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Linear Theory of Rotating Fluids Using Spherical Harmonics

Part I: Steady Flows

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It is shown that a systematic development of physical quantities using spherical harmonics provides analytical solutions to a whole class of linear problems of rotating fluids.

These solutions are regular throughout the whole domain of the fluid and are not much affected by the equatorial singularity of steady boundary layers in spherical geometries.

A comparison between this method and the one based on boundary layer theory is carried out in the case of the steady spin-up of a fluid inside a sphere.

KEY WORDS: Rotating fluids, spin-up, spherical harmonics.

1. INTRODUCTION

The classical way to attack problems of rotating fluids is to use boundary layer theory. Indeed, in most physical applications the Ekman number (the ratio of viscous force to Coriolis force) is very small giving rise to boundary layers near the boundaries of the fluid. This technique has been investigated extensively by Greenspan (1969). It turns out to be adequate when computing flows near planes orthogonal to the rotation axis, but becomes very tricky when the shape of the boundary departs from this geometry. This is particularly the case for fluids inside spherical or ellipsoidal containers. In such cases boundary layers are singular at some latitude and the boundary layer analysis then requires a careful study of the flow in the different regions of the container.

In this paper we show that a systematic expansion of physical quantities in spherical harmonics provides analytical solutions to a whole class of linear problems of rotating fluids well adapted for spherical or nearly spherical geometries. In fact, this new approach of the theory of rotating fluid runs parallel to the classical one based on the boundary layer theory, since, as we shall see, the radial solutions naturally split into two types of solutions describing respectively the boundary layers and the interior flow.

In the hydrodynamics of stars or planets, the use of such expansions is of general interest since the combination of rotation and spherical geometry is often met in these problems. In the literature, however, the use of this kind of analysis is still restricted to specific fields of hydrodynamics, in particular the geodynamo problem (Frazer, 1974; Cuong and Busse, 1981), or the problem of convection in stars or planets (Busse and Riahi, 1982); this present paper intends to show that it could be more widely used. Indeed, one of the main advantages of this method is that the problems caused by the equatorial singularity of the boundary layer that occurs in the case of a steady flow inside a rotating sphere or spheroid [see the discussion of this problem by Stewartson (1966), Dowden (1972) and Friedlander (1976)], are, in a way, removed, since the solutions derived are regular everywhere in the fluid (if the Ekman number is finite) (see the discussion in Section 3). On the contrary, in the boundary layer analysis, first order solutions are singular at the equator and such a singularity complicates dramatically the derivation of the full solution, especially if the container is not axisymmetric.

This paper is organized as follows. In Section 2 we consider the linear problems of steady flows of homogeneous fluids in a rotating container under the action of an external force field. For this case we present an expansion in spherical harmonics (Section 2.2) and give the general solutions of this problem (Section 2.3). In Section 3 we apply our formalism to the computation of the spin-up flow of a fluid inside a spherical shell and compare the results with those provided by the boundary layer theory. In Section 4, we consider the case of stratified fluids and show that, if the stratification is homogeneous (i.e. the Brünt–Väisälä frequency is constant throughout the fluid), analytical solutions can be found, revealing the existence of three types of layers.

2. THE FORMALISM

2.1 General equations

Let us consider first the flow of an incompressible fluid filling an almost spherical shell rotating with angular velocity ω. In a frame co-rotating with the shell, the linearized flow equations are

$$\partial \mathbf{v}/\partial t + 2\omega \times \mathbf{v} = -\rho^{-1}\nabla P + \nu\Delta\mathbf{v} + \mathbf{F}_{ext}, \quad (2.1)$$

$$\operatorname{div} \mathbf{v} = 0, \quad (2.2)$$

where P is the pressure, ρ the density, ν the kinematic viscosity and $\mathbf{F}_{ext}$ is an external body force. We scale the variables as follows:

$$\mathbf{u} = \mathbf{v}/V; \quad \mathbf{k} = \boldsymbol{\omega}/\omega; \quad \mathbf{x} = \mathbf{r}/R; \quad \tau = \omega t; \quad p = P/\rho\omega V,$$

where R is the mean radius of the shell. Equations (2.1) and (2.2) become

$$\partial \mathbf{u}/\partial \tau + 2\mathbf{k} \times \mathbf{u} = -\nabla p + E\Delta\mathbf{u} + \mathbf{f}_{ext}, \quad (2.3)$$

$$\operatorname{div} \mathbf{u} = 0, \quad (2.4)$$

where $E = \nu/\omega R^2$ is the Ekman number.

