Kinetic Turbulence in Magnetised Plasma

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with

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Forget Fluid Dynamics

Strictly speaking, fluid (hydro, MHD, two-fluid/Braginskii…) equations are only valid in the collisional limit:

\[ l \gg \lambda_{\text{mfp}} \quad \omega \ll v_{\text{ii}} \]

because they rely on particles being in local Maxwellian equilibrium, so they can be described by a few fluid moments: density, flow velocity, temperature (++)perhaps fields: magnetic, electric.

This means they are OK in dense, cold environments: wind in Chamonix valley, sodium dynamo, Earth mantle, solar convective zone, molecular clouds, some accretion discs…

They are NOT OK in hot, dilute astro and laboratory plasmas: solar wind, warm/hot ISM, intergalactic, tokamaks, LAPD, MPDX…
Large Scales and Small Scales

In fact, fluid description is perhaps OK at large scales, but virtually never on small (turbulence scales):

\[ f = F_0 + \delta f \]

Slow, collisionally enforced large-scale local equilibrium (Maxwellian)

Fast collisionless fluctuations (turbulence), often driven by gradients in the equilibrium profiles (\( \nabla T \), flow shear, etc…)

NB: even that is often not quite so, but I will not deal with non-Maxwellian equilibria in this lecture (see lectures by Kunz, Sulem, Passot on effects of pressure anisotropies: a difficult poorly chartered terrain, exciting area of current research)
Plasma Turbulence Extends to Collisionless Scales

Turbulence in the solar wind
[Sahraoui et al. 2009, PRL 102, 231102]

$\lambda_{mfp} \sim 10^8$ km ($\sim$1 AU)
$L \sim 10^5$ km
$\rho_i \sim 10^2$ km
Plasma Turbulence Extends to Collisionless Scales

Turbulence in the solar wind
[Alexandrova et al. 2009, PRL 103, 165003]

\( \lambda_{mfp} \sim 10^8 \text{ km (} \sim 1 \text{ AU) } \\
L \sim 10^5 \text{ km } \\
\rho_i \sim 10^2 \text{ km } \\

\begin{align*}
\frac{P(k)}{P_0} \text{ [nT}^8 \text{ km]} \\
&= \begin{cases}
10^{1.7} & \text{for } k \approx k_{\rho_i} \\
10^{2.8} & \text{for } k \approx k_{\lambda_i} \\
10^{-8} & \text{for } k \approx k_{\rho e} \\
10^{-8} & \text{for } k \approx k_\lambda
\end{cases}
\end{align*}

k = 2\pi f / V \text{ [km}^{-1}]
Plasma Turbulence Extends to Collisionless Scales

Interstellar medium: “Great Power Law in the Sky”

$L \sim 10^{13} \ km (~100 \ pc)$
$\lambda_{mfp} \sim 10^7 \ km$
$\rho_i \sim 10^4 \ km$
Plasma Turbulence Extends to Collisionless Scales

Intracluster (intergalactic) medium

$L \sim 10^{19} \text{ km} \ (\sim 1 \text{ Mpc})$
$\lambda_{mfp} \sim 10^{16} \text{ km} \ (\sim 1 \text{ kpc})$
$\rho_i \sim 10^4 \text{ km}$
What Is Turbulence in Such Systems?

Turbulence is always energy conversion from mean motions and fields into heat (internal energy).

In a weakly collisional plasma, it is a nontrivial question how this energy is partitioned between electrons, ions, minority admixtures, fast particles etc…
The $T_i/T_e$ Problem

- We know that $T_i \neq T_e$ is a non-equilibrium situation (entropy will increase if temperatures equalise)
- We know of no mechanism other than $i$-$e$ Coulomb collisions that would equalise temperatures (e.g., no instabilities or fluxes, like with gradients). But $v_{ie}$ is very small in weakly collisional plasmas
- Where we can measure both temperatures (lab, space), they tend to be within a factor of order unity of each other
- In extrasolar or extragalactic plasmas, we normally assume $T_i = T_e$ unless it is opportune to assume otherwise ($T_i >> T_e$ in some models of some accretion discs), but actually we have no idea what they are
- Fundamental question:
  Will turbulence equalise (or drive apart) $T_i$ and $T_e$? I.e.,
  - If $T_i > T_e$ then ion heating $<$ electron heating
  - If $T_i < T_e$ then ion heating $>$ electron heating
  If not, can we predict $T_i/T_e$ as a function of $\beta$?
Turbulence Is a Nonlinear Route to Dissipation

\[ \partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u + f, \quad \nabla \cdot u = 0. \]

\[ \frac{d}{dt} \int \frac{d^3r}{V} \frac{u^2}{2} = \varepsilon - \nu \int \frac{d^3r}{V} |\nabla u|^2 \]

\[ \varepsilon = (1/V) \int d^3r u \cdot f \]

If cascade is local, intermediate scales fill up \( \Rightarrow \) K41, spectra, and all that (see Boldyrev’s lecture)

\[ l_\nu \sim (\nu^3/\varepsilon)^{1/4} \sim L Re^{-3/4} \]

NB: viscosity can be arbitrarily small, but not zero, scales will adjust

If cascade is local, intermediate scales fill up \( \Rightarrow \) K41, spectra, and all that (see Boldyrev’s lecture)
Plasma Turbulence *Ab Initio*

\[
\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla f_s + \frac{q_s}{m_s} \left( E + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left( \frac{\partial f_s}{\partial t} \right)_c
\]

\[\nabla \cdot \mathbf{E} = 4\pi \sum_s q_s n_s,\]

\[\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} (\mathbf{j} + \mathbf{j}_{\text{ext}}),\]

\[\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{B} = 0.\]

