ÉCOLE DE PHYSIQUE
des HOUCHES

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Kinetic Turbulence in Magnetised Plasma

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AAS et al. 2008, *PPCF* **50**, 124024 [arXiv:0806.1069]
AAS et al. 2009, *Astrophys. J. Suppl.* **182**, 310 [arXiv:0704.0044]

Forget Fluid Dynamics

Strictly speaking, fluid (hydro, MHD, two-fluid/Braginskii…) equations are only valid in the collisional limit:

$$\boxed{l \gg \lambda_{\mathrm{mfp}} \qquad \omega \ll \nu_{ii}}$$

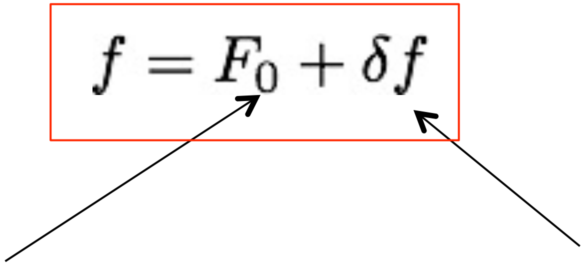
because they rely on particles being in local Maxwellian equilibrium, so they can be described by a few fluid moments: density, flow velocity, temperature (++perhaps fields: magnetic, electric)

This means they are OK in dense, cold environments:
wind in Chamonix valley, sodium dynamo, Earth mantle, solar convective zone, molecular clouds, some accretion discs…

They are NOT OK in hot, dilute astro and laboratory plasmas:
solar wind, warm/hot ISM, intergalactic, tokamaks, LAPD, MPDX…

Large Scales and Small Scales

In fact, fluid description is perhaps OK at large scales, but virtually never on small (turbulence scales):



$$\boxed{f = F_0 + \delta f}$$

Slow, collisionally enforced
large-scale local equilibrium
(Maxwellian)

Fast collisionless fluctuations
(turbulence),
often driven by gradients
in the equilibrium profiles
(∇T , flow shear, etc…)

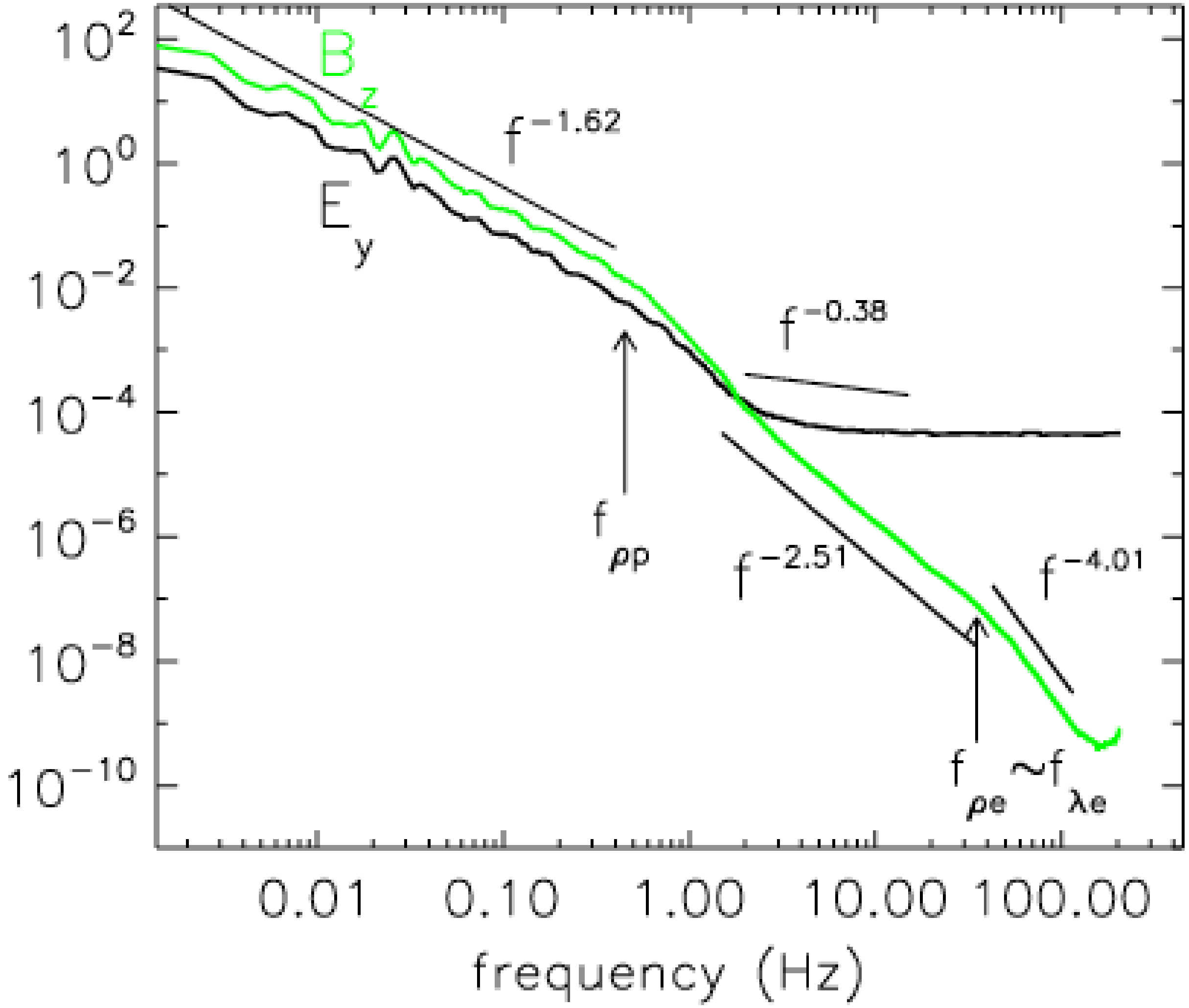
NB: even that is often not quite so,
but I will not deal with non-Maxwellian
equilibria in this lecture
(see lectures by Kunz, Sulem, Passot
on effects of pressure anisotropies:
a difficult poorly chartered terrain,
exciting area of current research)