Considering only steady flows and taking the curl of (2.3), we get the vorticity equation:

$$\nabla \times (2\mathbf{k} \times \mathbf{u}) = -E(\nabla \times)^3\mathbf{u} + \nabla \times \mathbf{f}_{ext}. \quad (2.5)$$

The boundary conditions on the velocity field are

$$\mathbf{u}=\mathbf{0} \quad \text{on} \quad x=x_s(\theta,\phi) \tag{2.6}$$

for a rigid boundary, and

$$\left.\begin{array}{ll}(a) & \mathbf{u}\cdot\mathbf{n}=0\\ (b) & ([\sigma]-[p])\mathbf{n}=\mathbf{0}\end{array}\right\} \quad \text{on} \quad x=x_s(\theta,\phi) \tag{2.7}$$

for a free boundary where no stress is applied ($[\sigma]$ and $[p]$ are the viscous and pressure stress tensor respectively, $\mathbf{n}$ is a unit vector normal to the boundary whose equation is $x=x_s(\theta,\phi)$). Note that (2.7b) can be transformed into a boundary condition on the velocity only,

$$\mathbf{n}\times[\sigma]\mathbf{n}=\mathbf{0}, \tag{2.7c}$$

which indicates that the tangential component of shear stress vanishes at the surface.

Equations (2.4), (2.5) together with (2.6) or (2.7) specify the velocity field completely. We shall solve this problem by expanding the velocity in terms of vectorial spherical harmonics.

2.2 The expansion in spherical harmonics

In spherical coordinates a vector field can be expanded as

$$\mathbf{u}(r,\theta,\phi)=\sum_{l=0}^{+\infty}\sum_{m=-l}^{+l} u_m^l(r)\mathbf{R}_l^m(\theta,\phi)+v_m^l(r)\mathbf{S}_l^m(\theta,\phi)+w_m^l(r)\mathbf{T}_l^m(\theta,\phi), \tag{2.8}$$

with

$$\mathbf{R}_l^m=Y_l^m\mathbf{e}_r;\quad \mathbf{S}_l^m=\nabla Y_l^m;\quad \mathbf{T}_l^m=\nabla\times\mathbf{R}_l^m$$

on a unit sphere; they are the vectorial spherical harmonics. $\mathbf{e}_r$ is the unit radial vector. [Hereafter we drop the summation sign and assume summation on repeated indices.] The Y_l^m are the usual normalized spherical harmonics satisfying

$$\partial^2 Y_m^l/\partial\theta^2+\cot\theta\cdot\partial Y_m^l/\partial\theta+(1/\sin^2\theta)\,\partial^2 Y_m^l/\partial\phi^2+l(l+1)\,Y_m^l=0. \tag{2.9}$$

The vectorial spherical harmonics are orthogonal on the sphere:

$$\int_{4\pi}\mathbf{R}_l^m\cdot\mathbf{S}_{l'}^{m'}\,d\Omega=\int_{4\pi}\mathbf{S}_l^m\cdot\mathbf{T}_{l'}^{m'}\,d\Omega=\int_{4\pi}\mathbf{T}_l^m\cdot\mathbf{R}_{l'}^{m'}\,d\Omega=0, \tag{2.10a}$$

$$\int_{4\pi}\mathbf{R}_l^m\cdot\mathbf{R}_{l'}^{m'*}\,d\Omega=\delta_{ll'}\,\delta_{mm'},$$

$$\int_{4\pi}\mathbf{S}_l^m\cdot\mathbf{S}_{l'}^{m'*}\,d\Omega=\int_{4\pi}\mathbf{T}_l^m\cdot\mathbf{T}_{l'}^{m'*}\,d\Omega=l(l+1)\,\delta_{ll'}\,\delta_{mm'}, \tag{2.10b}$$

where * denotes the complex conjugate. Note that $w_m^l(x)\mathbf{T}_l^m$ are often called toroidal fields while the combination $u_m^l(x)\mathbf{R}_l^m+v_m^l(x)\mathbf{S}_l^m$ is often referred to as a spheroidal field [which becomes poloidal if divergenceless, see e.g. Chandrasekhar (1961)].

These fields have the remarkable property of being easily transformed by the curl operator:

$$\nabla\times[u_m^l(x)\mathbf{R}_l^m]=x^{-1}u_m^l(x)\mathbf{T}_l^m, \tag{2.11a}$$

$$\nabla\times[v_m^l(x)\mathbf{S}_l^m]=-x^{-1}\{d[xv_m^l(x)]/dx\}\mathbf{T}_l^m, \tag{2.11b}$$

$$\nabla\times[w_m^l(x)\mathbf{T}_l^m]=x^{-1}l(l+1)w_m^l(x)\mathbf{R}_l^m+x^{-1}\{d[xw_m^l(x)]/dx\}\mathbf{S}_l^m. \tag{2.11c}$$

Therefore, once the viscous and the Coriolis forces have also been expanded in the same way as the velocity field (see the Appendix), the differential system governing the radial functions $u_m^l(x)$, $v_m^l(x)$ and $w_m^l(x)$ is easily derived from (2.4) and (2.5)

$$E'\Delta_l w_m^l+[l(l+1)]^{-1}imw_m^l=-A_{l-1}^l x^{l-1}\,\partial(x^{-l+2}u_m^{l-1})/\partial x$$
$$-A_{l+1}^l x^{-l-2}\,\partial(x^{l+3}u_m^{l+1})/\partial x-H_m^l, \tag{2.14a}$$

$$E'\Delta_l\Delta_l xu_m^l+[l(l+1)]^{-1}im\Delta_l xu_m^l=B_{l-1}^l x^{l-1}\,\partial(x^{-l+1}w_m^{l-1})/\partial x$$
$$+B_{l+1}^l x^{-l-2}\,\partial(x^{l+2}w_m^{l+1})/\partial x-l(l+1)x^{-1}[\partial(xG_m^l)/\partial x-F_m^l], \tag{2.14b}$$

$$x^{-2}[\partial(x^2u_m^l)/\partial x]-x^{-1}l(l+1)v_m^l=0, \tag{2.15}$$

where $A^l_{l-1} = A^{l-1}_l = l^{-2}\alpha^l_{l-1}$; $B^l_{l-1} = B^{l-1}_l = (l-1)(l+1)\alpha^l_{l-1}$; and where F^l_m, G^l_m and H^l_m are the radial functions of $\mathbf{f}_{ext}$ $(E' = E/2)$. α^l_k is defined in the Appendix.

