Mon. Not. R. astr. Soc. (1987) **227**, 295–314

Tidal heating in close binary stellar systems

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Accepted 1987 February 2. Received 1987 January 5; in original form 1985 November 7

Summary. Tidal heating of a low-mass star in a close binary system, resulting from the conjugate effect of angular momentum loss and tidal action, is investigated via a detailed study of the flow inside the secondary.

In the case of cataclysmic binaries we find that viscous dissipation is at most $10^{-3}\times$ the nuclear luminosity of the star; we thus confirm by a more exact model the result of Verbunt & Hut. It is shown, however, that the dissipation is very sensitive to the turbulent viscosity in the envelope of the secondary. The lack of reliable theory for such a viscosity, especially in presence of fast rotation, makes the estimation of total dissipation rather uncertain. This uncertainty leaves the possibility open that when the evolution is driven by magnetic braking, tidal dissipation may reach the luminosity of the star.

We also consider the case of very close pairs of white dwarfs. We show that such pairs, which are thought to be the progenitors of Type I Supernovae (Webbink), may dissipate a power as large as $10^{38}\,\mathrm{erg\,s^{-1}}$, provided that they reach synchronization; such a heating will strongly modify the conditions in which the nuclear explosion starts. We show that the detection of such objects is quite difficult.

1 Introduction

Close binary systems which are discussed in this paper are: cataclysmic variables, low mass X-ray binaries and very close pairs of white dwarfs.

The common property of these objects is that they lose angular momentum which in general brings them even closer together. For short periods ($P \leqslant 3$ hr) this is the effect of gravitational radiation; for longer periods, relevant to cataclysmic variables and low-mass X-ray binaries, it could be the consequence of the emission of a stellar wind by the low-mass main-sequence companion (Verbunt & Zwaan 1981).

Thus in the course of their evolution, the orbital period of these systems varies. In cataclysmic binaries or low-mass X-ray binaries, the low-mass companion (hereafter referred to as the secondary) is in general subjected to a strong tidal interaction which tends to synchronize its proper rotation with the orbital rotation. The consequence is that the variations of orbital period imply variations of the spin rotation; as the star is a fluid, variations of its spin rotation are realized through fluid flows. These flows may have important consequences in the evolution of the system, in particular because of the heat they can release in the secondary through viscous dissipation.

Viscous heating has already been considered in the literature in connection with magnetic braking acting in cataclysmic binaries (Eggleton 1983; Verbunt & Hut 1983). In this case, the mechanism of heating is little more complicated than the case in which gravitational radiation alone shrinks the orbit. When magnetic braking is at work in a cataclysmic binary, the secondary star is supposed to emit a strong stellar wind which is forced to corotate with the star by a magnetic field. Such a wind extracts angular momentum from the star and thus exerts a torque on it. The star is spun-down but tidal forces try to maintain corotation with the orbital motion. In a steady regime the tidal torque and the magnetic braking torque are almost in balance. However, the tidal torque is also exerted on the orbit, which in turn loses angular momentum. The consequences are that the orbit changes, the orbital rotations accelerates and an additional torque acting on the star appears. This last torque is the inertial torque due to the change of angular velocity of the tidal field. Therefore when magnetic braking drives the evolution of the system, the secondary is subject to the action of three torques [(i) magnetic braking torque, (ii) tidal torque and (iii) inertial torque] which trigger flows inside the star. When the evolution is driven by gravitational radiation only, angular momentum is lost directly from the orbit and only torques (ii) and (iii) remain.

Verbunt & Hut (1983) considered the dissipation in a system losing angular momentum via magnetic braking. They included only the action of torques (i) and (ii), which is correct since torque (iii) is much smaller than the stellar wind torque. To derive the dissipation, they first assumed that the flow is that of a solid body rotation; then using energy and angular momentum conservation laws, they derived that in a steady regime the power of the torque is the power dissipated in the fluid, i.e.

$$L_{\text{diss}}=(\omega_0-\omega_*)\,|\dot{J}_0|$$

where L_{diss} is the dissipation, $\dot{J}_0$ is the time derivative of the angular momentum of the orbital motion, ω_0 and ω_* the angular velocity of the orbital and spin rotation of the secondary. Then to estimate the asynchronicity $(\omega_0-\omega_*)$, they equate the magnetic braking and tidal torques and obtain:

$$\frac{\omega_0-\omega_*}{\omega_*}=\frac{\tau_{\text{syn}}}{\tau_{\text{BR}}}$$

where τ_{syn} and τ_{BR} are respectively the time-scales of synchronization and magnetic braking.

This approach of the problem *a priori* raises one question: how far is the assumption of solid body rotation justified? Such a velocity field cannot dissipate any energy in the star, nor can it meet the boundary conditions ($v_\perp=0$ at the surface) (see also Section 3.3). One may also question the use of τ_{syn} since this time-scale corresponds to a tidal torque produced by a transient flow, not by a steady flow as it is actually occurring.

One has therefore to consider the full hydrodynamics of these flows. Here we study the problem in a frame corotating with the tidal field. In such a frame, the flow is shown to be a solution of steady linear equations which can be solved by developing the fields into spherical harmonics.

In Section 2 of this paper, we derive the basic equations of the problem; we define the state of quasi-synchronism of a secondary and discuss the linearization of the equations using the simple model of a two-dimensional flow.

In Section 3, we work out the case of a spin-up flow in an homogeneous star (i.e. of constant density and viscosity), for which exact solutions can be found. These solutions are then used to show that, in the asymptotic case of very low viscosities which is relevant to cataclysmic binaries, the solutions can be simplified. This allows us to treat the realistic case of fluids in which the density and viscosity are not constant.

Section 4 is devoted to the applications of the method. We first consider the case of cataclysmic binaries, for which we show that dissipation reaches a maximum of 10^{30} erg s^{-1}. This is $\sim 50\times$ larger than the value computed using the Verbunt & Hut formula. In order to calculate the dissipation, we have used the standard mixing length theory, which neglects the effects of rotation although they can be very important. It appears that in principle, the dissipation can reach the luminosity of the star.

We then discuss a possible application of our results to very close binary white dwarfs which are thought to be the progenitors of Type I Supernovae (Webbink 1984). We show that under some conditions, very high dissipation rates may be reached.

2 Basic equations of the problem

2.1 HYDRODYNAMICAL EQUATIONS

In a frame corotating with the orbital motion, the mass and momentum conservation equations are:

$$\partial\varrho/\partial t + \mathrm{div}\,\varrho\mathbf{v} = 0; \qquad (2.1)$$

$$\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + 2\boldsymbol{\omega}\times\mathbf{v} + [\dot{\boldsymbol{\omega}}_{\mathrm{orb}} + \mathbf{B}(\mathbf{r})]\times\mathbf{r} = -\nabla W + \varrho^{-1}\mathbf{F}_{\mathrm{visc}}. \qquad (2.2)$$

In the momentum equation, we introduce the force field $\mathbf{B}(\mathbf{r})\times\mathbf{r}$ which represents the magnetic braking torque exerted on the star. Such a force field should be specified by a model of magnetic braking; as a first attempt we shall assume that $\mathbf{B}(\mathbf{r})$ can be written $B\mathbf{k}$ where B is a constant computed from the whole torque and $\mathbf{k}$ the unit vector along the rotation axis. Note that in such a case, we can use an 'effective' inertial acceleration $\dot{\omega}_{\mathrm{eff}} = \dot{\omega}_{\mathrm{orb}} + B$; when gravitational radiation is driving alone the evolution of the system, $\dot{\omega}_{\mathrm{eff}}$ reduces to $\dot{\omega}_{\mathrm{orb}}$ (hereafter $\dot{\omega}$ will refer to $\dot{\omega}_{\mathrm{eff}}$).

As we are dealing with low mass stars whose structure can be well approximated by a composite polytrope ($n=3$ and $n=3/2$), the pressure term $\varrho^{-1}\nabla P$ has been inserted in W, which is the sum of all the potentials acting within the star. More precisely, this potential function reads:

$$W = \Phi + U_{\mathrm{m}} + U_{\mathrm{c}} + f(\varrho), \qquad (2.3)$$

where Φ is the self-gravitation potential. Solution of Poisson's equation is:

$$\Delta\Phi + 4\pi G\varrho = 0 \qquad (2.4)$$

where $f(\varrho)$ is the pressure potential, while U_{m} and U_{c} are respectively the tidal and the centrifugal potential:

$$U_{\mathrm{m}} = -\frac{\omega^2 r^2}{2(1+q)}(1 - 3\sin^2\theta\cos^2\phi), \qquad (2.5)$$

$$U_{\mathrm{c}} = \tfrac{1}{2}\omega^2 r^2 \sin^2\theta, \qquad (2.6)$$

(q is the mass ratio of the secondary to the primary, $q \leqslant 1$).

