Astron. Astrophys. 259, 581–584 (1992)

ASTRONOMY AND ASTROPHYSICS

Ekman circulation and the synchronization of binary stars

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Received May 30, accepted December 13, 1991

Abstract. We show that large-scale flows driven by Ekman pumping in the spin-up/down of a tidally distorted star is not efficient enough to reduce the synchronization time. This latter time remains of the order of the viscous time, if the star is made of an incompressible viscous fluid.

The computation of the synchronization time scale of early-type binaries should follow the approach proposed by Zahn and recently improved by Rocca, and Goldreich and Nicholson.

Key words: hydrodynamics – stars: binaries: close – stars: imaging – stars: rotation

1. Introduction

Recently, Tassoul (1987), Tassoul & Tassoul (1990) proposed a new mechanism of tidal synchronization to explain the high degree of synchronism of early-type binaries. This mechanism is based on the Ekman advection of angular momentum. It is indeed a well-known result of rotating fluid theory that the spin-up (or down) of fluids in rigid containers is mainly achieved by the advection of fluid driven by Ekman layers (see Fig. 1). With such a mechanism, a fluid is spun up (or down) on a time scale much shorter than the viscous time scale.

In early-type binaries, the outer envelope is in radiative equilibrium and the viscosity therein is so small that the viscous time scale is much longer than the lifetime of the star. Synchronization of the rotation rates, thus, cannot be explained by a mere viscous damping of the fluid motions. To explain the observations which do show synchronism of that type of binaries, Tassoul (1987) invoked the above spin-up/down mechanism, which does indeed provide a much smaller time scale, compatible with the observations.

We shall show here that the mechanism proposed by Tassoul does not, in fact, exist. Indeed, the stress-free boundary conditions met by the fluid of a star imply that no momentum is transferred from the "wall" to the fluid, contrary to what happens when the fluid is contained in a rigid container.

In Sect. 2, using the case of an incompressible star, we show how the mechanism of spin-up works with stress-free boundary conditions. Then (Sect. 3) we consider a more realistic model of an early-type binary and show how density variations in the envelope modify the results. Finally, we discuss the possible mechanism of synchronization of early-type binaries.

2. The spin-up mechanism of an incompressible "star"

For the sake of simplicity, we shall first consider the case of an incompressible "star" (i.e. a self-gravitating ellipsoid of viscous constant-density fluid) in the tidal potential of a point mass companion[1]. The shape of the star is approximately a non-axisymmetric ellipsoid elongated along the line of centers of the system. We shall admit that initially the star is not strictly synchronized with the orbital motion of the companion so that in a frame corotating with the orbital rotation, there is a weak azimuthal flow. We assume the rotation axis of the star to be perpendicular to the orbital plane. The problem is then to determine the time scale on which this flow is damped out. Let us denote the angular velocity of the frame by Ω, the mean radius of the star by R and the kinematic viscosity of the fluid by ν. If R is the length scale and $(2\Omega)^{-1}$ the time scale, then the (linearized) dimensionless equations of the flow are:

$$\frac{\partial \boldsymbol{u}}{\partial \tau}+\boldsymbol{e}_z\times\boldsymbol{u}=-\nabla p+E\Delta\boldsymbol{u},$$

$$\operatorname{div}\boldsymbol{u}=0, \qquad (1)$$

where $E=\nu/2\Omega R^2$ is the Ekman number and p a scalar function containing all the potentials acting on the fluid.

This system should be completed by boundary conditions taken on the free surface of the star. Strictly speaking the surface of the star is modified by the flow itself; however, since we are working at linear approximation, we may neglect this extra deformation and assume that the surface of the star is its equilibrium one. The boundary conditions on the velocity are, thus,

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,$$

$$\boldsymbol{n}\times([\sigma]\boldsymbol{n})=\boldsymbol{0}, \qquad (2)$$

where $\boldsymbol{n}$ is the outer normal of the surface of the star and $[\sigma]$ is the viscous stress tensor. These boundary conditions express that the velocity field is tangent to the surface and that no tangential stress is applied on the surface of the star.