A few comments should be made about this system. First, it governs any steady flow occurring in a homogeneous fluid. Second, due to the axial symmetry of the coupling Coriolis force, solutions for different m are not coupled and each system (of given m) splits into two independent systems: one coupling u^{2p-1}_m and w^{2p}_m, the other coupling w^{2p-1}_m and u^{2p}_m (which are in fact two solutions of opposite parity in the symmetry $\mathbf{r}/-\mathbf{r}$).

2.3 The solutions

We now give the solutions of (2.14) when there is no external force field.

2.3.1 Eigensolutions The first type of solutions are given by the eigenfunctions of the operator Δ_l:

$$xu^l_m = U^l_m j_l(\mu x/E'^{1/2}) \quad \text{or} \quad xu^l_m = U^l_m y_l(\mu x/E'^{1/2}), \tag{2.16a}$$

$$w^l_m = W^l_m j_l(\mu x/E'^{1/2}) \quad \text{or} \quad w^l_m = W^l_m y_l(\mu x/E'^{1/2}). \tag{2.16b}$$

The j_l and y_l are respectively spherical Bessel funcions of the first and second kind, and are related to the general Bessel functions by

$$j_l(z) = (\pi/2z)^{1/2} J_{l+1/2}(z) \quad \text{and} \quad y_l(z) = (\pi/2z)^{1/2} Y_{l+1/2}(z)$$

[see Abramovitz and Stegun (1964) for basic properties].

The differential system (2.14) then becomes a linear system for U^l_m and W^l_m,

$$\begin{aligned} \{\mu^2 - [l(l+1)]^{-1} im\} W^l_m &= -A^l_{l-1} U^{l-1}_m + A^l_{l+1} U^{l+1}_m, \\ \{\mu^2 - [l(l+1)]^{-1} im\} U^l_m &= -B^l_{l-1} W^{l-1}_m + B^l_{l+1} W^{l+1}_m, \end{aligned} \tag{2.17}$$

which can be written in a matrixial form $[M]\mathbf{p} = \mu^2 \mathbf{p}$, where $[M]$ is a tridiagonal matrix of infinite order, and $\mathbf{p}$ is the eigenvector

associated with the eigenvalue μ^2. If we truncate the system to order N, one can show that $[M]$ has $N - |m| + 1$ eigenvalues which are:

i) distinct;

ii) purely imaginary;

iii) of modulus lower than unity: $0 \leqq |\mu^2| < 1$.

When N goes to infinity, the eigenvalues of $[M]$ tend to cover (if $m=0$) the interval $[-i; i]$ and are dense in it.

The form of the eigenvalues

$$\mu = |\mu|(1+i)/2^{1/2}$$

are reminiscent of the classical argument of the exponentials describing flows in Ekman layers; this shows that the eigensolutions are in fact the radial functions of the boundary layer solutions.

2.3.2 Polynomial solutions The eigensolutions, however, do not form a complete set of solutions for the system (2.14) [in (2.17) we eliminated the solutions with $\mu=0$]. If we limit the expansion in spherical harmonics to order $L=2N$ say, then the order of the differential system is $6N$ (in the axisymmetric case $m=0$) and it yields $6N$ independent solutions, while the eigensolutions form an independent set of $4N$ solutions. Thus, there remain $2N$ independent solutions which are polynomials. We denote them by $P^{j,m}$ and $Q^{j,m}$. Their general form is:

$$P^{j,m} = \begin{cases} u^{|m|}_m(x') \\ w^{|m|+1}_m(x') \\ \vdots \\ u^j_m(x) = x'^{j-1}, \\ 0 \\ \vdots \end{cases} \qquad Q^{j,m} = \begin{cases} 0 \\ \vdots \\ 0 \\ u^j_m(x') \\ w^{j+1}_m(x') \\ \vdots \end{cases},$$

where all the functions $u^l_m(x')$ and $w^l_m(x')$ are polynomials of degree $j-1$ in x' for the $P^{j,m}$ and in x'^{-1} for the $Q^{j,m}$ $(x' = x/E'^{1/2})$.

It should also be noted that the $P^{j,m}$ solutions are exact solutions of the infinite system, while the eigensolutions and the $Q^{j,m}$ are only exact solutions of the truncated system.