Plasma Turbulence Extends to Collisionless Scales

Turbulence in the solar wind

[Sahraoui et al. 2009, PRL **102**, 231102]

$\lambda_{mfp} \sim 10^8$ km (~1 AU)
$L \sim 10^5$ km
$\rho_i \sim 10^2$ km



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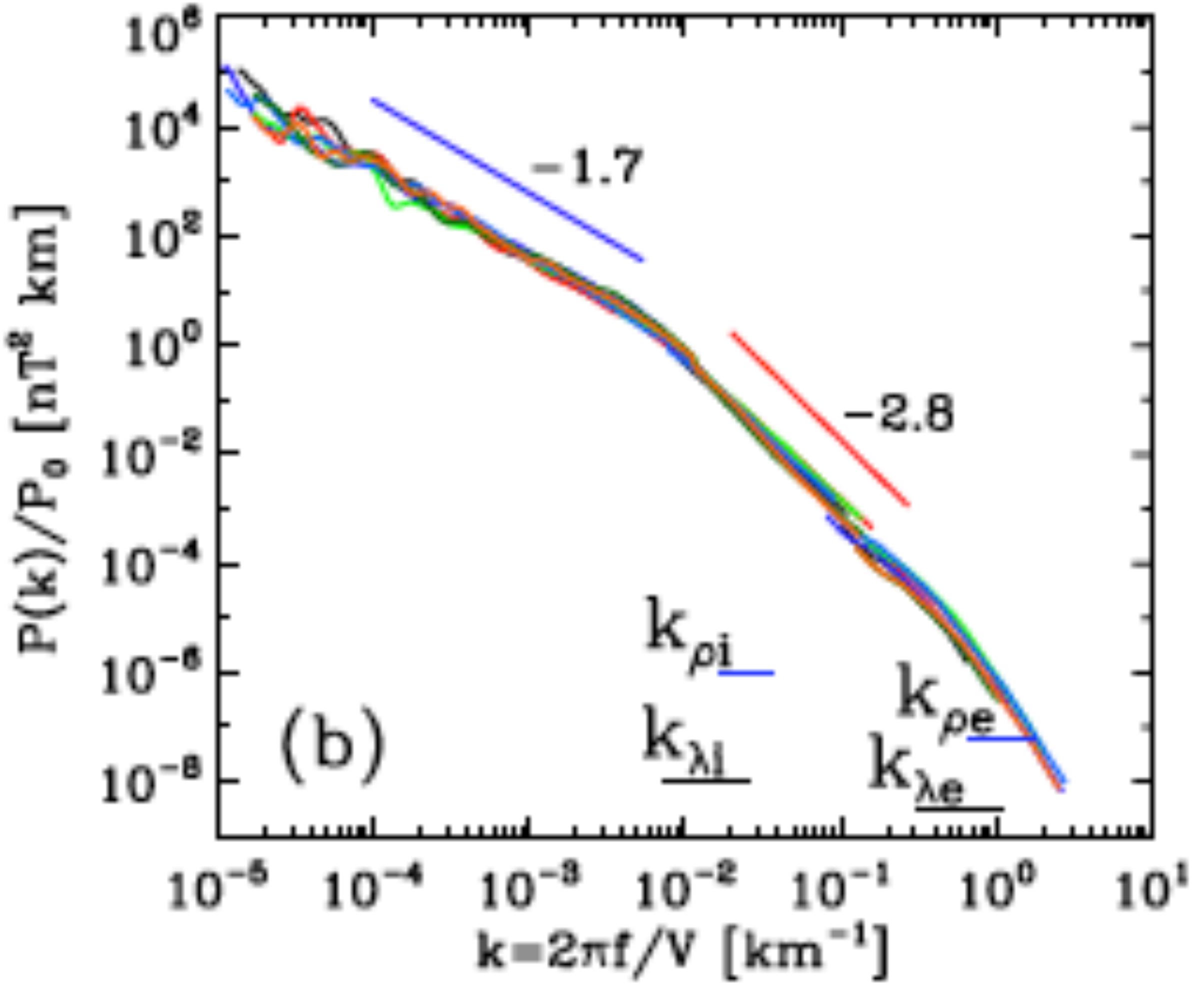
Turbulence in the solar wind