The set of equations (2.1), (2.2) has to be completed by boundary conditions. If

$$S(\mathbf{r}, t) = 0 \qquad (2.7)$$

is the equation of the surface of the star, these conditions read:

$$\partial S/\partial t+\mathbf{v}\cdot\nabla S=0, \tag{2.8}$$

$$(\boldsymbol{\sigma}-\mathbf{p})\,\nabla S=\mathbf{0}, \tag{2.9}$$

where $\mathbf{p}$ is the pressure tensor and $\boldsymbol{\sigma}$ is the viscous stess tensor which, for a Newtonian fluid of negligible bulk viscosity, reads:

$$\sigma_{ij}=\eta[\partial_i v_j+\partial_j v_i-2/3(\partial_k v^k)\,\delta_{ij}], \tag{2.10}$$

η being the dynamical viscosity.

Equation (2.8) expresses that no matter crosses the surface, while equation (2.9) expresses the continuity of the stress tensor across the surface of the star. It should also be noted that the actual surface of the star $S(\mathbf{r},\,t)$ is unknown and has to be determined when solving the problem.

The flow is now completely specified by equations (2.1), (2.2), (2.7), (2.8), (2.9). However, such a problem is of great complexity. To simplify it, we first assume that all transient flows have been damped and that the system is in a quasi-steady state. This is a natural assumption since transient flows are dampled in the synchronization time scale which is, in the cases we consider, much shorter than $\omega/\dot{\omega}$.

We show now that due to the weakness of the source term $\dot{\boldsymbol{\omega}}\times\mathbf{r}$ the problem is linear.

2.2 LINEARIZATION

We introduce the small parameter $\alpha=\dot{\omega}/\omega^2$ (which is the Rossby number of the flow), and expand the different physical quantities of the problem into powers of it. In astrophysical applications $\alpha\lesssim 10^{-10}$. We set

$$\mathbf{V}=\alpha\mathbf{V}_1+0(\alpha^2), \tag{2.11a}$$

$$W=W_e+\alpha W_1+0(\alpha^2), \tag{2.11b}$$

$$\varrho=\varrho_e+\alpha\varrho_1+0(\alpha^2), \tag{2.11c}$$

$$p=p_e+\alpha p_1+0(\alpha^2), \tag{2.11d}$$

where the index 'e' refers to the equilibrium configuration (when $\dot{\boldsymbol{\omega}}=\mathbf{0}$).

Inserting these developments into equations (2.1) and (2.2) and keeping the first order in α, we get simplified equations governing a linear stationary flow:

$$\operatorname{div}\varrho_e\mathbf{v}=0, \tag{2.12}$$

$$2\boldsymbol{\omega}\times\mathbf{v}+\dot{\boldsymbol{\omega}}\times\mathbf{r}=-\nabla W+\varrho_e^{-1}\mathbf{F}_{\text{visc}}. \tag{2.13}$$

Thus to the first order in α for the velocity field, we can use the equilibrium distribution of masses ϱ_e. As mentioned above, the surface of the secondary is modified by the flow of matter within it; however, as we may expect, to the first order in α, the boundary conditions can be taken on the equilibrium surface. Expanding the surface equation into powers of α, we can write:

$$S(r,\theta,\phi)=r-r_e(\theta,\phi)+\alpha r_1(\theta,\phi)+0(\alpha^2), \tag{2.14}$$

where $r=r_e(\theta,\,\phi)$ is the equation of the equilibrium shape (time dependence has been removed since we are looking for steady solutions). Such an expansion must also be performed in boundary conditions (2.8) and (2.9) and we easily get those met by the first-order approximation; they read:

$$\left\{\begin{array}{l}\mathbf{v}\cdot\mathbf{n}_e=0\\ \mathbf{n}_e\times\boldsymbol{\sigma}\mathbf{n}_e=\mathbf{0}\end{array}\right.\quad \text{at } r=r_e(\theta,\,\phi), \qquad (2.15),\ (2.16)$$

where $\mathbf{n}_e = \nabla[r - r_e(\theta, \phi)]$ is a vector normal to the equilibrium surface. Equation (2.16) is derived from (2.9) by eliminating the pressure; it expresses the lack of tangential stress on the surface.

The perturbation of the surface by the spin-up flow can be derived from the boundary condition on pressure; it gives:

$$r_1(\theta, \phi) = \left[(\sigma_1^{ri} n_c^i - p_l) \left(\frac{\partial p_e}{\partial r} \right)^{-1} \right]_{r=r_e}. \tag{2.17}$$

Note that in the case treated here, this equation is implicit for r_1, since the pressure p_1 also depends on the gravitational potential due to the modification of the surface.

It should also be noted that the true surface must really be very close to the equilibrium one: the distortion generated by the flow within the secondary must not exceed the tidal distortion, otherwise the hypothesis of quasi-synchronization breaks down. But we shall come back to this question below.

2.3 REFORMULATION

In view of the preceding simplifications we can now reformulate the problem in its essence.

The velocity field is governed by the mass conservation equation and the vorticity equation:

$$\operatorname{div} \varrho_e \mathbf{v} = 0, \tag{2.12}$$

$$\mathbf{curl}(2\boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r}) = \mathbf{curl}(\mathbf{F}_{\text{visc}}/\varrho_e). \tag{2.18}$$

This field satisfies the boundary conditions (2.15) and (2.16):

$$\begin{cases} \mathbf{v} \cdot \mathbf{n}_e = \mathbf{0} \\ \mathbf{n}_e \times \boldsymbol{\sigma}\, \mathbf{n}_e = \mathbf{0}. \end{cases}$$

This entirely specifies the velocity field generated by the spin-up of the star.

Such a state, described by equations (2.12), (2.18), (2.15) and (2.16), will be referred to as a state of quasi-synchronism. Quasi-synchronism will therefore describe situations where the surface of the star is very close to its instantaneous equilibrium one and where linear approximation is valid. We thus separate these states from those of (strong) asynchronism where the surface is much distorted and where non-linear terms are important.

Let us now specify the viscous dissipation function $\Phi(\mathbf{v})$ which can be computed from the velocity field. In the most general case this function reads:

$$\Phi(\mathbf{v}) = \eta/2[S_{ij}S^{ij} - 4/3(\partial_i v^i)^2] \tag{2.19}$$

where $\mathbf{S}$ is the shear stress tensor:

$$S_{ij} = \partial_i v_j + \partial_j v_i.$$

This function can also be written:

$$\Phi(\mathbf{v}) = \eta(\dot{\omega}/\omega)^2 \phi(\mathbf{r}, \eta, \varrho_e), \tag{2.20}$$

where ϕ is a dimensionless function.

This leads for the total dissipation in the whole star, to:

$$D = (\dot{\omega}/\omega)^2 \int_{(V)} \eta(\mathbf{r})\, \phi(\mathbf{r}, \eta, \varrho_e)\, d^3\mathbf{r}. \tag{2.21}$$

2.4 VALIDITY OF APPROXIMATIONS AND THE CONDITION OF QUASI-SYNCHRONISM

Before considering the general problem, we discuss the validity of the linearization in light of the very simple solutions which can be obtained when considering the two-dimensional flow in a 'flat star'. The velocity field is found to be (see Appendix A for derivation):

$$V_r=\frac{\dot{\omega}R^3}{4\nu\varepsilon}\left[\left(x-\frac{x^3}{3}\right)\sin 2\theta+O(\varepsilon)\right], \tag{2.22}$$

$$V_\theta=\frac{\dot{\omega}R^3}{4\nu\varepsilon}\left[-\frac{5x}{6\varepsilon}+\frac{1}{2}\left(x-\frac{2x^3}{3}\right)\cos 2\theta+O(\varepsilon)\right], \tag{2.23}$$

where ε is the tidal elongation of the secondary.

The most remarkable property of these solutions is that they are singular when $\varepsilon\rightarrow 0$. However, such a result is not a surprise: the solution we got here is a stationary solution; if the boundary is a sphere ($\varepsilon=0$) on which the fluid can slip ($\mathbf{v}\cdot\mathbf{n}=0$), nothing can remove from the fluid the angular momentum injected by the inertial 'rigid' torque $\dot{\boldsymbol{\omega}}\times\mathbf{r}$; the fluid will rotate faster and faster and no stationary state will be reached (or only as found, at infinite speed!).