To solve this problem one could use the method developed in Rieutord (1987, 1991). However, for deriving time scales the approach originally proposed by Greenspan (1969) is perfectly sufficient. The method consists in using an asymptotic expansion

[1] This is also the case considered by Tassoul & Tassoul (1990).

of the velocity using $\sqrt{E}$ as a small parameter and in decomposing the velocity in its interior part and boundary layer part:

$$\boldsymbol{u}=\boldsymbol{u}_0+\boldsymbol{b}_0+E^{1/2}(\boldsymbol{u}_1+\boldsymbol{b}_1)+E(\boldsymbol{u}_2+\boldsymbol{b}_2)+\cdots$$

where $\boldsymbol{b}$ stands for the boundary layer part of the flow while $\boldsymbol{u}$ is the interior part. $\boldsymbol{u}_0$ is usually called the geostrophic flow (see below). It is this flow whose damping rate we are looking for[2]. Its amplitude is of the order of $(\Omega_s-\Omega)R$ if Ω_s is the mean angular velocity of the star.

In order to present the mathematics as clearly as possible we shall use a set of ellipsoidal coordinates (ξ_1, ξ_2, ξ_3) adapted to the geometry of the star. ξ_1 is the equivalent of the radial coordinate in a system of spherical coordinate, and $\xi_1=1$ is the equation of the surface of the star.

2.1. *The geostrophic flow*

Let us first recall that the existence of the geostrophic flow depends on the shape of the container: a geostrophic flow can exist only if the container allows the existence of closed geostrophic contours [see Greenspan (1969) for definition]. For the ellipsoid, such contours are ellipses which are indeed closed curves.

This flow is then supposed quasi-steady, i.e., its time scale is much longer than the period of rotation. $\boldsymbol{u}_0$ is then the solution of

$$\begin{aligned}&\boldsymbol{e}_z\times\boldsymbol{u}_0=-\nabla P_0,\\ &\mathrm{div}\,\boldsymbol{u}_0=0,\\ &\boldsymbol{u}_0\cdot\boldsymbol{n}=0 \qquad \text{on} \quad \xi_1=1.\end{aligned} \tag{3}$$

Greenspan (1969) gave the solution of this system as

$$\boldsymbol{u}_0=\left(\frac{\partial}{\partial h}P_0(h, E^{1/2}\tau)\right)\boldsymbol{n}_b\times\boldsymbol{n}_t, \tag{4}$$

where the following notation has been introduced: The surface of the star is represented by two equations: $z=f(x, y)$ for the part above the equatorial plane and $z=-g(x, y)$ for the part below. $\boldsymbol{n}_b=\nabla(z-f(x, y))$ and $\boldsymbol{n}_t=-\nabla(z+g(x, y))$ are vectors normal to the top and the bottom surfaces, respectively. $h=f+g$ is the total height between the top and bottom surfaces.

Equation (4) shows that $\boldsymbol{u}_0$ is a two-dimensional flow in agreement with the Taylor–Proudman theorem. In this expression, the evolution time scale of $\boldsymbol{u}_0$ was assumed $\mathcal{O}(E^{-1/2})$ (as it was the case for Greenspan); we show below (Sect. 2.3) that it is longer, in fact $\mathcal{O}(E^{-1})$.

2.2. *The boundary layer flow*

The next step is to compute $\boldsymbol{b}_0$ so that $\boldsymbol{u}+\boldsymbol{b}$ matches the boundary conditions $\boldsymbol{n}\times([\sigma]\boldsymbol{n})=\boldsymbol{0}$ to zeroth order. The equation for $\boldsymbol{b}_0$ is given in Greenspan (1969) [Eq. (2.6.11)]:

$$\frac{\partial^2}{\partial\zeta^2}(\boldsymbol{n}\times\boldsymbol{b}_0+\mathrm{i}\boldsymbol{b}_0)=\mathrm{i}(\boldsymbol{n}\cdot\boldsymbol{e}_z)(\boldsymbol{n}\times\boldsymbol{b}_0+\mathrm{i}\boldsymbol{b}_0), \tag{5}$$

where ζ is the stretched normal coordinate:

$$\zeta=(1-\xi_1)/\sqrt{E}.$$

[2] For simplicity we shall suppose that no inertial waves are excited.