The complete set of solutions to order $2N$ can now be written down:

$$xu_m^l(x)=\sum_{i=|m|}^{+\infty} A_i^m U_l^{m,i} j_l(\mu_i^m x')+B_i^m U_l^{m,i} y_l(\mu_i^m x')$$

$$+\sum_{j=|m|,|m|+2,\dots}^{+\infty} C_j^m P_l^{j,m}(x')+D_j^m Q_l^{j,m}(x'), \tag{2.18a}$$

$$v_m^l(x)=[l(l+1)]^{-1}x^{-1}\,\partial(x^2u_m^l)/\partial x, \tag{2.18b}$$

$$w_m^l(x)=\sum_{i=|m|}^{+\infty} A_i^m W_l^{m,i} j_l(\mu_i^m x')+B_i^m W_l^{m,i} y_l(\mu_i^m x')$$

$$+\sum_{j=|m|,|m|+2,\dots}^{+\infty} C_j^m P_l^{j,m}(x')+D_j^m Q_l^{j,m}(x'). \tag{2.18c}$$

The $6N$ constants A_i^m, B_i^m, C_j^m and D_j^m must be computed using the boundary conditions (A.5) or (A.7). These solutions may describe, for instance, a flow between two spherical shells rotating about a common axis with slightly different angular velocities, a problem which has been considered many times in the literature in order to study shear layers (Proudman, 1956; Stewartson, 1966; Munson and Joseph, 1971).

Once the velocity field is known, it is a simple matter to derive the pressure field from the momentum equation (2.3); writing

$$p(\mathbf{x})=p_m^l(x)Y_l^m,$$

we obtain

$$p_m^l(x)=[l(l+1)]^{-1}[E'\,\partial(x\Delta_l xu_m^l)/\partial x+im(xu_m^l+xv_m^l)]$$

$$-l^{-1}(l-1)\alpha_{l-1}^l xw_m^{l-1}-(l+1)^{-1}(l+2)\alpha_{l+1}^l xw_m^{l+1}.$$

3. EXAMPLE: THE SPIN-UP OF A FLUID INSIDE A SPHERICAL SHELL

To illustrate these solutions we computed the very simple example of the steady spin-up of a fluid inside a spherical shell. In this case the angular velocity of the shell increases slightly with time and, in a frame co-rotating with the shell, the fluid is submitted to the bulk force $-\rho\dot{\omega}\times\mathbf{r}$ where $\dot{\omega}$ is the time derivative of the angular velocity of the shell; this force may be represented as a toroidal field:

$$-\rho\dot{\omega}\times\mathbf{r}=-\rho\dot{\omega}x(4\pi/3)^{1/2}\mathbf{T}_1^0. \tag{3.1}$$

In this case, we need only solutions regular at origin and they read

$$xu_m^l(x)=\sum_{i=|m|}^{+\infty} A_i^m U_l^{m,i} j_l(\mu_i^m x')$$

$$+\sum_{j=|m|,|m|+2,\dots}^{+\infty} C_j^m P_l^{j,m}(x')+(4\pi/5)^{1/2}\dot{\omega}x\delta_2^l\delta_m^0, \tag{3.2a}$$

$$w_m^l(x)=\sum_{i=|m|}^{+\infty} A_i^m W_l^{m,i} j_l(\mu_i^m x')$$

$$+\sum_{j=|m|,|m|+2,\dots}^{+\infty} C_j^m P_l^{j,m}(x'), \tag{3.2b}$$

where the constants A_i^m and C_i^m have to be computed by solving a linear system of order $3N$ (for a development to order $2N$ of the velocity field), given by the equations:

$$l=2,4,\dots,2N \qquad u_m^l=v_m^l=0, \tag{3.3a}$$

$$l=1,3,\dots,2N-1 \qquad w_m^l=0. \tag{3.3b}$$

Figure 1 shows the meridional circulation occurring in such circumstances for various values of the Ekman number. The number of spherical harmonics required to compute the flow is shown in Table I. This number has been derived by demanding that the local variations of dissipation (the square of the gradient of velocity) do not exceed the required precision "pr". If the quantity of interest is the velocity field, the conditions are less stringent and fewer harmonics are needed. This table clearly shows that the method is more adapted to solving problems with Ekman numbers which are not too small, and therefore appears to complement the boundary layer theory (hereafter BLT).