As it can easily be guessed, the singular behaviour of solutions when $\varepsilon\rightarrow 0$ will set a limit of the range of admissible values of ε. This limit will appear when we require the non-linear terms to be small compared to other terms or when we require the deformation of the surface generated by the fluid motion to be much smaller than the tidal one.

In order to compute the distortion generated by the flow, we use equation (2.17), in which we report the solutions (2.22), (2.23). We obtain:

$$\frac{\delta r}{R}=\frac{\dot{\omega}R^3}{E\varepsilon^2 GM}$$

where $E=\nu/2\omega R^2$ is the Ekman number of the flow, R being the average radius of the star.

Such a distortion must be much smaller than the tidal one ε; using the expression $\varepsilon\approx\omega^2R^3/GM$ where M is the secondary mass, we get the limit on ε:

$$\varepsilon\gg\varepsilon_l=\sqrt{\frac{\alpha}{E}}. \tag{2.24}$$

It can easily be seen that this equality is the same as the one we would obtain by demanding non-linear terms to be smaller than the Coriolis one.

The meaning of this limiting value of ε is enlightened when considering the velocity of the flow at $\varepsilon=\varepsilon_l$:

$$V_\theta=-\frac{\dot{\omega}R^3}{\nu\varepsilon_l^2}\approx-\omega R,$$

i.e. the star does not rotate in a frame at rest! Equation (2.24) is therefore the condition for quasi-synchronization to be achieved.

In the case of cataclysmic binaries which we study below, this condition is always fulfilled since $\varepsilon_1\approx 10^{-3}$ and the elongation of a star filling its Roche lobe is about 10 per cent.

Another condition must also be satisfied, however, for linear calculations to be valid: the non-linear terms must also be small compared to the viscous one, otherwise the flow may be turbulent. When the viscosity is very small this condition is usually not satisfied. It is well known that in such circumstances (see e.g. Landau & Lifshitz 1959), the mean velocity field of the turbulent flow can be described as a laminar flow but with a turbulent viscosity which is such that the Reynolds number of this flow is the one at which instabilities arise, that is to say, very roughly, near Re = 5000. This new viscosity which depends only on the strength of the source term ($\dot{\boldsymbol{\omega}} \times \mathbf{x}$) and on the geometry of the 'container', can be easily computed; with the previous assumptions and in the two-dimensional case, one finds:

$$\nu_t = (\dot{\omega} R^4 / 4\varepsilon^2 \, \mathrm{Re})^{1/2}, \qquad \mathrm{Re} = 5000, \tag{2.25}$$

which leads to the Ekman number $E_t = (\alpha / 16\varepsilon^2 \, \mathrm{Re})^{1/2}$; this number is usually much larger than the 'limit' one $E_l = \alpha\varepsilon^{-2}$ which leads to desynchronization. However, if α is large enough or, which is more likely, if ε is small enough, these two numbers may equalize. In this case, the turbulent viscosity is not large enough and the secondary never reaches quasi-synchronism.

We now show that condition (2.24) also implies that the rate of change of spin kinetic energy $\dot{E}_k$ is an upper limit for the viscous dissipation D. D can be estimated using the 2D model:

$$D = \eta \left(\frac{\dot{\omega}}{\omega} \right)^2 \frac{KR^3}{(\varepsilon E)^2}$$

where K is a dimensionless constant which has been adjusted to the 3D case of the next section ($K = 0.0231$).

$\dot{E}_k$ expresses as:

$$\dot{E}_k = kMR^2 \omega \dot{\omega}, \tag{2.26}$$

where $k = 0.132$ for a $n = 3/2$ polytrope (*cf.* Cox & Giuli 1968).

Then:

$$\delta = D / \dot{E}_k \approx 0.1\alpha / (\varepsilon^2 E).$$

This clearly demonstrates that as long as quasi-synchronism is achieved $\delta < 1$. Therefore, the viscous dissipation never exceeds the rate of change of kinetic energy and reaches this limiting value when the viscosity is just able to keep quasi-synchronization. This result is very interesting since $\dot{E}_k$ depends on very few parameters (i.e. the momentum of inertia of the secondary, the orbital pulsation ω and $\dot{\omega}$); we shall use it in the case of turbulent spin-up flows to determine the point of synchronization of white dwarfs (see Section 4).

3 Computation of the spin-up flow

We now have to solve equations (2.12) and (2.18) with the boundary conditions (2.15) and (2.16). The spin-up problem has been considered by many authors in particular by Greenspan (1969), in the context of a general study of rotating fluids and by Clark *et al.* (1971) and Friedlander (1976), in relation to the solar spin-down problem. These problems are quite cumbersome. The classical strategy is to use the boundary layer theory and to analyse the flow in the different regions of the container (here the gravitational potential).

However, we do not use this technique, not only because of its complexity, but because of the type of boundary conditions which have never been considered before. We have therefore developed another technique which is based on an expansion of the velocity field into vectorial spherical harmonics. Such an approach allows us to find analytic solutions in the case of a homogeneous fluid (Section 3.1). These solutions are then used to investigate the behaviour of the flow in the inviscid limit and to find a way of computing asymptotic solutions (Section 3.2). We then use these solutions to compute the dissipation in a polytropic star (Section 3.3).

3.1 SOLUTIONS IN THE HOMOGENEOUS CASE

In the case of an homogeneous fluid, the equations of the velocity field simplifies into:

$$\operatorname{div} \mathbf{v} = 0, \tag{3.1}$$

$$\mathbf{curl}(2\boldsymbol{\omega}\times\mathbf{v}+\dot{\boldsymbol{\omega}}\times\mathbf{r}) = \nu\,\mathbf{curl}\,\Delta v \tag{3.2}$$

with boundary conditions (2.12) and (2.18).

Equations (3.1) and (3.2) can be made dimensionless by setting:

$$\mathbf{v} = \frac{\dot{\omega}R}{\omega}\mathbf{u}; \qquad \mathbf{x} = \mathbf{r}/R; \qquad \mathbf{k} = \boldsymbol{\omega}/\omega$$

which gives:

$$\operatorname{div} \mathbf{u} = 0, \tag{3.3}$$

$$\mathbf{curl}(2\mathbf{k}\times\mathbf{u}+\dot{\mathbf{k}}\times\mathbf{x}) = 2E\,\mathbf{curl}\,\Delta\mathbf{u} \tag{3.4}$$

E is the Ekman number; its very small value ($E \leqslant 10^{-5}$) shows the presence of a thin layer of width $R\sqrt{E}$ where the velocity field adjusts to the boundary conditions; we note here that boundary layers usually imply high shear stresses, and so high dissipation, but in our case of free surface, we require no tangential stress at the surface, so that this boundary layer will be, as we shall see, a region of weak dissipation.

To solve the system of partial differential equations we expand the velocity field into vectorial spherical harmonics:

$$\mathbf{u} = \sum_{l,m} u_m^l(x)\,\mathbf{R}_l^m + v_m^l(x)\,\mathbf{S}_l^m + w_m^l(x)\,\mathbf{T}_l^m. \tag{3.5}$$

See Appendix B for the definition of the fields $\mathbf{R}_l^m$, $\mathbf{S}_l^m$ and $\mathbf{T}_l^m$. Using equation (3.5), we obtain a set of differential equations for u_m^l, v_m^l, w_m^l for which the following solutions can be found:

$$u_m^l(x) = \sum_{i=|m|}^{+\infty} A_m^i U_{i,l}^m j_l(\mu_m^i x)/\mu_m^i x + \sum_{j=|m|,|m|+2,\ldots}^{+\infty} C_j^m P_l^{j,m}(x), \tag{3.6}$$

$$v_m^l(x) = l^{-1}(l+1)^{-1}\frac{1}{x}\frac{\partial}{\partial x}(x^2 u_m^l), \tag{3.7}$$

$$w_m^l(x) = \sum_{i=|m|}^{+\infty} A_m^i W_{i,l}^m j_l(\mu_m^i x) + \sum_{j=|m|,|m|+2,\ldots}^{+\infty} C_j^m P_l^{j,m}(x), \tag{3.8}$$

where j_l are the spherical Bessel functions of first kind, related to the general Bessel functions by $j_l(z) = \sqrt{(\pi/2z)}\,J_{l+1/2}(z)$, and $P^{j,m}$ are polynomials of j-degree (see Appendix B for the definitions of $U_{i,l}^m$, $W_{i,l}^m$, μ_m^i, A_m^i, C_j^m).

It is interesting to note that such solutions naturally split into two kinds of solutions: boundary layer solutions (Bessel functions) and interior solutions (polynomials). We shall use this property to calculate the asymptotic ($E \to 0$) solutions.