PPCF 50, 124024 (2008)
Plasma Turbulence \textit{Ab Initio}

\[
\frac{d}{dt} \int \frac{d^3r}{V} \sum_s \int d^3\nu \frac{m_s v^2}{2} f_s = \int \frac{d^3r}{V} E \cdot j = \varepsilon - \frac{d}{dt} \int \frac{d^3r}{V} \frac{E^2 + B^2}{8\pi}
\]

Energy injection (model)

\begin{align*}
\nabla \cdot E &= 4\pi \sum_s q_s n_s, \\
\nabla \times B - \frac{1}{c} \frac{\partial E}{\partial t} &= \frac{4\pi}{c} (j + j_{\text{ext}}), \\
\n\frac{\partial B}{\partial t} &= -c \nabla \times E, \\
\n\nabla \cdot B &= 0.
\end{align*}

Work done

\[\varepsilon = -(1/V) \int d^3r E \cdot j_{\text{ext}}\]
Plasma Turbulence *Ab Initio*

\[
\frac{d}{dt} \int \frac{d^3r}{V} \sum_s \int d^3v \frac{m_s v^2}{2} f_s = \int \frac{d^3r}{V} E \cdot j = \varepsilon - \frac{d}{dt} \int \frac{d^3r}{V} \frac{E^2 + B^2}{8\pi}
\]

Work done

Entropy produced:

\[
\frac{dS_s}{dt} = \frac{d}{dt} \left[ -\int \frac{d^3r}{V} \int d^3v f_s \ln f_s \right] = -\int \frac{d^3r}{V} \int d^3v \ln f_s \left( \frac{\partial f_s}{\partial t} \right) \geq 0
\]

\[
f_s = F_{0s} + \delta f_s \quad F_{0s} = n_{0s}(\pi v_{th,s}^2)^{-3/2} \exp(-v^2/v_{th,s}^2)
\]

Boltzmann 1872

Then

\[
f_s \ln f_s = (F_{0s} + \delta f_s) \ln \left[ F_{0s} \left( 1 + \frac{\delta f_s}{F_{0s}} \right) \right]
\]

\[
\approx (F_{0s} + \delta f_s) \ln F_{0s} + \delta f_s - \frac{\delta f_s^2}{2F_{0s}} \quad \text{(to second order)}
\]

PPCF 50, 124024 (2008)
Plasma Turbulence \textit{Ab Initio}

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\frac{d}{dt} \int \frac{d^3r}{V} \sum_s \int d^3v \frac{m_s v^2}{2} f_s = \int \frac{d^3r}{V} E \cdot j = \varepsilon - \frac{d}{dt} \int \frac{d^3r}{V} \frac{E^2 + B^2}{8\pi}
\]

Work done

Entropy produced:

\[
T_{0s} \frac{dS_s}{dt} = \frac{d}{dt} \left[ \int \frac{d^3r}{V} \int d^3v \frac{m_s v^2}{2} (F_{0s} + \delta f_s) - \int \frac{d^3r}{V} \int d^3v \frac{T_{0s} \delta f_s^2}{2F_{0s}} \right]
\]

\[
= -\int \frac{d^3r}{V} \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c - n_{0s} \nu^{ss'}_{E} (T_{0s} - T_{0s'})
\]

\[
f_s \ln f_s = (F_{0s} + \delta f_s) \ln \left[ F_{0s} \left(1 + \frac{\delta f_s}{F_{0s}} \right) \right]
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PPCF \textbf{50}, 124024 (2008)
Plasma Turbulence \textit{Ab Initio}

\[
\frac{d}{dt} \int \frac{d^3r}{V} \sum_s \int d^3v \frac{m_sv^2}{2} f_s = \int \frac{d^3r}{V} E \cdot j = \varepsilon - \frac{d}{dt} \int \frac{d^3r}{V} \frac{E^2 + B^2}{8\pi}
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= -\int \frac{d^3r}{V} \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c -n_{0s} \nu_E^{ss'}(T_{0s} - T_{0s'})
\]

\[
\frac{d}{dt} \int \frac{d^3r}{V} \left[ \sum_s \int d^3v \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \frac{E^2 + B^2}{8\pi} \right]
\]

\[
= \varepsilon + \int \frac{d^3r}{V} \sum_s \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c
\]

\text{PPCF 50, 124024 (2008)}
Plasma Turbulence *Ab Initio*

\[
\frac{d}{dt} \int \frac{d^3r}{V} \sum_s \int d^3v \frac{m_s v^2}{2} f_s = \int \frac{d^3r}{V} E \cdot j = \varepsilon - \frac{d}{dt} \int \frac{d^3r}{V} \frac{E^2 + B^2}{8\pi}
\]

Work done

Energy injection (model)

\[
\varepsilon = -(1/V) \int d^3r E \cdot j_{\text{ext}}
\]

Heating:

\[
\frac{3}{2} n_{0s} \frac{dT_{0s}}{dt} = -\int \frac{d^3r}{V} \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c - n_{0s} \nu_E^{ss'} (T_{0s} - T_{0s}')
\]

Fluctuation energy budget:

\[
\frac{d}{dt} \int \frac{d^3r}{V} \left[ \sum_s \int d^3v \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \frac{E^2 + B^2}{8\pi} \right] = \varepsilon + \int \frac{d^3r}{V} \sum_s \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c
\]

PPCF 50, 124024 (2008)
“Energy” in Plasma Turbulence

\[ \frac{d}{dt} \int \frac{d^3r}{V} \left[ \sum_s \int d^3v \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \frac{E^2 + B^2}{8\pi} \right] \]

\[ = \varepsilon + \int \frac{d^3r}{V} \sum_s \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c \]

