3} = -\frac{2\gamma-1}{3(\gamma-1)} \\ m = -\frac{2q+7}{3} = -\frac{7\gamma-5}{3(\gamma-1)} \end{cases} \tag{21}$$

For $\gamma = 5/3$, one has $r = -7/6$. A direct consequence of Taylor's entrainment hypothesis (3) is that the plume radius increases linearly with depth. The half-angle β_0 of the cone is related to the adiabatic exponent:

$$\beta_0 = \frac{3\alpha}{q+2} = \frac{3\alpha(\gamma-1)}{2\gamma-1} \tag{22}$$

For the perfect monatomic gas, we have $\beta_0 = 6\alpha/7$. The entrainment coefficient α has been determined experimentally in various situations, and the value 0.083 is widely adopted (Turner

1986); we shall assume that this value also applies to our fully ionized gas. For later use we also give u_0 and R_0

$$\begin{cases} u_0 = \left(\dfrac{6\Gamma L}{q\beta_0^2}\right)^{1/3} \\ R_0 = \left(\dfrac{q+2}{3}\right)\dfrac{u_0^2}{\Gamma} \end{cases} \tag{23}$$

The careful reader has certainly noticed that one does not retrieve the incompressible case by letting the polytropic index tend to zero, $q \rightarrow 0$. The reason for this singular limit is the assumption made in the classical treatment (Morton et al. 1956) that the surrounding medium is isothermal, while in our stratified atmosphere the temperature grows linearly with depth. Let us emphasize however that our asymptotic solution (20) is rather special, since its focal point $\zeta = 0$ coincides with the "surface" of the atmosphere.

An interesting property of this asymptotic regime is the strict proportionality between the flux of kinetic energy and that of enthalpy:

$$R = \frac{\int \frac{1}{2}\rho V^3 r dr}{\int \rho c_p \, \delta T \, V r dr} = cst. \tag{24}$$

With the gaussian profile we have adopted for the plume, this ratio is given by

$$R = -\frac{2}{q+2} = -\frac{2\gamma - 2}{2\gamma - 1} \,. \tag{25}$$

Its value is 1 for the Boussinesq fluid (no density stratification) and decreases to 4/7 for the perfect monatomic gas.

In their recent numerical experiment Cattaneo et al. (1991) observed to their surprise that the strong downdrafts carried a flux of kinetic energy which very nearly cancelled the flux of enthalpy. Thus their downward plumes do not contribute much to the net transport of energy, which is due mainly to the quieter, interstitial medium. We find, on the contrary, that a diving plume transports a net amount of energy in the expected direction, namely towards the surface. The explanation, as also proposed by Cattaneo et al., is that the plumes of the numerical experiment are laminar structures which approximately satisfy Bernoulli's theorem.

2.2.2. Rising plumes

The case of rising plumes is radically different and more complex than that of diving plumes, contrary to one's naive expectation that similar approximations would lead to similar asymptotic solutions. The main reason is that the origin of the plume is no longer located in the vicinity of the "origin" of the atmosphere, and therefore they obey different scaling laws.

Let us start as above, and consider first a very light plume: imagine, for example, a plume of air rising from the bottom of the sea. We assume again that mixing is negligible. The approximation is now $\xi \ll 1$.

The momentum equation thus yields the very simple "free rise" solution

$$u^2 = 1 + 2\Gamma(1-\zeta) \,.$$

We now turn to the fully mixed situation where $|\xi - 1| \ll 1$. In this case the flow is still determined by (19) but $u < 0$ and $\xi < 1$ and the initial conditions are taken at $\zeta = 1$. An interesting point can be made if we cast this system into a second order differential equation for the momentum flux $P_m = A^{-2/3}\zeta^q\beta^2u^2$

$$\frac{d}{d\zeta}\left[\zeta^{5/3} P_m \frac{d}{d\zeta}\left(\frac{P_m}{\zeta^{2/3}}\right)\right] = \zeta^{q/2}\sqrt{P_m} \tag{26}$$

where $A = 4\alpha\Gamma F$. Since diving plumes obey this same equation, power law solutions exist: one verifies that

$$P_m = \left(\frac{9}{2q(q+2)}\right)^{2/3} \zeta^{(q+2)/3} \tag{27}$$

is a solution. However, a rising plume never approaches this solution. A closer look on (26) reveals that its solutions have movable branch points (i.e. which depend on the initial conditions); therefore by starting with different boundary conditions one obtains very dissimilar solutions (but see Ince (1926) for further details on these equations). Actually, the fact that the power law solution (27) cannot be reached by ascending plumes could have been guessed from the form of the constant R_0 in (23) which is always positive, while rising plumes should have $R_0 < 0$ because of (20).