3.2 ASYMPTOTIC SOLUTIONS

The existence of two different type of solutions makes the parallelism of the method used here with the boundary layer method obvious; this allows us to use one of the main results of this theory concerning the interior flow. In the boundary layer theory (see Greenspan 1969) the inviscid interior flow appears to be a superposition of modes (geostrophic and inertial waves, corresponding here to the P^js) the amplitude of which is fixed by their flux in the boundary layer. Therefore the computation of the boundary layer flow must be carefully achieved since the interior flow and thus the interior dissipation drastically depends on it. Indeed, contrary to the usual case of a fluid inside a rigid wall container, boundary layers do not mean high shear stress here since the free boundary conditions specify a vanishing tangential stress at the surface of the star (a fact that is reinforced by the vanishing of turbulent viscosity at the surface). The whole dissipation is thus produced by the interior flow.

To illustrate our statement we computed (see Fig. 1) the contribution to the whole dissipation of the different modes of the flow. It is clear that the main contribution occurs from the lowest order modes [i.e. $P(j=2, m=\pm 2)$] the amplitude of which must therefore be carefully computed. We also see the contribution of the Bessel functions confined in the narrow boundary layer. Fig. 2 shows the sum of the different contributions.

The way to compute the solutions in the general case where ϱ and ν are functions of the radial variable is therefore outlined. We shall replace the polynomials by interior inviscid solutions and keep, provided viscosity is rather constant near the surface, the Bessel solution for the boundary layer.

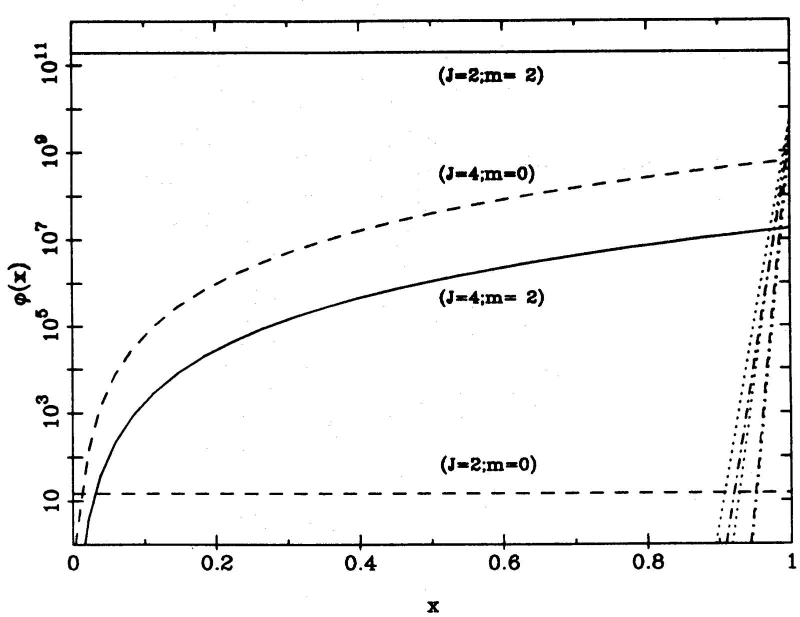

Figure 1. Dissipation $\phi(x)$ [*cf.* equation (2.20)] associated with the different modes as a function of the radial coordinate $x=r/R$, in an homogeneous star. The Ekman number is 10^{-5} and tidal elongation is $\varepsilon=0.05$.

3.3 SOLUTION OF THE GENERAL CASE

In the realistic case of a star, the full equations (2.12) and (2.18) must be solved. In this case ϱ and η are functions of (r, θ, ϕ); however, we shall assume that they are only functions of the radial variable 'r', thus neglecting coupling between solutions other than the surface one. Taking that into account would introduce an additional $O(\varepsilon)$ correction to the velocity field.

However, this implies that at the surface where boundary conditions are taken, neither the density nor the viscosity vanish, since this surface is defined as the sphere where the angular variations of density are of order of the density itself; this sphere is therefore of radius $R(1-\varepsilon)$ if R is the radius of the spherical star.

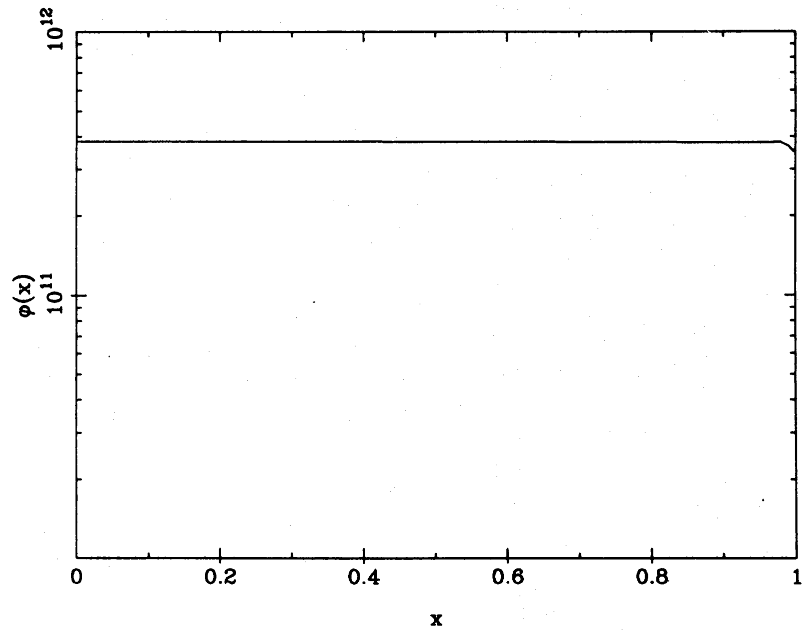

Figure 2. Total contribution of the modes to viscous dissipation as a function of the radial distance for the parameter given in Fig. 1.

Therefore assuming ϱ and η are radial functions, equations (2.12) and (2.18) can be transformed into their radial form:

$$h_m^l(x)=2\omega R^2\varrho\left\{-\frac{im}{l(l+1)}w_m^l-A_{l-1}^l\left[x^{l-1}\frac{\partial}{\partial x}\left(\frac{xu_m^{l-1}}{x^{l-1}}\right)+\varrho' u_m^{l-1}\right]\right.$$
$$\left.-A_{l+1}^l\left[x^{-l-2}\frac{\partial}{\partial x}(x^{l+2}xu_m^{l+1})+\varrho' u_m^{l+1}\right]+x\dot{\omega}\sqrt{\frac{\pi}{3}}\,\delta_l^1\delta_m^0\right\} \tag{3.9}$$

$$\frac{\varrho}{x}\frac{\partial}{\partial x}\left(\frac{xg_m^l}{\varrho}\right)-\frac{f_m^l}{x}=2\omega R^2\varrho\left\{-\frac{im}{l(l+1)}\left[\Delta_l xu_m^l+\frac{1}{x}\frac{\partial}{\partial x}(\varrho' xu_m^l)-\frac{l(l+1)\,\varrho' u_m^l}{x}\right]\right.$$
$$\left.+B_{l-1}^l x^{l-1}\frac{\partial}{\partial x}\left(\frac{w^{l-1}}{x^{l-1}}\right)+B_{l+1}^l x^{-l-2}\frac{\partial}{\partial x}(x^{l+2}w^{l+1})\right\}, \tag{3.10}$$

$$v_m^l(x)=l^{-1}(l+1)^{-1}\left[\frac{1}{x}\frac{\partial}{\partial x}(x^2u^l)+\varrho' u^l\right], \tag{3.11}$$

with

$$\varrho'(x)=\frac{x}{\varrho}\frac{d\varrho}{dx}$$

$$f_m^l(x)=2\frac{\partial\eta}{\partial x}\left[\frac{l(l+1)\,v_m^l-2u_m^l}{x}\right]+\frac{l(l+1)\,\eta z_m^l}{x}+\frac{4}{3}\frac{\partial}{\partial x}(\eta\, d_m^l) \tag{3.12a}$$

$$g_m^l(x)=\frac{1}{x}\frac{\partial}{\partial x}(x\eta\, z_m^l)+\frac{2}{x}\frac{\partial\eta}{\partial x}(u_m^l-v_m^l)+\frac{4\eta\, d_m^l}{3x} \tag{3.12b}$$

$$h_m^l(x)=\eta\Delta_l w_m^l+x\ \frac{\partial\eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{w_m^l}{x}\right);\qquad d_m^l(x)=\frac{\partial u_m^l}{\partial x}+\frac{2u_m^l-l(l+1)\,v_m^l}{x};$$

$$Z_m^l(x)=\frac{\partial v_m^l}{\partial x}+\frac{v_m^l-u_m^l}{x} \tag{3.12c}$$

which are completed by the boundary conditions (B.7) and (B.8).