Generalised energy = free energy of the particles + fields

Kruskal & Oberman 1958
Fowler 1968
Krommes & Hu 1994
Krommes 1999
Sugama et al. 1996
Hallatschek 2004
Howes et al. 2006
Schekochihin et al. 2007
Scott 2007
Abel et al. 2013

PPCF 50, 124024 (2008)
“Energy” in Plasma Turbulence

\[
\frac{d}{dt} \int \frac{d^3r}{V} \left[ \sum_s \int d^3v \frac{T_0 s \delta f_s^2}{2F_0 s} + \frac{E^2 + B^2}{8 \pi} \right] = \varepsilon + \int \frac{d^3r}{V} \sum_s \int d^3v \frac{T_0 s \delta f_s}{F_0 s} \left( \frac{\partial \delta f_s}{\partial t} \right)_c
\]

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Howes et al. 2006
Schekochihin et al. 2007
Scott 2007
Abel et al. 2013

NB: Landau damping is a redistribution between the e-m fluctuation energy and (negative) perturbed entropy (free energy). It was pointed out already by Landau 1946 that \( \delta f_s \) does not decay: “ballistic response” \( \delta f_s \propto e^{-i k \cdot v t} \)

PPCF 50, 124024 (2008)
Analogous to Fluid, But…

\[ \frac{d}{dt} \int \frac{d^3r}{V} \left[ \sum_s \int d^3v \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \frac{E^2 + B^2}{8\pi} \right] - T\delta S \]

\[ = \varepsilon + \int \frac{d^3r}{V} \sum_s \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c \]

\[ \partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u + f. \]

\[ \frac{d}{dt} \int \frac{d^3r}{V} \frac{u^2}{2} = \varepsilon - \nu \int \frac{d^3r}{V} |\nabla u|^2 \]

\[ \varepsilon = \frac{1}{V} \int d^3r \ u \cdot f \]

PPCF 50, 124024 (2008)
Analogous to Fluid, But…

\[
\frac{d}{dt} \int \frac{d^3r}{V} \left[ \sum_s \int d^3v \frac{T_{0s} \delta f_s^2}{2F_{0s}} + \frac{E^2 + B^2}{8\pi} - T \delta S \right]
= \varepsilon + \int \frac{d^3r}{V} \sum_s \int d^3v \frac{T_{0s} \delta f_s}{F_{0s}} \left( \frac{\partial \delta f_s}{\partial t} \right)_c
\]

small scales in 6D phase space

\[\nu_{ii} v_{thi}^2 \left( \frac{\partial}{\partial v} \right)^2 \sim \omega \Rightarrow \frac{\delta v}{v_{thi}} \sim \left( \frac{\nu_{ii}}{\omega} \right)^{1/2} \sim \frac{1}{\sqrt{k_{\parallel} \lambda_{mfp}}} \ll 1\]

Small scales in velocity space (phase mixing)

PPCF 50, 124024 (2008)
Linear Phase Mixing

\[ \frac{\partial \delta f}{\partial t} + v_\parallel \frac{\partial \delta f}{\partial z} + \text{stuff} = \left( \frac{\partial \delta f}{\partial t} \right)_e \]

Can think of this as shear in phase space

- particle streaming along magnetic field
- perpendicular nonlinearity, interaction with fields, etc. (to be dealt with later)
Linear Phase Mixing

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  (to be dealt with later)

“Ballistic response”: \( \delta f \propto e^{i k_\parallel v_\parallel t} \)
(always part of the Landau damping solution)

\[ \frac{\partial \delta f}{\partial v_\parallel} = i k_\parallel t \delta f \quad \Rightarrow \quad \frac{\delta v_\parallel}{v_\text{th}} \sim \frac{1}{k_\parallel v_\text{th} t} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]

So small-scale structure forms and is eventually wiped out by collisions (but we will see that there is a much faster nonlinear mechanism for that: turbulence)
Let us now see what happens nonlinearly: the turbulent cascade starts as MHD turbulence and gets to collisionless scales, then what?

\[ \delta B \]

\[ \delta u \]

energy injected

\[ \frac{1}{L} \quad k_{||} \lambda_{mp} \sim 1 \]
MHD Cascade Is Anisotropic

- **Strong anisotropy:**
  \[ \frac{k_{\parallel}}{k_{\perp}} \ll 1 \]
  In magnetised plasma, confirmed by numerics (MHD) and observations (solar wind, ISM)

- **Strong nonlinearity:**
  \[ \omega \sim k_{\parallel} v_A \sim k_{\perp} u_{\perp} \]
  Critical balance as a physical principle (see Boldyrev’s lectures)

- (2+1)D route through phase space:

[Solar wind diagram showing power spectra and wave numbers]

Turbulence Reaches Larmor Scales and Beyond

- **Strong anisotropy:** \( \frac{k_{||}}{k_{\perp}} \ll 1 \)

- **Strong nonlinearity:** \( \omega_{\text{linear}} \sim \omega_{\text{nonlinear}} \)
  
  **Critical balance as a physical principle**
  
  (see Boldyrev’s lectures)

- (2+1)D route through phase space:

  \[ k_{\perp} \rho_i \sim 1 \]

  \[ k_{||} \sim k_{\perp}^{2/3} \]

  \[ k_{||}^{-2} \]

  \[ k_{\perp}^{-5/3} \]

In magnetised plasma, confirmed by numerics (MHD) and observations (solar wind, ISM)

[Solar Wind]

Turbulence Reaches Larmor Scales and Beyond

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\frac{k_\parallel}{k_\perp} \ll 1
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In magnetised plasma, confirmed by numerics (MHD) and observations (solar wind, ISM)

• Strong nonlinearity: \[
\omega \sim k_\parallel v_A \sim k_\perp u_\perp
\]

\[\omega_{\text{linear}} \sim \omega_{\text{nonlinear}}\]