Nevertheless, just as diving plumes, rising plumes reach a "universal regime" after a transient depending on the initial conditions. Unfortunately, and contrary to diving plumes, such a regime cannot be expressed in terms of known functions. As we shall see in Fig. 3, the plume velocity is almost constant with height, in sharp contrast with the diving plume, where the flow slows down as the depth increases.

2.3. *Numerical solutions*

We have integrated the governing equations (10) for various initial conditions. Figure 2 displays the results for a diving plume in conditions similar to that of the Sun. The plot of the velocity clearly demonstrates the two asymptotic regimes described above. The fluid initially accelerates in almost free fall and the width of the plume shrinks rapidly, as required by the conservation of mass. As the atmosphere gets denser, entrainment becomes stronger and the flow slows down, while the width of the plume now increases; a glance at the density ratio shows that the plume is then well mixed.

From thereon, the flow approaches the asymptotic regime described in Sect. 2.2.1: it is almost self-similar, and its strength depends only on the energy flux carried by the plume. The initial conditions which specify the initial mass flux and momentum

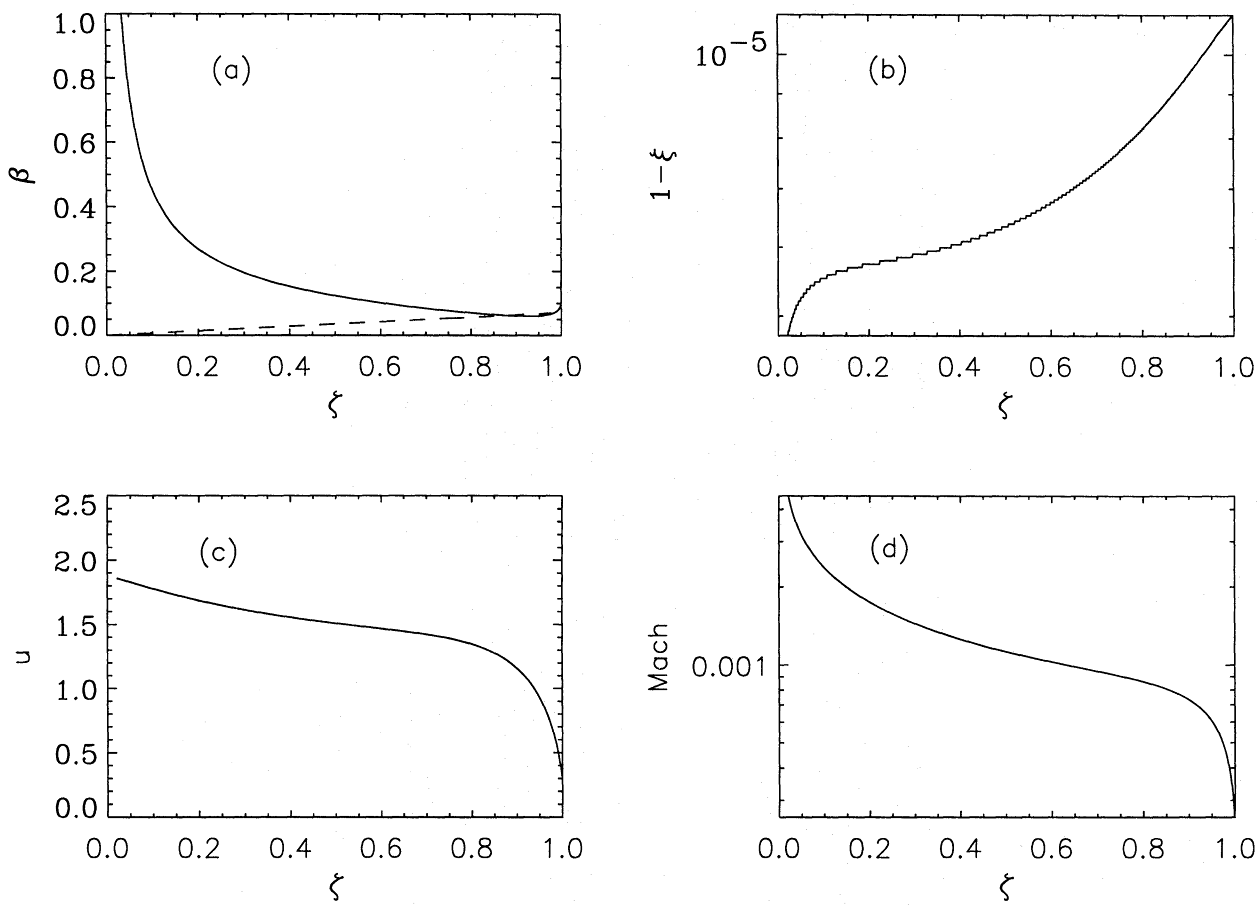

Fig. 3a–d. The solution for a rising plume when $\xi \sim 1$. The parameters of the flow, $A = 0.2$, $b = 0.1$ and $\Gamma = 10^7$, are representative of solar conditions. Note the difference in horizontal scale for the rising and diving plume (dashed line) in **a**

flux are thus "forgotten". As expected, the closer the initial conditions are to the self-similar solution, the faster the plume tends to its asymptotic shape; we found that this regime is obtained more quickly when the kinetic energy flux is neglected. The asymptotic regime is best illustrated in the linear increase of the radius β; note also that the monotonic decrease of the density contrast, which approaches the predicted power law.

Figure 3 shows the behavior of rising plumes. Again, as the plume rises, the density contrast decreases while the section of the plume increases, although its growth is no longer a linear function of height. But the velocity curve is more intricate. After the initial "free-rise", the flow begins to slow down, due to mass loading caused by entrainment. Thereafter, because the surrounding medium is less and less dense, entrainment is less effective while the momentum flux still increases; the consequence of this is an acceleration of the flow in the upper part of the atmosphere. We note that as for the diving plume flow, the rising plume is subsonic all the way.