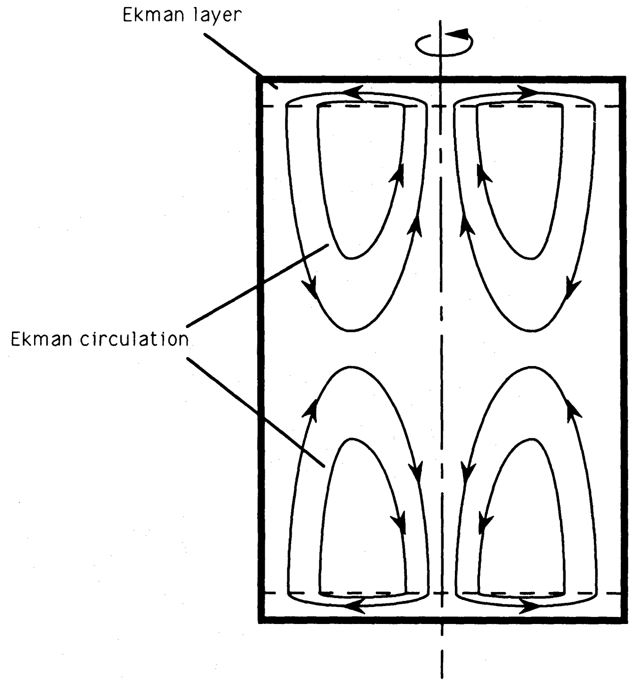

Fig. 1. Schematic view of the Ekman circulation resulting from the spin-up of a fluid inside a cylindrical container. The width of the boundary layer is not to scale

Note that Eq. (5) results from a combination of projections of the momentum equations in a plane tangent to the surface boundary and where only the depth dependence of the solution has been retained. The solution of Eq. (5) is

$$\boldsymbol{n}\times\boldsymbol{b}_0+\mathrm{i}\boldsymbol{b}_0=(\boldsymbol{n}\times\boldsymbol{b}_0+\mathrm{i}\boldsymbol{b}_0)_0\exp[-(\mathrm{i}\boldsymbol{n}\cdot\boldsymbol{e}_z)^{1/2}\zeta]. \tag{6}$$

The value of $\boldsymbol{b}_0$ at $\zeta=0$ must be such that

$$\begin{aligned}&\sigma_{12}(\boldsymbol{u}_0)+\sigma_{12}(\boldsymbol{b}_0)=0,\\ &\sigma_{13}(\boldsymbol{u}_0)+\sigma_{13}(\boldsymbol{b}_0)=0,\end{aligned} \tag{7}$$

where σ_{12} and σ_{13} are the two components of the tangential viscous stress.These quantities may be written as:

$$\begin{aligned}&\sigma_{12}(\boldsymbol{v})=\frac{\partial v_1}{\partial\xi_2}+\frac{\partial v_2}{\partial\xi_1}+\text{curvature} \quad \text{terms},\\ &\sigma_{13}(\boldsymbol{v})=\frac{\partial v_1}{\partial\xi_3}+\frac{\partial v_3}{\partial\xi_1}+\text{curvature} \quad \text{terms}.\end{aligned} \tag{8}$$

From these expressions and the one of $\boldsymbol{b}_0$ [Eq. (6)], it is easy to see that the amplitude of $\boldsymbol{b}_0$ is $\sqrt{E}$ smaller than that of $\boldsymbol{u}_0$. $\boldsymbol{u}_0$ being $\mathcal{O}(1)$, $\boldsymbol{b}_0$ is $\mathcal{O}(\sqrt{E})$ which is nothing but the next order of the expansion. Thus, we shall rename "$\boldsymbol{b}_0$" as $\boldsymbol{b}_1$, $\boldsymbol{b}_0$ being zero. We see here the main difference with problems dealing with rigid boundaries. When rigid boundaries are met, the boundary layer flow is of the same order as the geostrophic flow; with stress-free boundary conditions the boundary layer flow is just a small $\mathcal{O}(\sqrt{E})$ correction to the main flow; this is the crucial point. We may note that at this stage no Ekman circulation has as yet appeared. We show below that such circulation is described by the next order of the expansion.

2.3. Evolution time scale of the geostrophic flow

Before proceeding further, we discuss the evolution time scale of $\boldsymbol{u}_0$. We assumed above that this time scale was $\mathcal{O}(E^{-1/2})$ which is the classic spin-up time scale of fluids in rigid containers. We shall show now that such an assumption implies that $\boldsymbol{u}_0$ is independent of time.