This system is extremely complex; however, in view of the discussion of Section 3.2, it can be fairly simplified when the Ekman number is very small (in cataclysmic binaries $E<10^{-6}$). In that case, the left-hand side of (3.9) and (3.10) can be set to zero, thus giving interior solutions without their viscous corrections; the boundary solutions are the ones of the homogeneous case provided that we use the value of the viscosity at the boundary layer.

We therefore see that the dissipation depends drastically on the value of the viscosity at the surface where boundary conditions are taken, thus showing a strong difference with the synchronization problem considered by Zahn (1977), Campbell & Papaloizou (1983) where boundary conditions on the velocity field play a negligible part.

Fig. 3 shows the radial distribution of viscous heating in a completely convective secondary. It is seen that the flow dissipates much less energy near the centre of the star than elsewhere. This suggests that this region is much closer to solid rotation than the outer layers. This fact also appears in a projection on the rotation axis of the curl of the velocity field along the line $\theta=\pi/2$ and $\phi=0$ (see Fig. 4). A solid rotation would be represented by an horizontal line. The shape of the curve shows that only central regions ($x\leqslant 0.1$) rotate as a solid body.

The solution of the problem shown by Fig. 3 will not be strictly applicable when the flow is driven by the magnetic braking torque. In such a case, the flow is likely modified by the presence of the radiative core and especially by the magnetic field rooted in it. Taking this effect into account is much beyond the aim of this paper, and we shall assume next that the distribution of dissipation is the one shown by Fig. 3 even when magnetic braking occurs.

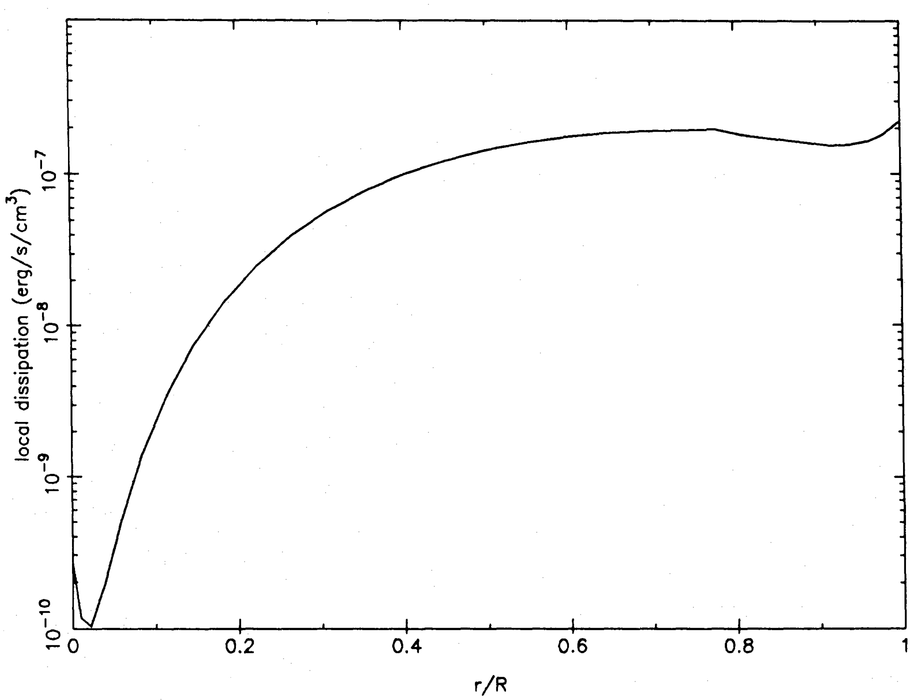

Figure 3. Distribution of viscous heating in the secondary during the cross of the gap (around $P\sim 2\,\mathrm{hr}$).

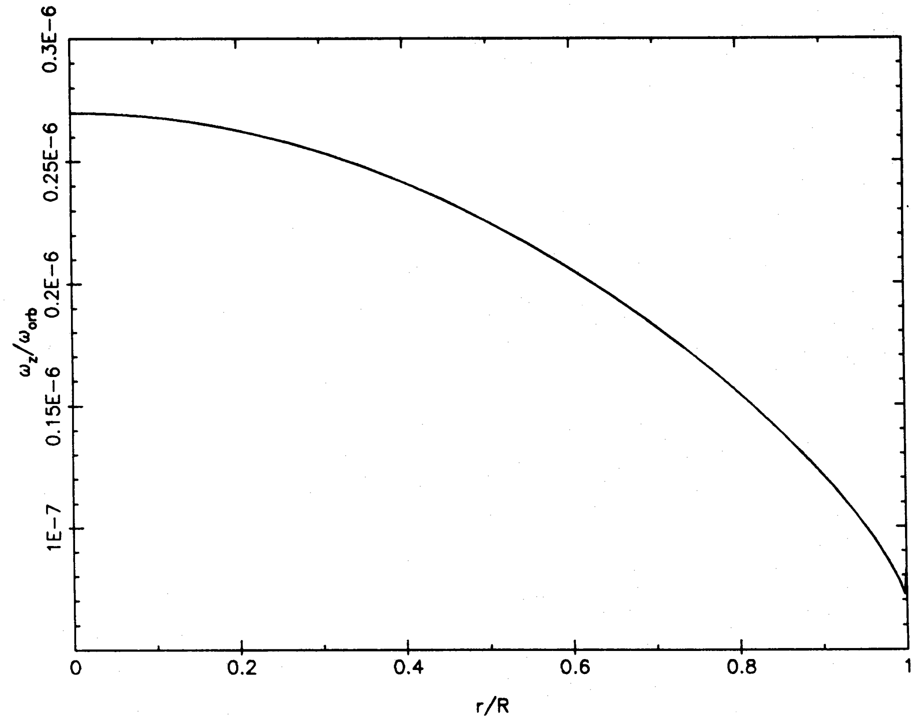

Figure 4. z-component of vorticity w_z along the line $\theta=\pi/2$, $\phi=0$, normalized by the orbital angular velocity $\omega_{\rm orb}$ (conditions are the same as in Fig. 3).

4 Applications to cataclysmic variables and Type I SN progenitors

We apply our calculations to the case of cataclysmic binaries where we show that tidal dissipation is of weak efficiency, and to the case of very close pairs of white dwarfs where, on the contrary, very important dissipation can be reached.

4.1 TIDAL HEATING IN CATACLYSMIC BINARIES

The current understanding of the evolution of cataclysmic binaries is that such systems once formed, lose a great amount of angular momentum through a mechanism, usually supposed to be magnetic braking (see Verbunt & Zwaan 1981), which explains the observed high values of mass transfer ($10^{-8}\,M_\odot\,{\rm yr}^{-1}$) from the low-mass main-sequence star to the compact object. It is usually assumed that, when the star becomes fully convective, this mechanism stops; the secondary then detaches from the Roche lobe and relaxes towards the main sequence. The system enters the observed period gap (between 2 and 3 hr). However, angular momentum losses via gravitational radiation are still at work, and the system contacts, but at a much lower rate. Mass exchange resumes only when the secondary again fills the Roche lobe, which happens for a period less than 2 hr. The system reaches a minimum period of 80 min when the star becomes degenerate, and expands until the secondary disappears.

This evolution has been modelled by Rappaport, Verbunt & Joss (1983) using for the mass-losing star a composite polytrope of index $n=3$ and $n=3/2$. We used this model in order to compute the viscous dissipation [equation (2.21)].

In all the evolution, we assume near corotation of the secondary; we shall not consider the other possibility of balance between tidal torque and stellar wind torque where the secondary is asynchronous (see van Paradijs 1986); indeed, it has been shown (Czerny & King 1986), that close binaries in such a state could not reflect the observed distribution of accretion rates versus orbital period of cataclysmic binaries.