In concluding this section, we would like to emphasize once more the difference between rising and diving plumes. This difference is manifest in the size of the plumes and in their velocity. Indeed, as shown by Fig. 3a the horizontal size of the rising plume is much larger than that of the diving plume, while its velocity remains well below that of the diving plume (compare Fig. 2c and Fig. 3c). For these reasons we do not expect the rising plumes to play a major role in the dynamics of the convective layer and we shall neglect them in the next section.

3. Turbulent plumes in the solar convection zone

In the preceding section we have presented the basic properties of rising or diving turbulent plumes in a horizontal plane-parallel layer. We now proceed to build a model of stellar convection zone which consists in an ensemble of diving plumes. We take the extreme view that all the convective flux is transported by these plumes, thus neglecting the contribution of the interstitial medium. As mentioned above, we ignore the possibility of rising plumes: presumably, these are difficult to initialize in the lower part of the convection zone, since there the temperature gradient is very close to adiabatic.

To be specific, we shall illustrate the possible role of turbulent plumes in the solar convection zone, whose basic parameters are recalled in Table 1.

Up to now, we have considered the case of a single plume in an infinite isentropic plane layer. Here we progress towards a more realistic situation: we now take into account the spherical

Table 1. Solar parameters used in the calculations

$g_\odot$	270 m s^{-2}
$R_\odot$	6.96 10^8 m
$L_\odot$	3.9 10^{26} W
η	0.7
T_0	1.86 10^6 K
ρ_0	190 kg m^{-3}

geometry of the fluid layer and include the variations of gravity with depth (neglecting however the mass of the envelope) so that the density and temperature field of the background are given by

$$\begin{cases} T(\zeta) = \dfrac{\eta}{1-\eta}\left(\dfrac{1}{x}-1\right) \\ \rho(\zeta) = [T(\zeta)]^q \\ x = 1-(1-\eta)\zeta \end{cases} \tag{28}$$

where η is the radius, scaled by the total radius, corresponding to the reference level. The set of Eqs. (28) is now replacing (9).