[Alexandrova et al. 2009, PRL **103**, 165003]



Plasma Turbulence Extends to Collisionless Scales

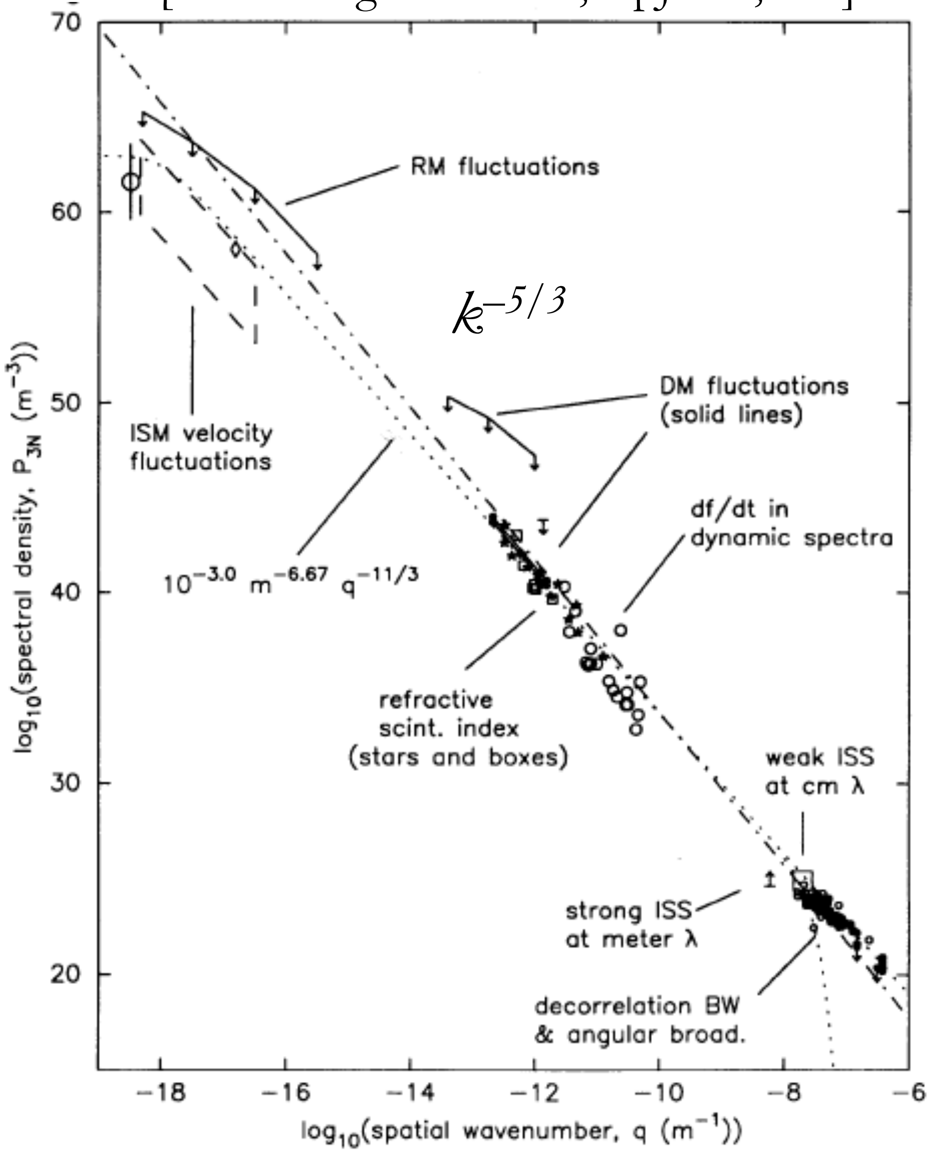
Interstellar medium: “Great Power Law in the Sky”