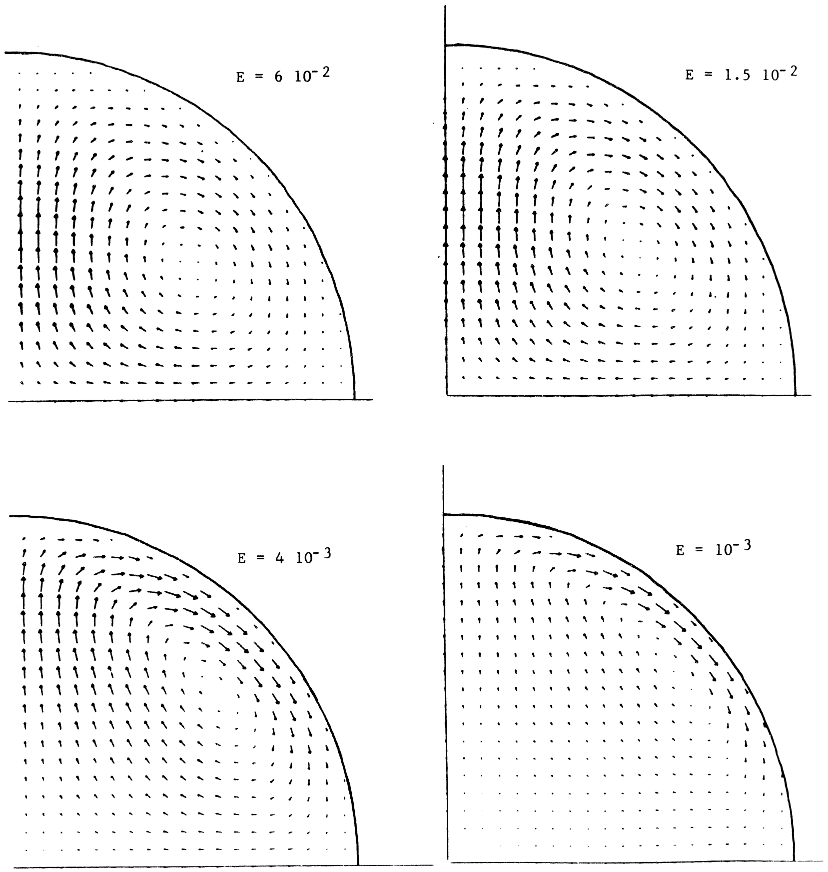

Figure 1 Meridional circulation in a spinning-up fluid inside a spherical shell computed from solutions (3.2). E is the Ekman number.

The lack of rapid convergence for small E, however, can be easily explained: this is the consequence of the form of the interior flow. If one computes, using BLT, the first order $O(E^{-1/2})$ of this flow, one finds

$$u_\phi = x \sin\theta \cdot E^{-1/2}(1 - x^2 \sin^2\theta)^{3/4}, \qquad (3.4)$$

which cannot be easily expanded in spherical harmonics for all the values of the radial coordinate "x". Expression (3.4) also shows that

Table I The number of spherical harmonics needed to compute the flow of Figure 1 as a function of the Ekman number (E) and the required precision (pr)

E \ Pr	1%	10%	50%
10^{-2}	6	4	4
10^{-3}	12	8	4
10^{-4}	20	14	6
10^{-5}	30	22	8
10^{-6}	42	30	8

when E is sufficiently small, the rate of convergence of the series is independent of E; therefore one may estimate that for $E \lesssim 10^{-6}$, 42 harmonics can describe the flow with a precision to within at least one percent.

As a result, it appears that the boundary layer is very well described by the series: only four harmonics are needed to describe the flow with a higher precision than a first order calculation with the BLT. A comparison between the two methods is given in Table II. We computed, using the two techniques, the viscous dissipation in

Table II Value of dissipation in the flow of Figure 1 at $x=1$, respectively computed with the spherical harmonics expansion D_{SH}) and the boundary layer theory (D_{BLT}). $\Delta D/D$ shows the relative difference between D_{SH} and D_{BLT}

E	D_{SH}	D_{BLT}	$\Delta D/D$
10^{-4}	$1.08\ 10^{7}$	$1.43\ 10^{7}$	25%
10^{-6}	$1.38\ 10^{11}$	$1.43\ 10^{11}$	3%
10^{-8}	$1.424\ 10^{15}$	$1.429\ 10^{15}$	$3\ 10^{-3}$
10^{-10}	$1.4281\ 10^{19}$	$1.4286\ 10^{19}$	$3\ 10^{-4}$
⋮	⋮	⋮	⋮
10^{-18}	$1.42857139\ 10^{35}$	$1.42857143\ 10^{35}$	$4\ 10^{-8}$

the boundary layer (at $x=1$) for a wide range of Ekman numbers. In the BLT calculation we included only the $O(E^{-1/2})$ term. The results are that, as expected, the value computed using spherical harmonics tends to the asymptotic value given by the BLT with a $O(E^{1/2})$ relative error. One may also note that this discrepancy is an order of magnitude larger than $E^{1/2}$; such a discrepancy betrays the influence of the $O(E^0)$ correction which is not negligible until $E \lesssim 10^{-6}$.

We shall now discuss the influence of the equatorial singularity on the convergence of the series. We noticed that it is in fact quite weak. This may be for two reasons: the first one is that, as stated by Friedlander (1976), "the interior flow is driven by the classical Ekman-layer circulation and is not modified to first order by equatorial effects"; thus, at very low Ekman numbers, the rate of convergence is almost independent of E (which is actually observed). The second reason is that at the other end of the scale, for large E, the series gives an accurate description of the flow. Therefore, we may expect some possible influence in an intermediate range. Actually some E-dependent fluctuations are observed in the range $10^{-10} < E < 10^{-7}$, but in any case they remain of small amplitude compared to those due to the form (3.4) of the asymptotic solution which is a good approximation for the interior flow in this range of Ekman numbers.

To conclude this section, we may say that if numerical accuracy is not of dramatic importance, then the use of an expansion in spherical harmonics provides a very correct description of the flow with a small number of harmonics (less than 10, say) in the whole range of values of Ekman numbers.