Critical balance as a physical princip.
(see Boldyrev’s lectures)

• (2+1)D route through phase space

• Stuff happens at \[k_\perp \rho_i \sim 1\]

[Sahraoui et al. 2009, PRL 102, 231102]
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NB: This transition is also key in fusion plasmas, whence comes much of the appropriate theoretical machinery (see Jenko’s lecture)

[Roach et al. 2009, PPCF 51, 124020]
Turbulence Reaches Larmor Scales and Beyond

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[Image of graph: \( n(k) \sim k^{-3} \) for \( B = 3.8 \) T and \( k^{-6} \) for \( B = 1.8 \) T]

[Hennequin et al. 2004, PPCF 46, B121]
Turbulence Reaches Larmor Scales and Beyond

- **Strong anisotropy:** \( \frac{k_{\parallel}}{k_{\perp}} \ll 1 \)

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  *Critical balance as a physical principle*

  (see Boldyrev’s lectures)

- **(2+1)D route through phase space**

- **Stuff happens at** \( k_{\perp} \rho_i \sim 1 \)

NB: This transition is also key in fusion plasmas, whence comes much of the appropriate theoretical machinery

(see Jenko’s lecture)

In magnetised plasma, confirmed by numerics (MHD) and observations (solar wind, ISM)

TOKAMAK DNS

\[ [\text{Görler & Jenko (2008), PoP 15, 102508}] \]
Critical Balance as an Ordering Assumption

- **Strong anisotropy:** $\epsilon \sim \frac{k_\parallel}{k_\perp} \ll 1$ (this is the small parameter!)

- **Strong nonlinearity:** $\omega \sim k_\parallel v_A \sim k_\perp u_\perp$
  
  (critical balance as an ordering assumption)
Gyrokinetics

- **Strong anisotropy:** \( \epsilon \sim \frac{k_{||}}{k_{\perp}} \ll 1 \) (this is the small parameter!)

- **Strong nonlinearity:** \( \omega \sim k_{||} v_A \sim k_{\perp} u_{\perp} \)
  (critical balance as an ordering assumption)

- **Finite Larmor radius:** \( k_{\perp} \rho_i \sim 1 \)

- **Weak collisions:** \( \frac{\omega}{\nu_{ii}} \sim \frac{k_{||} \lambda_{\text{mfp}}}{\sqrt{\beta_i}} \sim 1 \)

**GK ORDERING:** \( f_s = F_{0s} + \delta f_s \)

\[
\frac{\delta f_s}{F_{0s}} \sim \frac{\delta B}{B_0} \sim \frac{e\varphi}{T_e} \sim \frac{k_{||}}{k_{\perp}} \sim \frac{\rho_i}{\lambda_{\text{mfp}}} \sim \frac{\omega}{\Omega_i} \sim \epsilon
\]

Gyrokinetics: Kinetics of Larmor Rings

Particle dynamics can be averaged over the Larmor orbits and everything reduces to kinetics of Larmor rings centered at

\[ R_s = r + \frac{v \times \hat{z}}{\Omega_s} \]

and interacting with the electromagnetic fluctuations.

\[ \delta f_s = -q_s \varphi F_{0s}/T_{0s} + h_s(t, R_s, v_\perp, v_\parallel) \]

Boltzmann bit

distribution of rings

only two velocity variables, i.e., 6D → 5D

Gyrokinetics: Kinetics of Larmor Rings

Particle dynamics can be averaged over the Larmor orbits and everything reduces to kinetics of Larmor rings centered at

$$R_s = r + \frac{v \times \mathbf{\hat{z}}}{\Omega_s}$$

Catto transformation

and interacting with the electromagnetic fluctuations.

$$\delta f_s = - q_s \varphi F_{0s}/T_{0s} + h_s(t, R_s, v_\perp, v_\parallel)$$

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \left\{ \langle \chi \rangle_{R_s}, h_s \right\} = \frac{q_s F_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle_{R_s}}{\partial t} + \left( \frac{\partial h_s}{\partial t} \right)_c$$

$$\chi = \varphi - v \cdot A/c, \quad B = B_0 \mathbf{\hat{z}} + \delta B, \quad \delta B = \nabla \times A$$

+ Maxwell’s equations
quasineutrality and
Ampère’s law

$$\langle \chi(t, r, v) \rangle_{R_s} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \chi \left( t, R_s - \frac{v \times \mathbf{\hat{z}}}{\Omega_s}, v \right)$$

Gyrokinetics: Kinetics of Larmor Rings

Averaged gyrocentre drifts:
- $\mathbf{E} \times \mathbf{B}_0$ drift
- $\nabla \mathbf{B}$ drift
- motion along perturbed fieldline

Averaged wave-ring interaction

$$\left\langle \frac{d\mathbf{R}_s}{dt} \right\rangle_{\mathbf{R}_s} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s} - \left\langle \frac{d\mathcal{E}_s}{dt} \frac{\partial f_s}{\partial \mathcal{E}_s} \right\rangle_{\mathbf{R}_s}$$

$$\delta f_s = -q_s \varphi \frac{F_{0s}}{T_{0s}} + h_s(t, \mathbf{R}_s, v_\perp, v_\parallel)$$

$$\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \frac{c}{B_0} \left\{ \langle \chi \rangle_{\mathbf{R}_s}, h_s \right\} = \frac{q_s F_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle_{\mathbf{R}_s}}{\partial t} + \left( \frac{\partial h_s}{\partial t} \right)_c$$

$$\chi = \varphi - \mathbf{v} \cdot \mathbf{A}/c, \quad \mathbf{B} = B_0 \hat{z} + \delta \mathbf{B}, \quad \delta \mathbf{B} = \nabla \times \mathbf{A}$$