We shall also take into account the radiative flux by subtracting it from the total flux, thus (7) now reads

$$\frac{1}{2}\left(\frac{2\xi+1}{3}\right)\rho b^2 V\Delta h + \frac{1}{6}\left(\frac{3\xi+1}{4}\right)\rho b^2 V^3 = -(\mathscr{F}_{\rm tot}-\mathscr{F}_{\rm rad})/\pi\ . \tag{29}$$

The radiative flux is proportional to the temperature gradient; thus

$$\mathscr{F}_{\rm rad} = -\chi\left(\frac{\partial T}{\partial r}\right)_{\rm ad},$$

where the radiative conductivity χ is related to the opacity κ by

$$\chi = \frac{16\sigma T^3}{3\kappa\rho},$$

with σ being the Stefan constant. Using Kramers' law to express the conductivity in terms of density and temperature, and choosing the reference level where the radiative luminosity equals the total luminosity, i.e. where the convection zone should end according to the Schwarzschild criterion, we have

$$\frac{\mathscr{F}_{\rm rad}}{\mathscr{F}_{\rm tot}} \propto \frac{T^{6.5}}{\rho^2}. \tag{30}$$

With our assumption that all the convective energy is transported by the plumes, whose number is N, the total flux amounts to the luminosity of the star:

$$N\mathscr{F}_{\rm tot} = L_\odot. \tag{31}$$

3.1. Plumes interactions

Two types of interactions, presumably, will play a major role: interpenetration and coalescence. The first one is simply the confrontation of plumes of opposite signs (diving and rising) while the second one concerns plumes of the same sign (both rising or both diving). Much like vortices, plumes attract plumes of the same sign and repel or avoid those of the opposite sign (Moses et al. 1993).

3.1.1. Plumes diving in a rising counter flow

We shall not deal with the mutual interpenetration of diving and rising plumes since we banned the latter from our convection zone. However, the diving plumes themselves are always evolving in a counter flow: since the volume of the convection zone is finite, they generate this flow in order to ensure the conservation of mass.

Let us first estimate the maximum number of diving plumes which may coexist in the solar convection zone (SCZ), assuming that they all originate at the top and that all reach its bottom without being hindered by the counterflow. This number is given by the ratio of the area of the base of the SCZ to the section of a single plume at this depth:

$$N = 4\left[\frac{\eta}{\beta_0(1-\eta)}\right]^2 \tag{32}$$

where β_0 is given by (22). Taking $\eta = 0.7$, the result is

$$N_{\rm max} \sim 4300\ . \tag{33}$$

We see that even if the plumes were closely packed, they would not be very numerous. The important point to note here is that if the diving plume number equals $N_{\rm max}$, the filling factor f (fraction of the total area occupied by the plumes) will vary from zero at the top of the convection zone to unity at the bottom (disregarding the fact that discs cannot pave the sphere). Once f becomes larger than 1/2, at depth $z_0/\sqrt{2}$, the velocity of the backflow exceeds that of the plume.

To model the effect of this counterflow, we shall assume that the entrainment of mass into the plume is proportional to the difference between the velocity of the plume and that of the surrounding fluid. We thus write the mass equation as:

$$\frac{d}{dz}\left[\left(\frac{\xi+1}{2}\right)\rho b^2 V\right] = 2\alpha\,\rho b(V-V_u) \tag{34}$$

where V_u is the upward velocity of the surrounding fluid; since $V_u < 0$, entrainment is enhanced. The momentum equation needs also be completed by a term taking into account the entrainment of momentum into the plume, since the surrounding medium is no longer at rest. We thus transform (5) into

$$\frac{d}{dz}\left[\left(\frac{2\xi+1}{3}\right)\rho b^2 V^2\right] = 2gb^2\Delta\rho + 4\alpha\,\rho b V_u(V-V_u) \tag{35}$$