This result is very interesting when the container of the fluid is spheroidal. Indeed, in such a case a BLT calculation becomes very complex. If one considers, for instance, the physical problem of the steady spin-up of a star by a tidal field (see e.g. Rieutord and Bonazzola, 1987), one finds that such a problem is almost intractable with the BLT. Even the leading order of the solution cannot be derived (without the use of series expansions), since dramatic complications are introduced by the ellipsoidal surface of the star and by the free boundary conditions met by the fluid. By contrast, in such a case, the solutions (3.2) developed above can readily be used.

Finally, we would like to stress the fact that these solutions keep track of the differences between "interior solutions" (the polynomials) and "boundary layer solutions" (the Bessel functions). This point is important since it allows simplifications of the solutions when E is small. In such a case the viscous correction can be ignored in the

"polynomial solution". This is especially interesting, when one deals with heterogeneous fluids, like the one of a star where density, viscosity are radial functions.

4. SPHERICALLY STRATIFIED FLUIDS

The case of stratified fluids provides an interesting application of our method. Indeed, in the case of a fluid with a constant stratification (i.e. the Brünt-Väisälä frequency is constant), treated using the Boussinesq approximation, our formalism gives the full solution of steady flows.

The general equation of such a flow, written with dimensionless quantities, is

$$2\mathbf{k}\times\mathbf{u}+\nabla P-2N^2T\mathbf{x}=E\Delta\mathbf{u}, \tag{4.1a}$$

$$\operatorname{div}\mathbf{u}=0, \quad 2\mathbf{u}\cdot\mathbf{x}=E\mathscr{P}^{-1}\Delta T, \tag{4.1b, c}$$

where N is the dimensionless Brünt-Väisälä frequency and $\mathscr{P}$ is the Prandtl number. This system may be easily "translated" into radial equations; still eliminating the pressure, we obtain

$$E'\Delta_l w_m^l+[l(l+1)]^{-1}imw_m^l$$
$$=-A_{l-1}^l x^{l-1}\,\partial(x^{-l+2}u_m^{l-1})/\partial x-A_{l+1}^l x^{-l-2}\,\partial(x^{l+3}u_m^{l+1})/\partial x, \tag{4.2a}$$

$$E'\Delta_l\Delta_l xu_m^l+[l(l+1)]^{-1}im\Delta_l xu_m^l$$
$$=B_{l-1}^l x^{l-1}\,\partial(x^{-l+1}w_m^{l-1})/\partial x+B_{l+1}^l x^{-l-2}\,\partial(x^{l+2}w_m^{l+1})/\partial x$$
$$+l(l+1)N^2T_m^l, \tag{4.2b}$$

$$E'\mathscr{P}^{-1}\Delta_l T_m^l=xu_m^l, \tag{4.2c}$$

where we have set $T=T_m^l Y_l^m$.

The set of independent solutions of this system can be found in the same manner as in the previous case. One first searches for eigensolutions with Bessel functions:

$$xu_m^l = U_m^l j_l(\mu x) \quad \text{or} \quad xu_m^l = U_m^l y_l(\mu x), \tag{4.3a}$$

$$w_m^l = W_m^l j_l(\mu x) \quad \text{or} \quad w_m^l = W_m^l y_l(\mu x), \tag{4.3b}$$

$$T_m^l = \mathscr{T}_m^l j_l(\mu x) \quad \text{or} \quad T_m^l = \mathscr{T}_m^l y_l(\mu x). \tag{4.3c}$$

Such solutions form a complete set of solutions in the non-axisymmetric case ($m \neq 0$). If the parameters of the problem satisfy the inequality $N^2 \mathscr{P} E \ll 1$ (which is mostly the case in physical situations), then two types of eigenvalues appear:

$$\mu = (\pm i)^{1/2} \lambda / E^{1/2} \quad \text{or} \quad \mu = (\pm i)^{1/4} \lambda (N^2 \mathscr{P}/E)^{1/4}, \tag{4.4}$$

where λ is an $O(1)$ real number. These two types of eigenvalues correspond to two different layers: the classical Ekman layer of width $E^{1/2}$ and a thermal layer of width $(E/N^2\mathscr{P})^{1/4} \gg E^{1/2}$, which corresponds to a balance between the Coriolis force and the buoyancy force. In the axisymmetric case ($m=0$), we still have the Ekman layer eigenvalue but the second type of eigenvalue degenerates: some eigenvalues vanish while others take the form

$$\mu = \pm i \lambda N \mathscr{P}^{1/2}. \tag{4.5}$$

This last type of eigenvalue may give rise to a third type of layer if $N\mathscr{P}^{1/2} \gg 1$. The vanishing eigenvalues in fact betray the existence of axisymmetric polynomial solutions which can be written

$$w_0^{2n+1} = x^{2n+1}, \qquad T_0^{2n} = -[(4n+3)(2n+2)/(2n+1)N^2]\alpha_{2n+1}^{2n} x^{2n},$$

or (4.6)

$$w_0^{2n-1} = x^{-2n}, \qquad T_0^{2n} = [(4n-1)(2n-1)/(2n)N^2]\alpha_{2n-1}^{2n} x^{-2n-1}.$$

These results may be applied to study the steady spin-up of stratified fluids. For instance, if we consider (like Clark *et al.*, 1971), the spin-up of a strongly stratified fluid in an axisymmetric container in which $E^{1/2} \ll (N\mathscr{P}^{1/2})^{-1} \ll 1$, we can state immediately that the flow will consist in an interior flow given by (4.6) and two boundary layer flows: the usual Ekman layer (of width $E^{1/2}$) and a thermal layer [of width $(N\mathscr{P}^{1/2})^{-1}$]. We also note that these solutions clearly show

that the Ekman pumping will remain between the Ekman layer and the thermal layer, since the interior solutions (4.6) cannot convect the fluid as their velocity field is purely azimuthal.