+ Maxwell’s equations
  (quasineutrality and Ampère’s law)

$$\langle \chi(t, \mathbf{r}, \mathbf{v}) \rangle_{\mathbf{R}_s} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \chi \left( t, \mathbf{R}_s - \frac{\mathbf{v} \times \hat{z}}{\Omega_s}, \mathbf{v} \right)$$

GK Phase Mixing (Entropy Cascade)

linear phase mixing  
\( \frac{\partial h_i}{\partial t} + v_\parallel \frac{\partial h_i}{\partial z} + \frac{c}{B_0} \{ \langle \varphi \rangle_{R_i}, h_i \} - \left( \frac{\partial h_i}{\partial t} \right)_c = \frac{\partial}{\partial t} \frac{Ze\langle \varphi \rangle_{R_i}}{T_{0i}} F_{0i} \)

nonlinearity  
\( \langle v_E \rangle_{R_i} \cdot \nabla h_i \)

collisions

- Gyroaveraged fluctuations mix \( h_i \) via this term, so \( h_i \) develops small (perpendicular) scales in the gyrocenter space: \( k_\perp \rho_i \gg 1 \)

PPCF 50, 124024 (2008)
GK Phase Mixing (Entropy Cascade)

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- Gyroaveraged fluctuations mix \( h_i \) via this term, so \( h_i \) develops small (perpendicular) scales in the gyrocenter space: \( k_\perp \rho_i \gg 1 \)
- In this limit, free energy conservation is

\[ \frac{d}{dt} \int d^3 R_i d^3 v \frac{h_i^2}{2F_{0i}} = \int d^3 R_i d^3 v \frac{h_i}{F_{0i}} \left( \frac{\partial h_i}{\partial t} \right)_c \leq 0 \]

This is (minus) the entropy of the perturbed distribution; it is damped only by collisions (Boltzmann!), so \( h_i \) must be phase mixed to small scales in velocity space. HOW?

PPCF 50, 124024 (2008)
GK Phase Mixing (Entropy Cascade)

linear phase mixing
\[ \frac{\partial h_i}{\partial t} + v_{||} \frac{\partial h_i}{\partial z} + \frac{c}{B_0} \{ \langle \varphi \rangle_{R_i}, h_i \} - \left( \frac{\partial h_i}{\partial t} \right)_c = \frac{\partial}{\partial t} \frac{Z e \langle \varphi \rangle_{R_i}}{T_{0i}} F_{0i} \]

(collisions)

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- Gyroaveraged fluctuations mix \( h_i \) via this term, so \( h_i \) develops small (perpendicular) scales in the gyrocenter space: \( k_\perp \rho_i \gg 1 \)

- Two values of the gyroaveraged potential \( \langle \varphi \rangle_{R_i}(v) \) and \( \langle \varphi \rangle_{R_i}(v') \) come from spatially decorrelated fluctuations if

\[ \frac{v_\perp}{\Omega_i} - \frac{v'_\perp}{\Omega_i} \sim \frac{1}{k_\perp} \Rightarrow \frac{\delta v_\perp}{v_{thi}} \sim \frac{1}{k_\perp \rho_i} \]

[This perpendicular nonlinear phase-mixing mechanism was anticipated by Dorland & Hammett 1993]

PPCF 50, 124024 (2008)
GK Phase Mixing (Entropy Cascade)

\[ \frac{\partial h_i}{\partial t} + v_{\parallel} \frac{\partial h_i}{\partial z} + \frac{c}{B_0} \left\{ \langle \varphi \rangle_{R_i}, h_i \right\} - \left( \frac{\partial h_i}{\partial t} \right)_c = \frac{\partial}{\partial t} \frac{Ze\langle \varphi \rangle_{R_i}}{T_{0i}} F_{0i} \]

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\]

[This perpendicular nonlinear phase-mixing mechanism was anticipated by Dorland & Hammett 1993]

[PPCF 50, 124024 (2008)]

[Tatsuno et al. 2009, PRL 103, 015003]
Entropy Cascade

• The cascade is now in phase space, involving both position and velocity, because entropy must get to small scales in velocity

• G. Plunk has developed a spectral formalism to quantify perpendicular velocity-space structure via Hankel transforms:

\[ \hat{h}_i(k, p, v_\parallel) = 2\pi \int dv_\perp v_\perp J_0(pv_\perp) h_i(k, v_\perp, v_\parallel) \]

\[ E(k, p) = p \langle |\hat{h}_i(k, p)|^2 \rangle \]

• T. Tatsuno found the cascade along the \((k, p)\) diagonal in his 2D GK DNS


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Entropy Cascade

• The cascade is now in phase space, involving both position and velocity, because entropy must get to small scales in velocity
• **Kolmogorov-style constant-flux argument gives**
  
  \[ h_i \sim k^{-4/3}_\perp \]
  
  \[ \phi \sim k^{-10/3}_\perp \]

  [PPCF 50, 124024 (2008)]

• 2D GK DNS by **T. Tatsuno** confirm these scalings
  
Entrophy Cascade

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spectrum of $h_i \sim k^{-4/3}_\perp$
  spectrum of $\varphi \sim k^{-10/3}_\perp$

PPCF 50, 124024 (2008)

• 2D GK DNS by T. Tatsuno confirm these scalings

• It is attractive to think of this as a universal theory of sub-Larmor turbulence and, for example, attribute to it the sub-Larmor scalings seen in 3D GK DNS of tokamak turbulence by Görler & Jenko (2008)

[Görler & Jenko (2008), PoP 15, 102508]
Entropy Cascade

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  *PPCF 50, 124024 (2008)*

- 2D GK DNS by **T. Tatsuno** confirm these scalings

- **G. Plunk** has developed a full Kolmogorov-style theory for the 2D version of this turbulence, complete with **third-order exact laws**, **direct and inverse cascades**, **scalings in phase space** etc.
  [Plunk et al. 2010, *JFM* 664, 407]
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  [Plunk et al. 2010, *JFM* 664, 407]