[Armstrong et al. 1995, ApJ **443**, 209]

$L \sim 10^{13}$ km ($\sim$100 pc)

$\lambda_{mfp} \sim 10^7$ km

$\rho_i \sim 10^4$ km



Plasma Turbulence Extends to Collisionless Scales

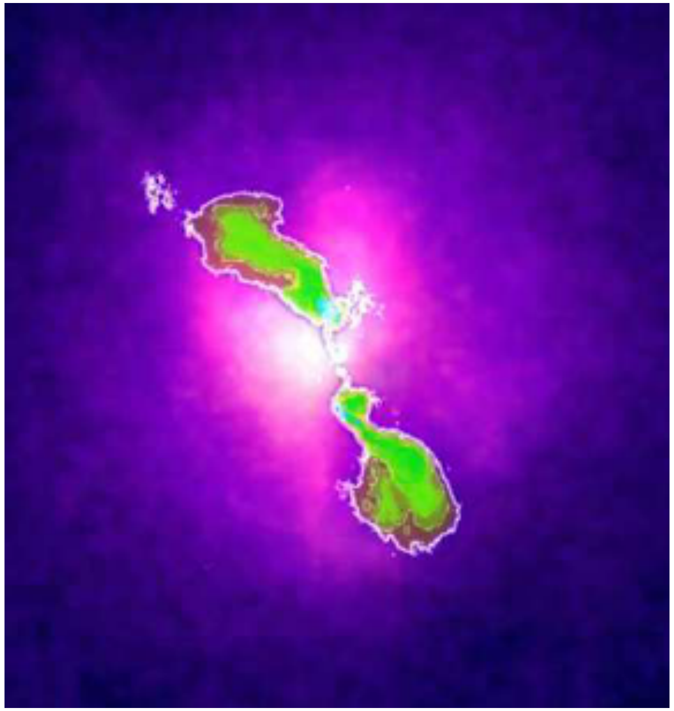
Intracluster (intergalactic) medium

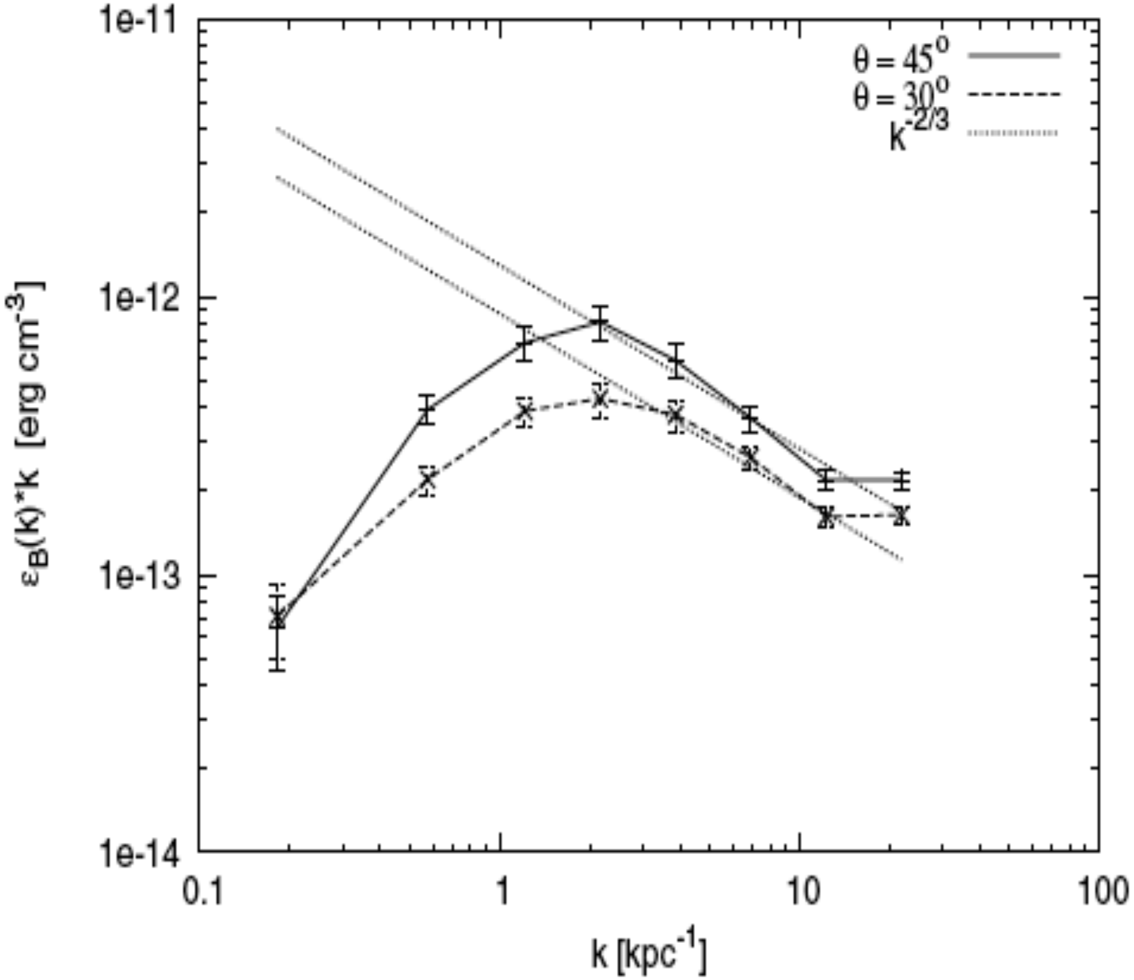
Hydra A cluster [Vogt & Enßlin 2005, A&A **434**, 67]

$L \sim 10^{19}$ km (~1 Mpc)

$\lambda_{mfp} \sim 10^{16}$ km (~1 kpc)