Friedlander (1976) considered the case where $E^{1/2} \ll N^2 \mathscr{P} \ll 1$ as the conditions met in the solar interior. In such a situation the eigensolutions, with the eigenvalues $\mu = \lambda N \mathscr{P}^{1/2}$, no longer form a boundary layer since the argument of the Bessel functions is very small; these eigensolutions are thus equivalent to polynomials and therefore represent a flow driven by diffusion. Hence, we can with Friedlander readily draw the conclusion that in such a case a steady spin-up flow is a combination of an Ekman suction and a thermal diffusion.

In another respect, the solutions for the stratified fluid can also be used to study the marginal stability of fluids in spheroidal containers. Indeed, a knowledge of the analytical solutions of the equations of motions greatly simplifies this kind of problem, since the frontiers of stability are given by the roots of the determinant of the linear system expressing the boundary conditions. We note that in the study of such problems, we are not limited to axisymmetric containers, unlike Chandrasekhar (1961).

5. CONCLUSION

We shall first recall the main points of the method.

1) The expansion of fields in spherical harmonics gives analytical solutions of flows occurring in incompressible or homogeneously stratified rotating fluids.

2) The equatorial singularity that one meets when using the boundary layer theory in spherical geometries appears to have little effect on the convergence of the series.

3) The case of flows in non-axisymmetric containers meeting free boundaries, is much more tractable with this method than with the use of boundary layer theory.

4) These solutions keep the distinction between boundary layer solutions and interior ones, and in general do not mask the physics of the flow.

5) The numerical accuracy of the method is greatest when the Ekman number is not too small ($E \gtrsim 10^{-6}$); however, due to its good description of boundary layers, this technique gives correct order of magnitude solutions over the whole range of values of the Ekman number with a small number of spherical harmonics.

Finally, this new approach to the theory of rotating fluid in spherical geometries is interestingly completed by the previous approach based on the boundary layer theory, since the direct manipulation of the original vectorial equations is still necessary to interpret the results in terms of the balance of forces. However, as far as quantitative results are concerned, the technique presented here can give accurate results to some problems, like those mentioned in point (3) above, where the developments required by the boundary layer approach yield inextricable calculations.

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Appendix

In this appendix, we give the development into spherical functions of (a) the viscous and Coriolis forces that are necessary to derive the radial equations (2.14); (b) the boundary conditions (2.7); and (c) the viscous stress tensor; we also give (d) the mean radial dissipation function which has been used to compare the accuracy of this method with that based on the BLT.

a) Expansion of forces

Viscous force If one notes that for an incompressible fluid $\Delta\mathbf{u} = -\nabla\times\nabla\times\mathbf{u}$, then with relations (2.11), it is direct to derive:

$$\Delta\mathbf{u} = x^{-1}\Delta_l x u_m^l \mathbf{R}_l^m + x^{-1}\{\partial[x\Delta_l x u_m^l/l(l+1)]/\partial x\}\mathbf{S}_l^m + \Delta_l w_m^l \mathbf{T}_l^m, \quad (A.1)$$

where $\Delta_l = \partial^2/\partial x^2 + 2x^{-1}\,\partial/\partial x - x^{-2}l(l+1)$.

The Coriolis force is a bit more tedious to derive; however, if we introduce

$$\cos\theta\, Y_l^m = \alpha_l^k Y_k^m; \quad \sin\theta\cdot\partial Y_l^m/\partial\theta = \beta_l^k Y_k^m, \quad (A.2a)$$

$$\alpha_k^l = [(l^2-m^2)/(4l^2-1)]^{1/2}\delta_{k,l-1}$$
$$+\{(l+1)^2-m^2]/[4(l+1)^2-1]\}^{1/2}\,\delta_{k,l+1}, \quad (A.2b)$$

$$\beta_l^k = -(l+1)\alpha_k^l\,\delta_{k,l-1} + l\alpha_k^l\,\delta_{k,l+1}, \quad (A.2c)$$

$$\gamma_l^k = \{[k(k+1)+l(l+1)-2]/2k(k+1)\}\alpha_l^k, \quad (A.2d)$$

then we obtain

$$\mathbf{2k\times u} = -2([imv_m^l - w_m^k\beta_k^l]\mathbf{R}_l^m + \{[l(l+1)]^{-1}im(u_m^l + v_m^l) - w_m^k\gamma_k^l\}\mathbf{S}_l^m$$
$$+\{[l(l+1)]^{-1}(imw_m^l + u_m^k\beta_l^k) + v_m^k\gamma_k^l\}\mathbf{T}_l^m). \quad (A.3)$$

b) Radial boundary conditions

The system of differential equations must be completed by the boundary conditions on the radial functions. Let us suppose that the shape of the boundary is not far from spherical and that it can be written as

$$x = x_s(\theta, \phi) = 1 + \varepsilon_s f(\theta, \phi) \tag{A.4}$$

with $\varepsilon_s \ll 1$.