To compute the turbulent viscosity generated in the convective envelope, we use the mixing length theory of convection. However, this model is far from satisfying since it assumes that the star is non-rotating. In our case the rotation of the system is very rapid and the Coriolis force affecting any motion of matter is very strong. Indeed, one can show that the ratio of buoyancy force to Coriolis force is very small:

$$f_{\rm buoy}/f_{\rm Cor}=12E(\Lambda), \tag{4.1}$$

where $E(\Lambda)$ is the Ekman number $(\nu/2\omega\Lambda^2)$ associated to the mixing length Λ (ν being the turbulent viscosity). This number may be evaluated, for a $n=3/2$ polytrope, using the mixing length theory described by Cox & Guili (1968) and the mass–radius, mass–luminosity relations for this kind of star (see Rappaport, Joss & Webbink 1982; Lang 1980):

$$R/R_\odot=0.76\,(M/M_\odot)^{0.78}; \qquad L/L_\odot=(M/M_\odot)^{2.8} \tag{4.2}$$

we derive

$$E(\Lambda)=2.8\times10^{-6}\,(M/M_\odot)^{0.08}J^{-2/3}\,P_{\rm h}m(x)l(x)^{1/3}/(x^{8/3}\,\theta^{3/2}) \tag{4.3}$$

where J is the ratio of the mixing length to the pressure height scale, $P_{\rm h}$ the period in hours, $l(x)$, $m(x)$ are the reduced mass and luminosity at radius $x=r/R$, and θ is the Lane–Emden $n=3/2$ function. This expression shows that, except in a thin layer close to the surface, the Coriolis force is everywhere dominating the convective flow. We therefore expect that the coefficient of turbulent viscosity is largely anisotropic. This may thus result in large uncertainties on the viscosity and therefore on the dissipation.

Nevertheless, we computed the global dissipation using this viscosity model. We also computed the time derivative of the spin kinetic energy which was shown (Section 2.4) to be an upper bound for the dissipation; as this quantity depends only on the inertia momentum of the secondary, ω and $\dot\omega$, it determines a safe limit to viscous dissipation. We plotted on Fig. 5 the viscous dissipation (D), the time-derivative of spin kinetic energy ($\dot E_k$) and luminosity as functions of the period. One can recognize the different stages of evolution: first contact, the gap, the minimum period.

This figure shows that dissipation is weak compared to the luminosity of nuclear origin ($D\sim10^{-3}\,L$). However, Fig. 5 also shows that during the magnetic braking phase, total power of torques applied to the star is comparable to the luminosity of the star. This leaves in principle the possibility for dissipation to reach the luminosity of the star; it can be the case if, for instance, the actual value of viscosity is 10^{-3} times smaller than the one given by the mixing length theory; another possibility of enhanced dissipation (actually pointed out by Verbunt & Hut 1983) may come from the likely heterogeneous distribution of torque density inside the star. Indeed, the magnetic fields guiding the wind out of the system seem to be rooted in the transition region between the core and the envelope of the secondary. This would imply that the magnetic braking torque is mainly applied on the interior of the star while tidal action is more important in the outer layers. This phenomenon likely enhances the differential rotation of the star and hence the viscous dissipation.

Fig. 5 shows that, when the evolution of the cataclysmic variable is driven by gravitational radiation, dissipation remains very weak (at maximum $10^{23}\,{\rm erg\,s^{-1}}$) and cannot have any influence on the evolution of the system.

We also plotted the dissipation according to the previous approach of Verbunt & Hut (1983) (formula 21). The time-scale of synchronization is the one proposed by the authors during the phase of magnetic braking ($P\geqslant3$ hr), and the one computed by Campbell & Papaloizou (1983) during the phase driven by gravitational radiation. The curve clearly shows that the formula of Verbunt & Hunt (1983) underestimated dissipation by at least a factor 10. This difference is in main part the consequence of the fact that the amplitude of the spin-up flow is controlled by

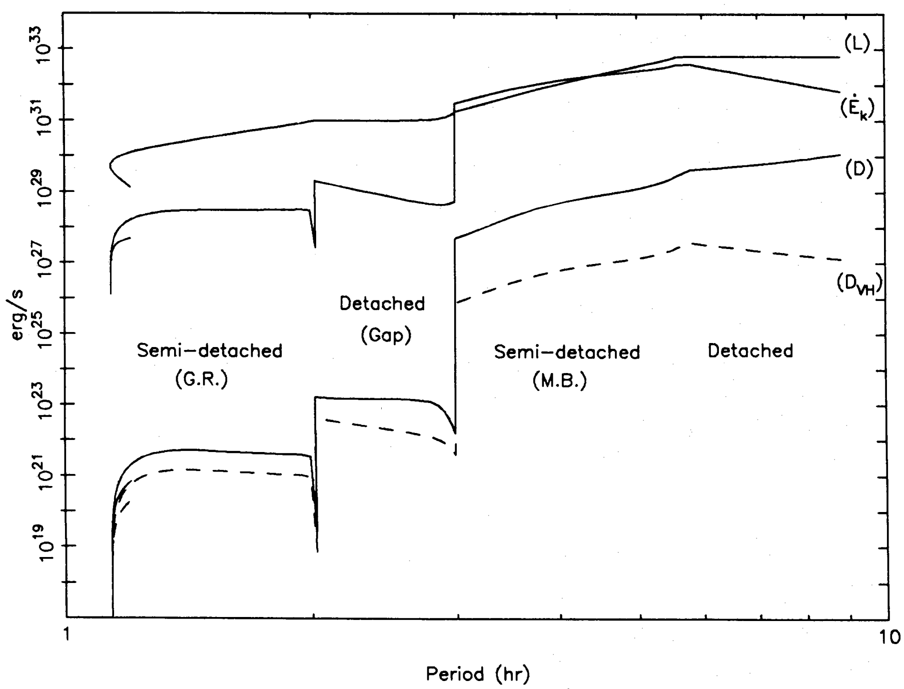

Figure 5. Dissipation (D) during the evolution of a cataclysmic binary; $\dot{E}_k$ and L are respectively the time derivative of spin kinetic energy (the upper bound of viscous dissipation) and luminosity of the star. D_{VH} is a plot of dissipation according to the previous approach of Verbunt & Hut (see text). The initial masses of the secondary and the primary are 0.7 and $1\,M_{\odot}$, respectively. It is assumed that all the transferred matter is ejected by the primary during novae explosions.

boundary layers and especially the viscosity therein. The slightly different behaviour of the dissipation curves when the system is detached comes from the different dependence of the two formulae towards tidal elongation ε.

To conclude, we may say that when the evolution of cataclysmic binaries or low-mass X-ray binaries is driven by magnetic braking, the influence of tidal heating still cannot be ruled out. The next step will be to include the effects of rotation on turbulent viscosity and the heterogeneous distribution of magnetic braking torque.

4.2 MAGNETIC FIELDS

Viscous heating is not the only consequence that may result from the presence of the spin-up flow, however. Indeed, the shear stress of this flow can generate by dynamo effect a surface magnetic field. Assuming for the fluid a negligible resistivity, we may estimate the maximum intensity of the magnetic field by assuming that the magnetic pressure gradient is as intense as the viscous force near the surface:

$$\boldsymbol{\nabla} H^2 \approx 8\pi\varrho \mathbf{F}_{\mathrm{visc}}$$

which yields $|H| \sim 10^2$ G, which is comparable to surface fields strength required in the case of magnetic braking (Verbunt 1984).

4.3 THE CASE OF VERY CLOSE PAIRS OF WHITE DWARFS

The main reason of the weakness of viscous heating in cataclysmic binaries is the smallness of $\dot{\omega}/\omega$ which is about $10^{-15}\,\mathrm{s}^{-1}$. As this quantity depends mainly on the period (in $P^{-8/3}$ in the case of gravitational radiation), it is natural to expect stronger effects on systems with shorter period.

Such systems can be pairs of white dwarfs, which, due to their small size, can get very close together. Such systems have already been discussed by Webbink (1984) and Iben & Tutukov (1984) as possible progenitor of Type I Supernovae. In this model the coalescense of the two white dwarfs is the origin of the explosion. Many calculations have already been conducted on the nuclear explosion triggered by high-mass transfers (see e.g. Nomoto & Iben 1985).

In view of the strength of the tidal field, hydrodynamical flows are likely to be very important before the collapse, thus generating high dissipation rates. We shall therefore see how much heating may result from the rise of a spin-up flow.

For such a mechanism to work, however, one of the stars must be quasi-synchronous. This is far from obvious since the viscosity of the gas of a white dwarf is very low (see e.g. Durisen 1973), making the viscous time-scale much larger than the lifetime of the system. However, this also suggests that the fluid may easily be subject to shear instabilities developing into turbulence; such a tidally induced turbulence has already been discussed in literature by Horedt (1975), Seguin (1976) and Zahn (1977), but it is still not clear whether or not the tidal flow becomes unstable and yields turbulence, since in these works only a linear analysis of instabilities was made.