- **Dissipation scale in phase space** (cf. Kolmogorov scale vs. Re)
  
  \[
  \frac{\delta v_{\perp c}}{v_{thi}} \sim \frac{1}{k_{\perp c} \rho_i} \sim D_o^{-3/5}
  \]

  **Dorland Number**

  \[
  D_o = \frac{1}{\nu_{ii} \tau \rho_i} \text{ characteristic time at the ion gyroscale}
  \]
Entropy Cascade

- The cascade is now in phase space, involving both position and velocity, because entropy must get to small scales in velocity
- **Kolmogorov-style constant-flux argument gives**
  
  - spectrum of $h_i \sim k_{\perp}^{-4/3}$
  - spectrum of $\varphi \sim k_{\perp}^{-10/3}$

- Just last week a PRL by **E. Kawamori** came out claiming a laboratory measurement that confirms the entropy cascade:

  [Kawamori (2013), PRL 110, 195001]
Linear vs. Nonlinear (GK) Phase Mixing

Since cascade is nonlinear, mixing occurs in one turnover time (fast)

\[ \frac{\delta v_{\perp}}{v_{\text{thi}}} \sim \frac{1}{K_{\perp c} \rho_i} \sim D_0^{-3/5} \]

\[ D_0 = \frac{1}{\nu_{ii} \tau_{\rho_i}} \gg 1 \]

characteristic time at the ion gyroscale
Linear vs. Nonlinear (GK) Phase Mixing

**NONLINEAR**

\[
\frac{\delta \nu_{\perp c}}{\nu_{thi}} \sim \frac{1}{k_{\perp c} \rho_i} \sim D_0^{-3/5}
\]

\[
D_0 = \frac{1}{\nu_{ii} \tau_{\rho_i}} \gg 1
\]

Since cascade is nonlinear, mixing occurs in one turnover time (fast)

**LINEAR**

\[
\frac{\partial h_i}{\partial t} + v_{||} \frac{\partial h_i}{\partial z} + \ldots = 0
\]

“ballistic response”: \( h_i \propto e^{ik_{||}v_{||}t} \)

\[
\frac{\delta v_{||}}{\nu_{thi}} \sim \frac{1}{k_{||} \nu_{thi} t} \sim 1
\]

after one turnover time if “critical balance” holds, so linear phase mixing is slow
So How Do MHD and GK Tie Together?

Gyrokinetics is valid (almost) everywhere

As \( k_\parallel \ll k_\perp \), the cyclotron frequency is only reached deep in the dissipation range.

\[
\begin{align*}
\delta B & \\
\delta E & \\
n_{\text{energy injected}} & \\
\text{collisional (fluid)} & \quad \text{collisionless (kinetic)}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{L} & \\
k_\parallel \lambda_{\text{mfp}} & \sim 1 \\
k_\perp \rho_i & \sim 1 \\
k_\perp \rho_e & \sim 1
\end{align*}
\]

GYROKINETICS

FLUID THEORY

Gyrokinetics: Long-Wavelength Limit

\[ k_{\perp} \rho_i \ll 1 \] RMHD strictly valid

for \textit{Alfvén waves}:

\[
\frac{\partial}{\partial t} \nabla^2_\perp \Psi + \{ \Psi, \nabla^2_\perp \Psi \} = v_A \frac{\partial}{\partial z} \nabla^2_\perp \Psi + \{ \Psi, \nabla^2_\perp \Psi \}
\]

\[
\frac{\partial \Psi}{\partial t} + \{ \Phi, \Psi \} = v_A \frac{\partial \Phi}{\partial z}
\]

[Strauss 1976, Phys. Fluids 19, 134]

\[ \delta E \]

\[ \delta B \]

Alfvénic fluctuations are \textbf{not Landau damped}
(no parallel electric field)
and \textbf{do not know about collisions}
because they are Maxwellian.
So it’s all like in MHD
(see Boldyrev’s lecture).

\[ k \sim 5/3, 3/2 \]

\[ \frac{1}{L} \]

\[ k_{||} \lambda_{mfp} \sim 1 \]

\[ k_{\perp} \rho_i \sim 1 \]

\[ k_{\perp} \rho_e \sim 1 \]

\textbf{GYROKINETICS}

\textbf{FLUID THEORY}

Gyrokinetics: Long-Wavelength Limit

For Alfvén waves:

\[
\frac{\partial}{\partial t} \nabla^2 \Phi + \{\Phi, \nabla^2 \Phi\} = v_A \frac{\partial}{\partial z} \nabla^2 \Psi + \{\Psi, \nabla^2 \Psi\}
\]

\[
\frac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_A \frac{\partial \Phi}{\partial z}
\]

Alfvénic fluctuations are **not Landau damped** (no parallel electric field) and do not know about collisions because they are Maxwellian. So it’s all like in MHD (see Boldyrev’s lecture).

\[ k_{\perp} \rho_i \ll 1 \text{ RMHD strictly valid} \]

\[ k_{\perp} \rho_i \sim 1, \quad k_{\perp} \rho_e \sim 1 \]

\[ \frac{1}{L} \quad k_{\parallel} \lambda_{mfp} \sim 1 \]

**GYROKINETICS**

Compressive fluctuations do not seem to be Landau damped, even though one might think they ought to be (a bit of a puzzle, A. Kanekar, work in progress).
Gyrokinetics: Short-Wavelength Limit

$\delta B$ + compressive fluctuations

$\delta E$

energy injected

$\frac{1}{L} = k_{||} \lambda_{mfp} \sim 1$

$k_{\perp} \rho_i \sim 1$

$k_{\perp} \rho_e \sim 1$

$\mathbf{KAW}$

$K^ {\sim 5/3, 3/2}$?