To demonstrate this point we first write

$$\frac{\partial \boldsymbol{u}_0}{\partial \tau}=E^{1/2}\frac{\partial \boldsymbol{u}_0}{\partial \tau'}, \qquad \tau'=\sqrt{E}\tau$$

as done by Greenspan. $\partial \boldsymbol{u}_0/\partial \tau'$ appears in the equation for $\boldsymbol{u}_1$ as:

$$\frac{\partial \boldsymbol{u}_0}{\partial \tau'}+\boldsymbol{e}_z\times\boldsymbol{u}_1=-\nabla P_1,$$

$$\operatorname{div}\boldsymbol{u}_1=0,$$

$$(\boldsymbol{u}_1+\boldsymbol{b}_1)\cdot\boldsymbol{n}=0 \qquad \text{on} \quad \xi_1=1. \tag{9}$$

Following Greenspan [see his Eqs. (2.6.14)–(2.6.18)], we get

$$\int_{\mathscr{S}_t}\boldsymbol{n}_t\cdot\boldsymbol{u}_1\,\mathrm{d}\Sigma_t=\oint_{\mathscr{C}} f\frac{\partial \boldsymbol{u}_0}{\partial \tau'}\cdot\mathrm{d}\boldsymbol{s}+\oint_{\mathscr{C}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s},$$

$$\int_{\mathscr{S}_b}\boldsymbol{n}_b\cdot\boldsymbol{u}_1\,\mathrm{d}\Sigma_b=\oint_{\mathscr{C}} g\frac{\partial \boldsymbol{u}_0}{\partial \tau'}\cdot\mathrm{d}\boldsymbol{s}-\oint_{\mathscr{C}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{s}, \tag{10}$$

where $\mathscr{C}$ denotes a closed geostrophic contour which surrounds the surfaces $\mathscr{S}_t$ and $\mathscr{S}_b$, respectively, on the top and bottom boundaries of the volume.

From Eq. (9) and the definition of $\boldsymbol{b}_1$ we have

$$\boldsymbol{n}_t\cdot\boldsymbol{u}_1=\boldsymbol{n}_b\cdot\boldsymbol{u}_1=0$$

so that

$$\oint_{\mathscr{C}}(f+g)\frac{\partial \boldsymbol{u}_0}{\partial \tau'}\,\mathrm{d}\boldsymbol{s}=0 \Leftrightarrow \oint_{\mathscr{C}}\frac{\partial \boldsymbol{u}_0}{\partial \tau'}\,\mathrm{d}\boldsymbol{s}=0.$$

From Eq. (4) one now finds

$$\frac{\partial \boldsymbol{u}_0}{\partial \tau'}=0.$$

This shows that, on the assumed time scale, $\boldsymbol{u}_0$ is constant or, equivalently, that the time scale of evolution is longer and belongs to the next order $\mathcal{O}(E^{-1})$ and $\boldsymbol{u}_1$ is not excited. The correct scaling is thus,

$$\frac{\partial \boldsymbol{u}_0}{\partial \tau}=E\frac{\partial \boldsymbol{u}_0}{\partial \tau''}; \qquad \tau''=E\tau,$$

which means that $\boldsymbol{u}_0$ is damped on the viscous time scale. To proceed, we need now to investigate the $\mathcal{O}(E)$ quantities.

2.4. The Ekman circulation

The Ekman circulation is driven by the boundary layer flow $\boldsymbol{b}_1$. Indeed, $\boldsymbol{b}_1$ does not, in general, fulfill the continuity equation. However, this flow has been completely determined and we cannot change it. The only possibility is to add a flow of the next order ($\boldsymbol{b}_2$ here), such that $\operatorname{div}(\boldsymbol{b}_1+\boldsymbol{b}_2)=0$. Since, physically, $\boldsymbol{b}_1$ is a horizontal flow (parallel to the boundary), to insure mass conservation $\boldsymbol{b}_2$ must be a vertical flow that is a flux of matter between the boundary layer and the interior fluid. In addition, $\boldsymbol{b}_2$ being a boundary layer flow, it has the same exponential dependence with respect to ξ_1 as $\boldsymbol{b}_1$ in Eq. (6). Writing $\boldsymbol{b}_2=b_2\boldsymbol{n}$, $\boldsymbol{n}$ being the outer normal of the surface, b_2 is given by:

$$\frac{\partial b_2}{\partial \zeta}=\lambda b_2+\operatorname{div}\boldsymbol{b}_1, \tag{11}$$

where λ is an $\mathcal{O}(1)$ coefficient.