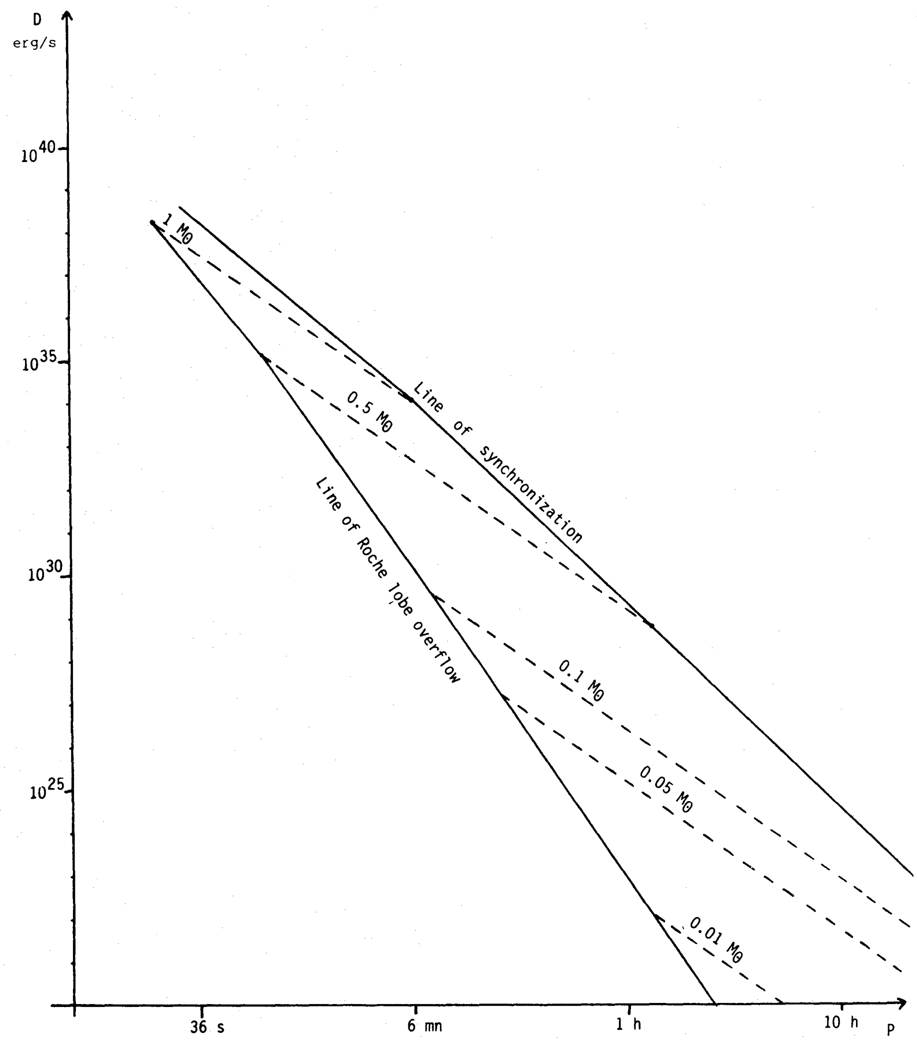

Figure 6. Viscous heating in a synchronized white dwarf before the start of mass exchange, as a function of the period and the mass of the lightest dwarf. The mass of the primary is always 1 $M_\odot$. The mass–radius relation used is the one given by Ritter (1985).

In the case where quasi-synchronization has been achieved, the spin-up is still turbulent. A simple model which can be derived in the way presented in Section 2.4, gives for the dissipation:

$$D = K' M R^2 \dot{\omega}^{3/2} / \varepsilon, \tag{4.4}$$

with $K' = 3.4$.

It is easy to compute tidal dissipation in this case. However, it must not exceed $\dot{E}_k$; the point where $D = \dot{E}_k$ will be assumed to the point where quasi-synchronism can occur. On Fig. 6, we plotted the viscous dissipation during this phase of 'approach', until the beginning of mass exchange. This figure shows that for systems composed of two white dwarfs of one solar mass, dissipation can reach the impressive value of 10^{38} erg s^{-1}. Such an energy will raise the temperature of the medium of about 10^6–10^7 K in a time-scale ~1000 yr, thus changing the conditions in which nuclear reactions are expected to occur when mass exchange starts.

One may also note that if only 1 per cent of this energy is radiated, such systems may become very bright much before the nuclear explosion, producing white dwarfs ~10^6 times brighter than isolated ones. The observational consequences of such a phenomenon should be very important since one can expect to observe progenitors of Type I Supernovae. The luminosity of such an object would range from 10^{32} erg s^{-1} at synchronization to 10^{36} erg s^{-1} at the end of the detached phase. This implies that the surface temperature would rise from 2.6×10^4 to 2.6×10^5 K for a blackbody emission. The corresponding wavelengths of maximum emission are respectively at 1100 and 110 Å (~0.1 keV), which is nearly the band where the interstellar medium is most opaque! However, when the luminosity of the dwarfs reaches 10^{36} erg s^{-1}, the soft X-ray's flux (0.1 keV) is quite intense; such sources may be detectable at a distance of a few hundred parsecs (including interstellar absorption), by an instrument such as the XUV wide field camera of the future ROSAT X-ray telescope. But the short duration of this emission (~40 yr) makes these objects quite rare (~0.4 in our Galaxy if we assume a rate of SN I explosion of 0.01 yr^{-1}). At synchronization the lifetime of the system is about 10^5 yr, but except in the last 40 yr, the X-ray emission is completely negligible. In the other spectral band these objects may be observable during the whole 10^5 yr and so may be a bit more numerous; let us consider for instance the *U*-band (3650 Å), at 1 kpc their magnitude ranges from 23 to 19 (still including interstellar absorption). With the birthrate of SN I indicated above and an assumed uniform distribution in the Galaxy, we find only ~10 objects that are closer than 1 kpc to the Earth; in view of such figures, we can conclude that, despite their characteristic very blue spectrum, their detection will be quite difficult.

However, we would like to insist that it is necessary to check the basic hypothesis, that pairs of white dwarfs can reach the state of quasi-synchronism by a tidally induced turbulence; in particular new problems may arise with the presence of magnetic fields.

In the opposite case where no mechanism has managed to synchronize the star, one should consider the excitation of eigenmodes of a white dwarf by the time periodic potential. Resonances may then generate shock waves which, breaking at this surface, may heat it up.

5 Conclusion

In this paper we computed the flow that occurs in the secondary of a close binary as the consequence of the spin-up induced by the conjugate effect of angular momentum losses and tidal effects.

It appears that in close binary systems like cataclysmic variables or low-mass X-ray binaries, viscous dissipation associated with these flows is always negligible when gravitational radiation is driving their evolution. However, when a mechanism like magnetic braking drives the evolution current model of turbulent viscosity this leads to a maximum dissipation of 10^{30} erg s^{-1} which is also small compared to the luminosity of the star ($L \sim 6 \times 10^{32}$ erg s^{-1}); however, we showed that, in principle, dissipation can reach the luminosity of the star if the actual viscosity is less than the one given by the mixing length. The definite conclusion about the possible importance of tidal heating in cataclysmic variables and low-mass X-ray binaries driven by magnetic braking thus relies on a more elaborate model of the internal structure of the secondary star in the conditions met in these systems.

The case of close pairs of white dwarfs is different since, due to the short periods these systems can reach, the spin-up is very important if the dwarfs are quasi-synchronous. In such a case, dissipation as large as 10^{38} erg s^{-1} can be reached; this would have serious consequences on the evolution of what is thought to be progenitors of Type I Supernovae. However, the hypothesis of quasi-synchronization is still not demonstrated and further work is needed to understand this mechanism. It is the case that, long before the close pairs of white dwarfs collapse, violent hydrodynamical effects can occur, that are probably able to change the initial conditions of the supernova explosion.

Acknowledgments

The authors thank Jean-Marie Hameury and Jean-Pierre Lasota for helpful discussions and comments on the manuscript, and the referee for useful criticism.

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Appendix A

In this appendix we calculate the expression of the velocity field of a spin-up flow in a 2D configuration, which can be made fully explicit.

Thus leaving the third dimension z, we treat the case of a 'flat star' all in the orbital plane (we keep the 3D form of the potentials, however).

As in all 2D problems, we introduce a current function ψ and, using polar coordinates (r, θ), we set:

$$V_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad V_\theta=-\frac{\partial\psi}{\partial r}. \tag{A.1}$$

The equation of vorticity (2.18) then becomes:

$$\Delta\Delta\psi=-2\dot{\omega}/\nu, \tag{A.2}$$

where ν is the kinematic viscosity.