Equations (34) and (35) are completed by

$$(4\pi r^2 - N\pi b^2)\rho V_u + N\pi b^2\rho V = 0 \tag{36}$$

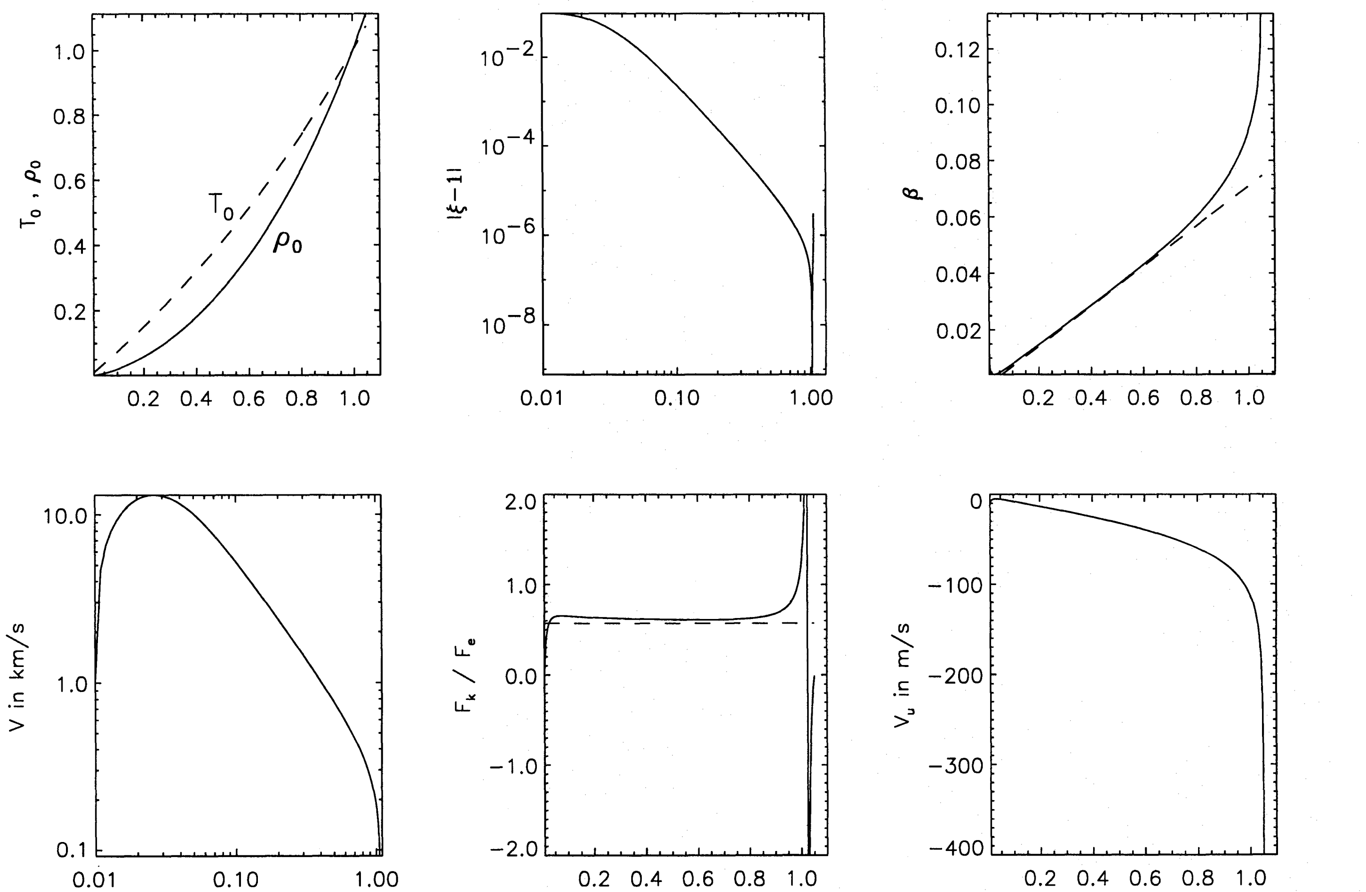

Fig. 4. Profiles of characteristic quantities of a diving plume for typical solar conditions as a function of depth. The number of plumes is 1000. Note the change of sign of the density fluctuation $\Delta\rho$ which induces buoyancy braking and the reversal of the enthalpy flux. The ratio of the kinetic energy flux to the enthalpy flux is also plotted along the asymptotic value of 4/7 (dashed line)

which expresses the conservation of mass. The consequence of these additional effects is that diving plumes will reach the bottom of the convection zone only if they are strong enough and not too numerous. We shall return to this question when discussing penetration at the base of the convection zone.

We integrated numerically the modified governing equations (34), (35), together with (36) and (29). The solution for typical solar parameters is presented in Fig. 4. From this integration it turns out that only 1000 plumes are able to reach simultaneously the bottom of the SCZ.

3.1.2. Plume coalescence

At the top of the SCZ, the number of diving plumes probably equals that of the granules, which is considerably larger than 1000. Hence a drastic reduction of the plumes number must take place as depth increases. This is achieved by two means: by the adverse effect of the upflow and through the merging of plumes.

As one may note in Fig. 4, the velocity of the upflow is increasing with depth. Thus the diving plumes will be stopped at a depth which depends on their strength: only the 1000 strongest reach the bottom, while the millions generated by the granules do not go below 10 Mm (say). Once it has come to rest, the cold fluid of the plume may well dissolve in the (turbulent) backflow, and disappear as such. However it can also have another fate. Just imagine that the plume which stops at some depth has in its vicinity a slightly stronger companion. Because this companion is accreting the surrounding medium, the cold fluid which remains from the stopped plume will soon flow into the neighboring plume, thereby rendering it even stronger.