This flow does not satisfy the first boundary condition $\boldsymbol{u}\cdot\boldsymbol{n}=0$ and, hence, drives an interior flow $\boldsymbol{u}_2$ which satisfies the inviscid equations

$$\frac{\partial \boldsymbol{u}_0}{\partial \tau''}+\boldsymbol{e}_z\times\boldsymbol{u}_2=-\nabla P_2,$$

$$\operatorname{div}\boldsymbol{u}_2=0,$$

$$\boldsymbol{u}_2\cdot\boldsymbol{n}=-\boldsymbol{b}_2\cdot\boldsymbol{n} \qquad \text{on} \quad \xi_1=1. \tag{12}$$

$\boldsymbol{u}_2$ is usually referred to as the secondary circulation or Ekman circulation and $\boldsymbol{b}_2$ is the Ekman pumping. We see that it is a second-order correction to the main flow.

We could now continue by transforming the system (12) into a system similar to (10) ($\boldsymbol{u}_2$ replacing $\boldsymbol{u}_1$), and derive the time differential equation for $\boldsymbol{u}_0$. However, we already know the time scale of evolution; this further calculation would just give the detailed dependence of the damping rate on the geometry of the container.

At this point, one should note that the above calculations apply to a broad class of containers since the precise reference to the ellipsoidal geometry was never used. The only constraint is that the shape of the container must allow closed geostrophic contours.

2.5. Summary and conclusions

In view of the above demonstration the complete solution for the spin-up of an incompressible star should be written as

$$\boldsymbol{u}=\boldsymbol{u}_0+E^{1/2}\boldsymbol{b}_1+E(\boldsymbol{u}_2+\boldsymbol{b}_2)+\cdots. \tag{13}$$

We see here the difference with the case of a fluid in a container with rigid boundaries. When rigid boundaries are used the conditions (7) should be replaced by

$$\boldsymbol{u}_{\text{tangential}}=\boldsymbol{0}$$

or

$$\boldsymbol{u}_0+\boldsymbol{b}_0=\boldsymbol{0}.$$

Thus, in this case $\boldsymbol{b}_0$ is nonzero. Then $\boldsymbol{b}_1$ is the Ekman pumping and $\boldsymbol{u}_1$ is the Ekman circulation. The expansion of the solution in powers of $\sqrt{E}$ is, thus,

$$\boldsymbol{u}=\boldsymbol{u}_0+\boldsymbol{b}_0+E^{1/2}(\boldsymbol{u}_1+\boldsymbol{b}_1)+\cdots.$$

The solution with rigid boundaries, thus, shows that the spin-up of the interior fluid (that is most of the fluid) will be achieved by the Ekman circulation. This flow is indeed advecting the angular momentum from the container to the interior. Its role is, thus, crucial in the spin-up of fluids in containers: it makes the time scale $\mathcal{O}(E^{-1/2})$ which is much shorter than the viscous one $\mathcal{O}(E^{-1})$ (in dimensionless units). In contrast, when the container is not rigid but is just the gravitational potential of the star, we see that the Ekman circulation is reduced by a factor $\sqrt{E}$; the time scale is then lengthened by $1/\sqrt{E}$ and turns out to be of the order of the viscous time scale. In other words, walls for which

$\boldsymbol{n} \cdot \Omega \neq 0$, impose their (constant) vorticity to the fluid which reacts by forming a boundary layer. These walls, thus, impose an important horizontal stress, which disappears when stress-free boundary conditions are used.

To be complete, we now give the time scale for the spin-up/down, assuming the elongation of the ellipsoid is ε. From Eq. (1) it is straigthforward to derive the equation of kinetic energy:

$$\frac{\partial}{\partial \tau}\left(\int_v \tfrac{1}{2} u^2 \,\mathrm{d}V\right) = -E \int_v (\mathscr{D}\boldsymbol{u})^2 \,\mathrm{d}V,$$

where $E(\mathscr{D}\boldsymbol{u})^2$ represents the local viscous dissipation, $E_c = \int_v \frac{1}{2} u^2 \,\mathrm{d}V$ is the kinetic energy of the star and $\boldsymbol{u}$ is given by (13). Each component of $\boldsymbol{u}$ can be expanded in powers of ε:

$$\boldsymbol{u}_0 = \boldsymbol{u}_{00} + \varepsilon \boldsymbol{u}_{01} + \cdots.$$

Since $\boldsymbol{u}_{00}$ is nothing but solid rotation, $(\mathscr{D}\boldsymbol{u}_{00}) = 0$, it follows that

$$\frac{1}{E_c}\frac{\partial E_c}{\partial \tau} = -E\varepsilon^2 K,$$

where K is an $\mathscr{O}(1)$ constant. The associated time scale is, thus,

$$t_{\mathrm{sync}} = \frac{(2\Omega)^{-1}}{E\varepsilon^2} \sim \frac{R^2}{\nu}\left(\frac{a}{R}\right)^6 \tag{14}$$

as previously obtained by Zahn (1977) (a is the distance between centers of mass).