Rigid conditions These are $\mathbf{u}(x_s) = \mathbf{0}$, which in our harmonic analysis must be written as

$$\int_{4\pi} \mathbf{u}(x_s)\cdot\mathbf{R}_L^M \, d\Omega = \int_{4\pi} \mathbf{u}(x_s)\cdot\mathbf{S}_L^M \, d\Omega = \int_{4\pi} \mathbf{u}(x_s)\cdot\mathbf{T}_L^M \, d\Omega = 0. \tag{A.5}$$

In the general case these equations must be expanded in powers of ε_s and $f(\theta, \phi)$ as a sum of spherical harmonics; each projection on the spherical function (L, M) gives three equations, usually coupling many independent solutions of the fundamental system (2.14). When the surface is a sphere, they simply read

$$u_M^L(1) = v_M^L(1) = w_M^L(1) = 0. \tag{A.6}$$

Free conditions The conditions (2.7a) and (2.7c) give

$$\int_{4\pi} \mathbf{u}\cdot\mathbf{n}(x_s) Y_L^M \, d\Omega = \int_{4\pi} \mathbf{n}\times[\sigma]\mathbf{n}(x_s)\cdot\mathbf{S}_L^M \, d\Omega = \int_{4\pi} \mathbf{n}\times[\sigma]\mathbf{n}(x_s)\cdot\mathbf{T}_L^M \, d\Omega = 0. \tag{A.7}$$

In the case of a sphere these conditions reduce to

$$u_M^L(1) = \partial(x^{-1}v_M^L)/\partial x\big|_{x=1} = \partial(x^{-1}w_M^L)/\partial x\big|_{x=1} = 0. \tag{A.8}$$

c) Viscous stress tensor

The viscous stress tensor for an incompressible fluid is

$$\sigma_{ij} = \eta(\partial_i v_j + \partial_j v_i) = \eta s_{ij}. \tag{A.9}$$

Its spherical components can be expressed as follows

$$s_{rr} = 2(\partial u_l^m/\partial r) Y_l^m, \tag{A.10}$$

$$s_{\theta\theta} = r^{-1}\{[2u_l^m - l(l+1)v_l^m] Y_l^m + v_l^m X_l^m + w_l^m Z_l^m\}, \tag{A.11}$$

$$s_{\phi\phi} = r^{-1}\{[2u_m^l - l(l+1)v_l^m] Y_l^m - v_l^m X_l^m - w_l^m Z_l^m\}, \tag{A.12}$$

$$s_{\theta\phi} = r^{-1}(v_l^m Z_l^m - w_l^m X_l^m), \tag{A.13}$$

$$\begin{pmatrix} s_{r\theta} \\ s_{r\phi} \end{pmatrix} = s_m^l \mathbf{S}_l^m + t_m^l \mathbf{T}_l^m, \tag{A.14}$$

where we have introduced the radial functions

$$s_m^l = \partial v_m^l/\partial r + r^{-1}(u_m^l - v_m^l), \tag{A.15}$$

$$t_m^l = r\,\partial(r^{-1}w_m^l)/\partial r, \tag{A.16}$$

and the spherical functions

$$X_l^m(\theta, \phi) = 2\,\partial^2 Y_l^m/\partial\theta^2 + l(l+1) Y_l^m, \tag{A.17}$$

$$Z_l^m(\theta, \phi) = 2(\partial/\partial\theta)[(1/\sin\theta)\,\partial Y_l^m/\partial\phi], \tag{A.18}$$

which verify the orthogonality relations

$$\int_{4\pi} X_l^m X_{l'}^{m'*} + Z_l^m Z_{l'}^{m'*} \, d\Omega = (l-1)l(l+1)(l+2)\,\delta_{ll'}\,\delta_{mm'}, \tag{A.19}$$

$$\int_{4\pi} X_l^m Z_{l'}^{m'} - X_{l'}^{m'} Z_l^m \, d\Omega = 0. \tag{A.20}$$

d) The mean radial dissipation function

This function is

$$d(x) = \tfrac{1}{2}\eta \int_{4\pi} [s_{rr}^2 + s_{\theta\theta}^2 + s_{\phi\phi}^2 + 2(s_{r\theta}^2 + s_{r\phi}^2 + s_{\theta\phi}^2)] \, d\Omega \tag{A.21}$$

and can be reduced to

$$d(x)=\eta \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} \{2|\partial u_m^l/\partial x|^2+|[2u_m^l-l(l+1)v_m^l]/x|^2$$
$$+l(l+1)(|s_m^l|^2+|t_m^l|^2)+(l-1)l(l+1)(l+2)x^{-2}(|v_m^l|^2+|w_m^l|^2)\}. \tag{A.22}$$