But the merging of two plumes may proceed in a smoother way, as observed in the numerical simulations of Stein & Nordlund (1989) (see also Spruit et al. 1990). Two neighboring plumes are advected by each other, since they both accrete surrounding fluid, and they are pulled to each other until they finally merge. The merging depth can be estimated if we reduce the plume to its central line and assume the flow to be in the asymptotic regime. We give in the Appendix the details of the calculations; the result is

$$z_m = \frac{d}{\sqrt{2\alpha\beta_0}} \qquad (37)$$

where d is the initial separation of the two plumes, which we assumed here to be identical. If we use the values of β for the monatomic ideal gas and $\alpha = 0.083$, then

$$z_m \sim 9.2\, d\,.$$

Such coalescences then repeat at increasing depths, until the survivors reach the base of the convection zone. This is a sort of inverse cascade with a large scale flow building up from smaller scales. Interestingly enough, this phenomenon has recently been observed in the case of laminar plumes by Moses et al. (1993).

3.2. Penetration and overshoot

3.2.1. Penetration at the base of a convection zone

When the plumes reach the bottom of the unstable domain, they still possess a finite velocity which enables them to penetrate some distance into the stable, subadiabatic region, where they establish a nearly adiabatic stratification by releasing their entropy when they come to rest. A first attempt to estimate the extent of penetration of such plumes was made by Schmitt et al. (1984). However, they did not include the return flow, and they had to impose both the velocity V and the filling factor f at the base of the unstable domain because they did not solve the plume equations above that level. They found empirically that the penetration depth varies as $f^{1/2}V^{3/2}$. This scaling was explained by Zahn (1991), who also gave a crude estimate of the extent of penetration: a fraction (of order unity) of the scale-height of the radiative conductivity, which is here $z_0/3.5$ with Kramers' law.

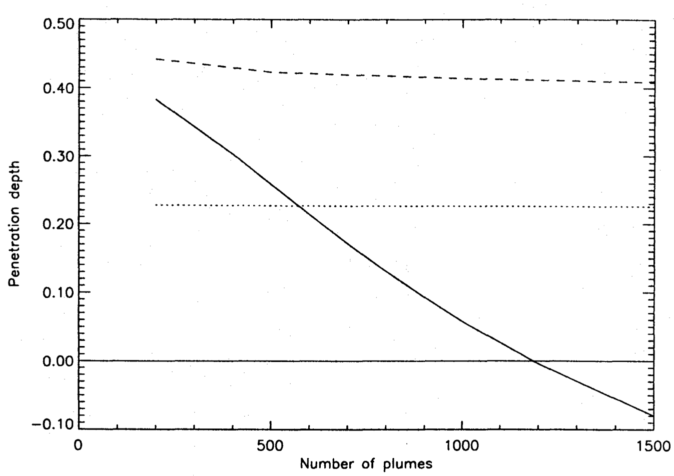

Fig. 5. The extent of penetration, scaled by the thickness of the unstable domain, as a function of the number of plumes. The dashed line traces the case when the upward motion of the background is neglected: the penetration depth is then almost independent of N. The dotted line is for the same case but when the kinetic energy flux is neglected: the difference between these two lines shows the important role played by the kinetic energy in the penetration process. When the backflow is incorporated (solid line), the penetration depth decreases monotonically with N. The zero level is the boundary given by Schwarzschild criterion

In our approach we compute the penetration in a more consistent way, by integrating the governing equations from the top of the convection zone until the plumes come to rest; the results are displayed in Fig. 5. We consider first the case where the effect of the backflow is ignored. As a consequence, the penetration depth is almost independent of the plume number, as was predicted by Zahn 1991; however the limiting value is very large and incompatible with the helioseismic observations (Dziembowski 1993). In the second case, we include the interaction of plumes with the backflow. The penetration depth depends almost linearly on the plume number; indeed as this number increases, the backflow strengthens the loading of the plume flow with upward momentum.

This dependence shows that a much more sophisticated model is needed to describe the termination of the plume flow. In particular, the loss of mass by the plume in this region, or "detrainment", and its transfer to the upward motion, is not accounted for by the equations we use. A way of dealing with this was presented by Abramovich (1963) for the case of a jet in a dead-end channel. It consists in connecting the turbulent entraining flow to a potential flow. His results show however only qualitative agreement with the experiments.