$K^ {\sim 1/3, -2/3}$?

$K^ {\sim 7/3, 8/3}$?

$K^{-7/3, 8/3}$?

$\mathbf{KAW}$

$K^{-5/3, 3/2}$?

$K^{-1/3, -2/3}$?

$K^{-7/3, 8/3}$?

$\mathbf{KAW}$

$K^{-1/3, -2/3}$?

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$K^{-7/3, 8/3}$?

$\mathbf{KAW}$

$K^{-5/3, 3/2}$?

$\mathbf{KAW}$

$K^{-1/3, -2/3}$?

$K^{-7/3, 8/3}$?
Gyrokinetics: Larmor-Scale Transition

Alfvén waves + compressive fluctuations

$\delta B$

$\delta E$

energy injected

$k_{||} \lambda_{mfp} \sim 1$

$k_{\perp} \rho_i \sim 1$

$k_{\perp} \rho_e \sim 1$

$1/L$

ion Landau damping

electron Landau damping

$KAW$

$k^{-1/3,-2/3}$?

$k^{-7/3, 8/3}$?


Fluid Theory
Gyrokinetics: Larmor-Scale Transition

Alfvén waves
+ compressive fluctuations

δE

δB

δf_i

energy injected

$k_\parallel \lambda_{mfp} \sim 1$

$k_\perp \rho_i \sim 1$

$k_\perp \rho_i \sim (\nu_{ii}\tau_{\rho_i})^{-3/5}$

$1/L$

KAW

k^{-4/3}

ion Landau damping

Entropy cascade

electron Landau damping

δf_i

GYROKINETICS

FLUID THEORY

Free Energy Cascade

The splitting of the cascade at the ion gyroscale determines relative heating of the species

The free energy invariant splits into these different bits

\[ \delta E, \delta B, \delta f_i \]

\[ k_{\parallel} \lambda_{\text{mfp}} \sim 1 \]

\[ k_{\perp} \rho_i \sim (v_{ii} / \rho_i)^{-3/5} \]

Alfvén waves + compressive fluctuations

Ion Landau damping

Entropy cascade

Electron entropy cascade

Electron heating

Ion heating

\[ \text{ApJS 182, 310 (2009)} \]
Larmor Transition: 3D GK DNS (by G. Howes)

Alfvén waves + compressive fluctuations

δE
δB
δf_i

δf_i

δB

δE

Entrophy cascade

Landau damping

Landau damping

$k_\perp \rho_i \sim 1$

$k_\perp \rho_i \sim (\nu_{ii} \tau_{\rho i})^{-3/5}$

[Howes et al. 2008, PRL 100, 065004]

Sub-Larmor Cascade: 3D GK DNS (by G. Howes)

Alfvén waves + compressive fluctuations

δB
δE
δf_i

KAW

Landau damping
Entropy cascade
electron Landau damping

[Howes et al. 2011, PRL 107, 035004]

KAW Fluctuations

Start with GK, consider the scales such that \( k_\perp \rho_i \gg 1, \quad k_\perp \rho_e \ll 1 \)

This is not a very wide interval, but an important one:

\[
\sqrt{\frac{m_i}{m_e}} \approx 42
\]

(answer to the general question of life, Universe and everything)
KAW Fluctuations

Start with GK, consider the scales such that \( k \perp \rho_i \gg 1, \ k \perp \rho_e \ll 1 \)

\[
\begin{align*}
\frac{\partial \Psi}{\partial t} &= v_A (1 + Z/\tau) \hat{b} \cdot \nabla \Phi, \\
\frac{\partial \Phi}{\partial t} &= -\frac{v_A}{2 + \beta_i (1 + Z/\tau)} \hat{b} \cdot \nabla (\rho_i^2 \nabla^2 \Psi)
\end{align*}
\]

This is the anisotropic version of EMHD [Kingsep et al. 1990, Rev. Plasma Phys. 16, 243], which is derived (for \( \beta_i \gg 1 \)) by assuming magnetic field frozen into electron fluid and doing a RMHD-style anisotropic expansion:

\[
\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi e n_{0e}} \nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}]
\]

\[
\begin{align*}
\frac{\delta \mathbf{B}}{B_0} &= \frac{1}{v_A} \hat{z} \times \nabla \perp \Psi + \hat{\mathbf{z}} \frac{\delta B \parallel}{B_0} \\
\frac{\delta n_e}{n_{0e}} &= -\frac{Z e \phi}{T_{0i}} = -\frac{2}{\sqrt{\beta_i \rho_i v_A}} \frac{\Phi}{v_A} \\
\frac{\delta B \parallel}{B_0} &= \beta_i \left(1 + \frac{Z}{\tau}\right) \frac{Z e \phi}{T_{0i}} \sqrt{\beta_i} \left(1 + \frac{Z}{\tau}\right) \frac{\Phi}{\rho_i v_A} \\
u_{\parallel e} &= \frac{c}{4\pi e n_{0e}} \nabla_\parallel^2 A_\parallel = -\frac{\rho_i \nabla_\perp^2 \Psi}{\sqrt{\beta_i}}
\end{align*}
\]

Boltzmann ions

Pressure balance

\[
B_0 \delta B \parallel = -\delta p_i - \delta p_e = -T_{0i} \delta n_i - T_{0e} \delta n_e.
\]