3. Applications to early-type binaries

We wish now to know how the preceding results are modified when more realistic models of early-type binaries are considered.

The main difference with the incompressible case is that we need to take into account the density and viscosity variations in the star. Equations of motion may be written

$$\frac{\partial \boldsymbol{u}}{\partial \tau} + \boldsymbol{e}_z \times \boldsymbol{u} = -\nabla p + E \boldsymbol{F}_{\rho,\nu}(\boldsymbol{u}),$$

$$\mathrm{div}\, \rho \boldsymbol{u} = 0, \tag{15}$$

where we assumed a barotropic equation of state for gas of the star (thus, all the potentials are gathered in the function p); ρ is the density, E an Ekman number and $\boldsymbol{F}_{\rho,\nu}$ a second-order linear operator depending on the functional form of ρ and ν. Boundary conditions are still Eq. (2).

Essentially the same analysis can be conducted on this system. The main difference comes from the width of the surface boundary layer. Indeed, in such stars, the envelope is radiative and can be approximated by an $n=3$ polytrope in which the (radiative) dynamical viscosity behaves as $\mu = \mu_0(1-\xi_1)$ close to the surface, while the density behaves as $\rho = \rho_0(1-\xi_1)^3$. The fact that both viscosity and density vanish at the surface where the boundary conditions are imposed, makes the operator $\boldsymbol{F}_{\rho,\nu}$ singular there and the boundary layer wider. We refer the reader to Tassoul & Tassoul (1982) for the boundary layer analysis. The width of the boundary layer is (see Tassoul 1987)

$$\delta = \left(\frac{\mu_0}{\rho_0 \Omega R^2}\right)^{1/4},$$

which can be evaluated as

$$\delta/R = 3.4\ 10^{-5} P_{\mathrm{d}}^{1/4} \left[\frac{(L/L_\odot)}{(M/M_\odot)}\right]^{1/4},$$

where L and M are, respectively, the luminosity and the mass of the star, P_{d} is the orbital period in days. This formula shows that the boundary layer is extremely thin; in fact, it is of the order of the thickness of the atmosphere of the star. However, as in the incompressible case, neither it nor the associated Ekman circulation play any part in the synchronization process.

4. Conclusions

We have shown that, in the process of synchronization of binary stars, boundary layers play a very minor part contrary to what happens for fluids in rigid containers. Ekman circulation is a negligible part of the flow.

Synchronization results from the dissipation occurring in the main flow. An evaluation of the synchronization time scale due to viscosity is given by Eq. (14) but one should be aware that all the mechanisms of dissipation ought to be taken into account to get a reliable time scale. The works of Zahn (1975, 1977), Rocca (1987, 1989), and Goldreich & Nicholson (1989a, b) give a more complete approach of this problem, and we refer the reader to these articles for the latest developments.

Acknowledgements. I thank particularly Jean-Paul Zahn for the enlightening discussions we had together and his kind reading of the manuscript. I also thank Jean-Louis Tassoul for useful criticism.

References

Goldreich P., Nicholson P., 1989a, ApJ 342, 1075
Goldreich P., Nicholson P., 1989b, ApJ 342, 1079
Greenspan H.P., 1969, The Theory of Rotating Fluids. Cambridge University Press, Cambridge, UK
Rieutord M., 1987, Geophys. Astrophys. Fluid Dyn. 39, 163
Rieutord M., 1991, Geophys. Astrophys. Fluid Dyn. 59, 185
Rocca A., 1987, A&A 175, 81
Rocca A., 1989, A&A 213, 114
Tassoul J.-L., 1987, ApJ 322, 856
Tassoul J.-L., Tassoul M., 1982, ApJS 49, 317
Tassoul M., Tassoul J.-L., 1990, ApJ 359, 155
Zahn J.-P., 1975, A&A 41, 329
Zahn J.-P., 1977, A&A 57, 383