3.2.2. Overshoot at the top of a convective envelope

The problem of convective overshoot in the atmosphere of the Sun is not, unfortunately, just the mirror image of that of penetration. The main reason is that in the top layers the horizontal scale of the coherent structures, namely the granules, is so small that the Péclet number is order of unity and thus radiative cooling becomes important. Moreover, these thermally coherent structures do no longer behave as long-lived plumes, but rather as ephemeral thermals, and their dynamics cannot be described by stationary flows.

The realm of the plumes is thus the bulk of the convection zone, with its penetrative extension below. Near the surface, another approach must be taken, if one wishes to significantly improve beyond the mixing-length formalism. But this region is also in much closer reach of the numerical simulations, which yield there excellent results, as it has been demonstrated convincingly by Å. Nordlund and his coworkers.

4. Discussion

Let us summarize the few results we have obtained in this first exploration of the role that turbulent diving plumes may play in stellar convective envelopes.

If Taylor's parametrization of the entrainment of surrounding fluid into turbulent plumes holds in the hot plasma of stars (this point will be discussed in a forthcoming paper by Bonin & Rieutord 1994), then a limited number of such plumes can reach the base of the convection zone.

To first approximation, these plumes behave much as if they were traversing an isentropic atmosphere, with no feedback at all, and ignoring the radiation flux. Shortly after their start, they reach an asymptotic regime where their size increases linearly with depth. Due to the density stratification, the cone angle is somewhat smaller than in the Boussinesq case treated so far: it is $6\alpha/7$ instead of $6\alpha/5$, with α being the entrainment constant (whose experimental value is $\alpha = 0.083$).

Another non-Boussinesq effect is the importance of the kinetic energy flux, which is directed downwards. This adverse flux could well, in principle, neutralize the action of the plume, as was observed in the numerical simulations of Cattaneo et al. (1991). We find however that the enthalpy flux always exceeds the kinetic flux: the ratio between the two is constant in the asymptotic regime, and its value is 4/7, if one assumes that the horizontal profile of the plume is gaussian.

These asymptotic properties are strictly valid only in a plane-parallel atmosphere and if self-similarity of the flow is preserved (see Bonin & Rieutord 1994); they are modified somewhat in a spherical envelope. Another step towards a more realistic description is to take into account the quiescent backflow which is generated by the plumes. It operates as a severe selection mechanism: the weakest plumes will be stopped at some intermediate level, if they have not merged before with a stronger neighbor. This backflow causes also some departure from the asymptotic regime of the ideal plume, at large depth.

Assuming that the solar convection zone is superadiabatic over a depth of 200,000 km, we have calculated the maximum number of plumes that may reach this level, where the convective flux reverses its sign. Taking the counterflow into account, we found that number to be around 1000, which means that the spacing between plumes would be about 60 Mm. Note however that this evaluation is based on a very strong assumption: namely that the totality of the convective energy flux (enthalpy plus kinetic) is carried by the diving plumes. Under these same conditions, the extent of penetration into the stable region beneath would depend on the number of plumes, but neither can be predicted by this simple model.

All these results are independent of the top boundary conditions on the mass flux and on the initial momentum flux carried by the plumes. What determines the flow is the convective energy flux near the surface, which would be conserved along the plume if there were no radiative transport.

Implicitly, we made another assumption, namely that these plumes behave as plumes, i.e. that their life time largely exceeds the flow travel time. We thus postulated that the downflows are stationary, although it is not obvious that they will ever reach their steady regime in the adverse upflow.

But our plume model suffers from a far more severe shortcoming: it is not applicable to the upper layers of the convection zone, where radiative leaking becomes important, and where we know from the observations that the turbulent eddies behave much more like thermals than as plumes. Thus our model is unable to predict the depth of the convection zone, which depends heavily on the transition from the radiative regime at the surface to the nearly adiabatic conditions below.

It thus appears that a complete phenomenological model of a stellar convection zone must have a dual character: it should include both the advective transport by a collection of diving plumes and the turbulent diffusion of the interstitial medium, whose stratification is not strictly adiabatic. We feel that it would be worthwhile to build such a model, in order to test its properties. In the meanwhile, the mixing length approach is better suited to deal with the upper boundary layer, even though it depends on a free parameter.