The general solution of this equation, regular at origin, are:

$$\psi(r, \theta)=\sum_{m=0}^{+\infty} a_n r^{n+2}\sin(n\theta+\phi_n)+b_n r^n \sin(n\theta+\chi_n), \tag{A.3}$$

where the constants a_n, b_n, ϕ_n, χ_n have to be determined by boundary conditions (2.15) and (2.16). In the 2D form they read:

$$V_r\frac{\partial S}{\partial r}+\frac{V_\theta}{r}\frac{\partial S}{\partial\theta}=0, \tag{A.4}$$

$$\left(\sigma^{r\theta}\frac{\partial S}{\partial r}+\frac{\sigma^{\theta\theta}}{r}\frac{\partial S}{\partial\theta}\right)\frac{\partial S}{\partial r}=\left(\sigma^{rr}\frac{\partial S}{\partial r}+\frac{\sigma^{r\theta}}{r}\frac{\partial S}{\partial\theta}\right)\frac{1}{r}\frac{\partial S}{\partial\theta}. \tag{A.5}$$

As already discussed the surface can be assumed to be the equilibrium one the equation of which can be written (in its simplest form):

$$S(r,\theta)=r-R(1+\varepsilon\cos^2\theta)=0. \tag{A.6}$$

In order to handle the problem in a simple way, we develop the solutions into powers of the tidal elongation, ε and retain in this development only the two first orders. These solutions will than fulfil the boundary conditions to the first order in ε:

$$V_r(R)+\varepsilon\left[\left(\frac{\partial V_r}{\partial r}\right)_R R\cos 2\theta+\sin 2\theta\, V_\theta(R)\right]=0, \tag{A.7}$$

$$\sigma^{r\theta}(R)+\varepsilon\left[\left(\frac{\partial\sigma^{r\theta}}{\partial r}\right)_R R\cos 2\theta-4\sin 2\theta\,\sigma^{rr}(R)\right]=0, \tag{A.8}$$

where the components of the viscous stress σ read:

$$\sigma^{rr}=2\eta\frac{\partial V_r}{\partial r}=-\sigma^{\theta\theta}, \tag{A.9}$$

$$\sigma^{r\theta}=\sigma^{\theta r}=\eta\left[\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)+\frac{1}{r}\frac{\partial V_r}{\partial\theta}\right], \tag{A.10}$$

(A.3), (A.7) and (A.8) give the required solutions:

$$V_r=\frac{\dot{\omega}R^3}{4\nu\varepsilon}\left[\left(x-\frac{x^3}{3}\right)\sin 2\theta+O(\varepsilon)\right], \tag{A.11}$$

$$V_\theta=\frac{\dot{\omega}R^3}{4\nu\varepsilon}\left[-\frac{5x}{6\varepsilon}+\frac{1}{2}\left(x-\frac{2x^3}{3}\right)\cos 2\theta++O(\varepsilon)\right]. \tag{A.12}$$

The dissipation function corresponds to this velocity field:

$$\Phi(x)=\eta\left(\frac{\dot{\omega}}{\omega}\right)^2(8E\varepsilon)^{-2}(1-x^2)^2. \tag{A.13}$$

where $x=r/R$ and $E=\nu/2\omega R^2$.

Appendix B

We present in this appendix the derivation of the solutions of equations (3.3) and (3.4).

The field used in the expansion (3.5) has the following definitions:

$$\mathbf{R}_l^m=Y_l^m(\theta,\phi)\,\mathbf{e}_r;\qquad \mathbf{S}_l^m=r\nabla Y_l^m;\qquad \mathbf{T}_l^m=\mathbf{curl}(r\mathbf{R}_l^m) \tag{B.1}$$

where $\mathbf{e}_r$ is the unit radial vector and Y_l^m are the usual normalized spherical harmonics verifying:

$$\frac{\partial^2 Y_l^m}{\partial\theta^2}+\operatorname{cotg}\theta\frac{\partial Y_l^m}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2 Y_l^m}{\partial\phi^2}+l(l+1)\,Y_l^m=0. \tag{B.2}$$

The fields $\mathbf{R}_l^m$, $\mathbf{S}_l^m$, $\mathbf{T}_l^m$ are orthogonal on the sphere, i.e.

$$\int_{4\pi}\mathbf{R}_l^m\cdot\mathbf{S}_{l'}^{m'}\,d\Omega=\int_{4\pi}\mathbf{S}_l^m\cdot\mathbf{T}_{l'}^{m'}\,d\Omega=\int_{4\pi}\mathbf{T}_l^m\cdot\mathbf{R}_{l'}^{m'}\,d\Omega=0, \tag{B.3}$$

$$\int_{4\pi}\mathbf{R}_l^m\cdot\mathbf{R}_{l'}^{m'*}\,d\Omega=\delta_l^{l'}\delta_m^{m'};\qquad \int_{4\pi}\mathbf{S}_l^m\cdot\mathbf{S}_{l'}^{m'*}\,d\Omega=\int_{4\pi}\mathbf{T}_l^m\cdot\mathbf{T}_{l'}^{m'*}\,d\Omega=l(l+1)\,\delta_l^{l'}\delta_m^{m'}. \tag{B.4}$$

Replacing (3.5) in (3.3) gives the relation (3.7) between u_m^l and v_m^l. The equation of vorticity (3.4) can be transformed after some tricky calculations in the following differential system:

$$E\Delta_l w_m^l+\frac{imw_m^l}{l(l+1)}=-A_{l-1}^l x^{l-1}\frac{\partial}{\partial x}\left(\frac{u_m^{l-1}}{x^{l-2}}\right)-A_{l+1}^l x^{-l-2}\frac{\partial}{\partial x}(x^{l+3}u_m^{l+1})+x\sqrt{\frac{\pi}{3}}\delta_l^1\delta_{\mathrm{m}}^0, \tag{B.5}$$

$$E\Delta_l\Delta_l xu_m^l+\frac{im\,\Delta_l xu_m^l}{l(l+1)}=B_{l-1}^l x^{l-1}\frac{\partial}{\partial x}\left(\frac{w_m^{l-1}}{x^{l-1}}\right)+B_{l+1}^l x^{-l-2}\frac{\partial}{\partial x}(x^{l+2}w_m^{l+1}), \tag{B.6}$$

with

$$A_{l-1}^l=A_l^{l-1}=\alpha_{l-1}^l/l^2;\qquad B_{l-1}^l=B_l^{l-1}=(l-1)(l+1)\,\alpha_{l+1}^l,\qquad \Delta_l=\frac{\partial^2}{\partial x^2}+\frac{2}{x}\frac{\partial}{\partial x}-\frac{l(l+1)}{x^2},$$

$$\alpha_{l-1}^l=\alpha_l^{l-1}=\sqrt{\frac{l^2-m^2}{4l^2-1}}$$

and $E=\nu/2\omega R^2$.

The solutions of this system can be separated in two kinds: eigensolutions and polynomial solutions. Eigensolutions are of the form:

$$xu_m^l=U_m^l j_l(\mu x),$$

$$w_m^l=W_m^l j_l(\mu x),$$

the j_l being the eigenfunctions of the operator Δ_l. When replaced in (B.5) and (B.6), the system becomes a linear system for the constants U_m^l, W_m^l which appear to be eigenvectors of this system associated to the eigenvalue μ.

These solutions, however, do not form a complete set since, if we stop the expansion into spherical harmonics to order, say $L=2N$, then the order of the differential system is (in the $m=0$ case) $6N$, which requires $3N$ independent solutions regular at origin, while the eigensolutions form a set of $2N$ independent solutions. The N remaining solutions are polynomials which are computed in the following way. One starts by setting $u^j(x)=x^{j-1}$ $w_m^l=u_m^l=0$ if $l>j$, the other functions with $l\leq j$ are computed by recurrence with (B.5) and (B.6); this gives the $P_l^{j,m}(x)$ solution.

Boundary conditions (2.15) and (2.16) must be projected in the following way:

$$\int_{4\pi} \mathbf{u}\cdot\mathbf{n}_e(x_s)\, Y_L^M\, d\Omega = \int_{4\pi} (\mathbf{n}_e \times \boldsymbol{\sigma}\mathbf{n}_e)(x_s)\cdot \mathbf{S}_L^M\, d\Omega = \int_{4\pi} (\mathbf{n}_e \times \boldsymbol{\sigma}\mathbf{n}_e)(x_s)\cdot \mathbf{T}_L^M\, d\Omega \tag{B.7}$$

where $x=x_s(\theta, \phi)$ is the equilibrium surface equation.