[Ions more or less an immobile neutralising background]
Kinetic Alfvén Waves

Start with GK, consider the scales such that \( k_\perp \rho_i \gg 1, \ k_\perp \rho_e \ll 1 \)

\[
\begin{align*}
\frac{\partial \Psi}{\partial t} &= v_A \left(1 + \frac{Z}{\tau}\right) \hat{b} \cdot \nabla \Phi, \\
\frac{\partial \Phi}{\partial t} &= -\frac{v_A}{2 + \beta_i \left(1 + \frac{Z}{\tau}\right)} \hat{b} \cdot \nabla \left(\rho_i^2 \nabla^2 \Psi\right)
\end{align*}
\]

Linear wave solutions:

\[
\omega = \pm \sqrt{\frac{1 + \frac{Z}{\tau}}{2 + \beta_i \left(1 + \frac{Z}{\tau}\right)}} \ k_\perp \rho_i k_\parallel v_A
\]

- Right-hand elliptically polarized
- \( \delta E \sim k_\perp \phi \propto k_\perp \delta B \)
- Landau-damped

\[ \text{ApJS 182, 310 (2009)} \]
Kinetic Alfvén Waves

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\end{align*}
\]

Linear wave solutions:

\[
\omega = \pm \sqrt{\frac{1 + Z/\tau}{2 + \beta_i (1 + Z/\tau)}} k_{\perp} \rho_i k_{\parallel} v_A
\]

\[\text{ApJS 182, 310 (2009)}\]
KAW Turbulence

Start with GK, consider the scales such that \( k_\perp \rho_i \gg 1, \ k_\perp \rho_e \ll 1 \)

\[
\begin{align*}
\frac{\partial \Psi}{\partial t} &= v_A (1 + Z/\tau) \hat{b} \cdot \nabla \Phi, \\
\frac{\partial \Phi}{\partial t} &= -\frac{v_A}{2 + \beta_i (1 + Z/\tau)} \hat{b} \cdot \nabla \left( \rho_i^2 \nabla^2 \Psi \right)
\end{align*}
\]

Linear wave solutions:

\[
\omega = \pm \sqrt{\frac{1 + Z/\tau}{2 + \beta_i (1 + Z/\tau)}} k_\perp \rho_i k_\parallel v_A
\]

• There is a cascade of KAW,
  \[
  \frac{\delta B_\parallel}{B_0} \sim \frac{\Phi}{\rho_i v_A} \sim \frac{k_\perp \Psi}{v_A} \sim \frac{\delta B_\perp}{B_0}
  \]

• Critical balance + constant flux argument à la K41/GS95 give
  \[
  k_\perp^{-7/3}
  \]
  spectrum of magnetic field with anisotropy
  \[
  k_\parallel \sim k_\perp^{1/3}
  \]


• Electric field has \( k_\perp^{-1/3} \) spectrum because \( \delta E \sim k_\perp \phi \propto k_\perp \delta B \)

• Recent modification of the theory by Boldyrev amends the spectrum to \( k_\perp^{-8/3} \) by restricting cascade to 2D sheets [arXiv:1204.5809]

• NB: none of this takes into account Landau damping
Sub-Larmor Cascade: 3D GK DNS (by G. Howes)

Alfvén waves + compressive fluctuations

$\delta B$

$\delta E$

$\delta f_i$

$E_{E_i}(k_\perp) \propto k_\perp^{-1/3}$

$E_{B_i}(k_\perp) \propto k_\perp^{-7/3}$

$E_{B_i}(k_\perp) \propto k_\perp^{-2.8}$

$E_{B_i}(k_\perp) \propto k_\perp^{-1}$

$\kappa_\perp \rho_i \sim 1$

$\kappa_\perp \rho_i \sim (v_{ii}\tau_{\rho_i})^{-3/5}$

[Howes et al. 2011, PRL 107, 035004]

Sub-Larmor Cascade: 3D GK DNS (by G. Howes)

1. Power law spectra all the way to electron gyroscale despite electron Landau damping

[Howes et al. 2011, PRL 107, 035004]
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1. Power law spectra all the way to electron gyroscale despite electron Landau damping
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4. Magnetic spectrum ~ $k_{\perp}^{-2.8}$ which is steeper than KAW standard result $-7/3$ (perhaps closer to Boldyrev’s $-8/3$) and in close agreement with solar wind data

[Howes et al. 2011, PRL 107, 035004]
Sub-Larmor Cascade: Solar Wind

1. Power law spectra all the way to electron gyroscale despite electron Landau damping
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4. Magnetic spectrum $\sim k_\perp^{-2.8}$ which is steeper than KAW standard result $-7/3$ (perhaps closer to Boldyrev’s $-8/3$) and in close agreement with solar wind data

[Alexandrova et al. 2009, PRL 103, 165003]
Sub-Larmor Cascade: Solar Wind

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Conclusions

- **Kinetic turbulence is a free-energy cascade in phase space towards collisional scales**
- **Gyrokinetics is a good approximation for magnetised turbulence**
- In gyrokinetic turbulence, a fast **nonlinear perpendicular** phase-mixing mechanism allows small-scale structure to emerge simultaneously in physical and velocity space ("entropy cascade")
- We still need to understand how **linear (||)** and **nonlinear (⊥)** phase mixing compete/coexist
- The free energy cascade **splits** into various channels:
  - AW + compressive above ion gyroscale ("inertial range")
  - KAW + entropy cascade below ion gyroscale ("dissipation range")
- The splitting at the ion gyroscale determines the relative heating of the two species ➔ energy partition
- **Structure of kinetic cascades** (KAW turbulence, entropy cascade, compressive cascade) is interesting in its own right (and measurable!)

Free Energy Cascade

The splitting of the cascade at the ion gyroscale determines relative heating of the species.

The free energy invariant splits into these different bits.

Alfvén waves
+ compressive fluctuations

Ion heating

Electron heating

Entropy cascade

KAW

Electron entropy cascade

The splitting of the cascade at the ion gyroscale determines relative heating of the species.