An important feature of the solar convection zone, which has been ignored in this paper, is its rotation. Since the ambiant angular momentum will be entrained into the plumes, these will spin up with depth, and they will concentrate both vorticity and helicity. Owing to their robustness, such vortices could well be the cause of the observed differential rotation, especially of its peculiarity of being quasi-independent of depth (Brown et al. 1989; Dziembowski et al. 1989). Furthermore, we can easily imagine that plumes play a major role in the generation of magnetic fields. Since they are places of high helicity they will also be the seat of strong alpha-effect. Recent dynamo simulations of Brandenburg et al. (1993) have shown that laminar diving plumes were indeed playing an important part in the field generation and its storage at the base of the convective layer. Such a role is also expected for the turbulent plumes.

These aspects, which are of great importance for stellar activity, will be addressed in a future paper.

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Appendix A

A.1. Flux equations for plumes dynamics

We give here the governing equations (10) in the form where they have been integrated numerically.

The conservation of mass equation is given by

$$\left(\frac{\xi+1}{2\xi+1}\right)\frac{dM}{d\zeta} - \frac{M}{(2\xi+1)^2}\frac{d\xi}{d\zeta} = \frac{4\alpha}{3}\sqrt{\frac{3P\zeta^q}{2\xi+1}}; \qquad \text{(A1)}$$

likewise, the momentum equation reads

$$\frac{dP}{d\zeta} = \Gamma(\xi-1)M^2/P, \qquad \text{(A2)}$$

while the energy equation becomes:

$$\xi^2[3u^2/2\Gamma - 2\zeta + 4L/M] + \xi[u^2/2\Gamma + \zeta + 2L/M] + \zeta = 0. \qquad \text{(A3)}$$

By differentiation we get

$$-\left[\frac{3\xi+1}{2\xi+1}\frac{u^2}{\Gamma M} + \frac{2L}{M^2}\right]\frac{dM}{d\zeta} + \left[\frac{u^2}{(2\xi+1)^2 2\Gamma} - \frac{\zeta}{\xi^2}\right]\frac{d\xi}{d\zeta} = \left(1-\frac{1}{\xi}\right) - \frac{3\xi+1}{2\xi+1}\frac{P}{\Gamma M^2}\frac{dP}{d\zeta} \qquad \text{(A4)}$$

here $u = P/M$ is the velocity. Equation (A4) combined with (A1) and (A2) yields the evolution of M.

A.2. Merging of two plumes

We consider two diving plumes in the self-similar regime whose velocities are expressed as

$$V_{z,1} = V_1\zeta^{-n}$$

$$V_{z,2} = V_2\zeta^{-n}.$$

They both induce an radial accretion flow

$$V_{r,12} = -\alpha\beta_0 V_1\zeta^{1-n}/x_{12}(\zeta)$$

$$V_{r,21} = -\alpha\beta_0 V_2\zeta^{1-n}/x_{12}(\zeta),$$

where $V_{r,12}$ is the radial velocity imposed by plume 1 on plume 2 and vice-versa for $V_{r,21}$, β_0 is the half-angle of the cone and x_{12} is the distance between the axes of the two plumes. If x is the coordinate along the line joining their centers and x_2 the position of the second plume, the trajectory of this plume verifies

$$\frac{dx_2}{d\zeta} = -\frac{\alpha\beta_0\zeta}{x_{12}}\left(\frac{V_1}{V_2}\right)$$

while for the other plume we have

$$\frac{dx_1}{d\zeta} = -\frac{\alpha\beta_0\zeta}{x_{12}}\left(\frac{V_2}{V_1}\right)$$

The distance between the two plumes $x_{12} = x_2 - x_1$ is a solution of

$$\frac{dx_{12}}{d\zeta} = -\frac{\alpha\beta_0\zeta}{x_{12}}\left(\frac{V_1}{V_2}+\frac{V_2}{V_1}\right) \quad \text{(A5)}$$

If d is the initial distance between the plumes, then we have

$$x_{12} = \sqrt{d^2 - \alpha\beta_0\left(\frac{V_1}{V_2}+\frac{V_2}{V_1}\right)\zeta^2} \quad \text{(A6)}$$

$x_{12} = 0$ gives the depth at which the two plumes merge:

$$\zeta_m = \frac{d}{\sqrt{\alpha\beta_0\left(V_1/V_2 + V_2/V_1\right)}} \quad \text{(A7)}$$

which yields (37) when $V_1 = V_2$.

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