$\rho_i \sim 10^4$ km

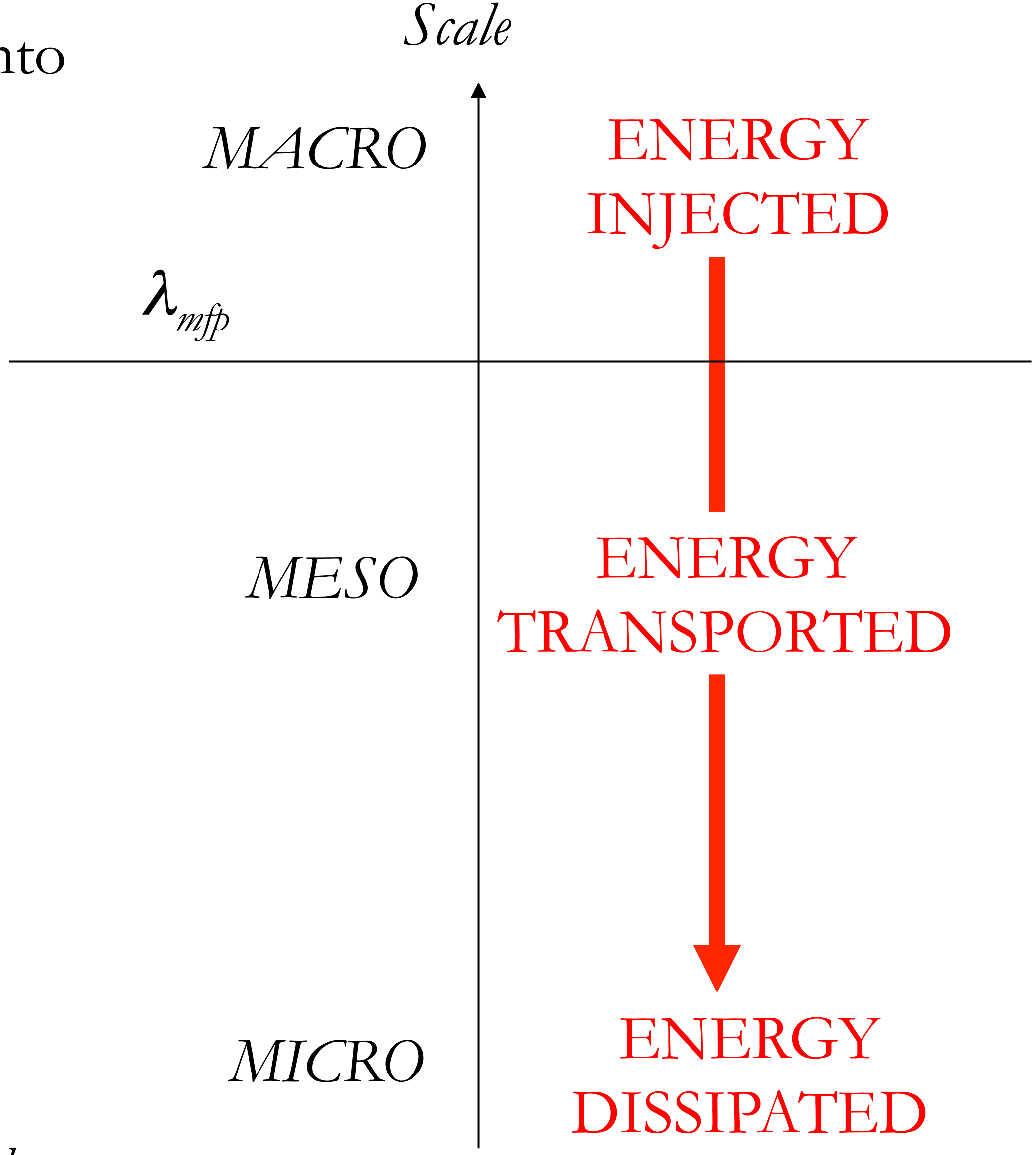




What Is Turbulence in Such Systems?

Turbulence is always energy conversion from mean motions and fields into heat (internal energy)

In a weakly collisional plasma, it is a nontrivial question how this energy is partitioned between electrons, ions, minority admixtures, fast particles etc…



Scale

MACRO

ENERGY INJECTED

λ_{mfp}

MESO

ENERGY TRANSPORTED

MICRO

ENERGY DISSIPATED

The T_i/T_e Problem

- We know that $T_i \neq T_e$ is a non-equilibrium situation (entropy will increase if temperatures equalise)
- We know of no mechanism other than *i-e* Coulomb collisions that would equalise temperatures (e.g., no instabilities or fluxes, like with gradients). But ν_{ie} is very small in weakly collisional plasmas
- Where we can measure both temperatures (lab, space), they tend to be within a factor of order unity of each other
- In extrasolar or extragalactic plasmas, we normally assume $T_i = T_e$ unless it is opportune to assume otherwise ($T_i >> T_e$ in some models of some accretion discs), but actually we have no idea what they are
- **Fundamental question:**

Will turbulence equalise (or drive apart) T_i and T_e? I.e.,

- If $T_i > T_e$ then ion heating < electron heating
- If $T_i < T_e$ then ion heating > electron heating

If not, can we predict T_i/T_e as a function of β?

Turbulence Is a Nonlinear Route to Dissipation

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

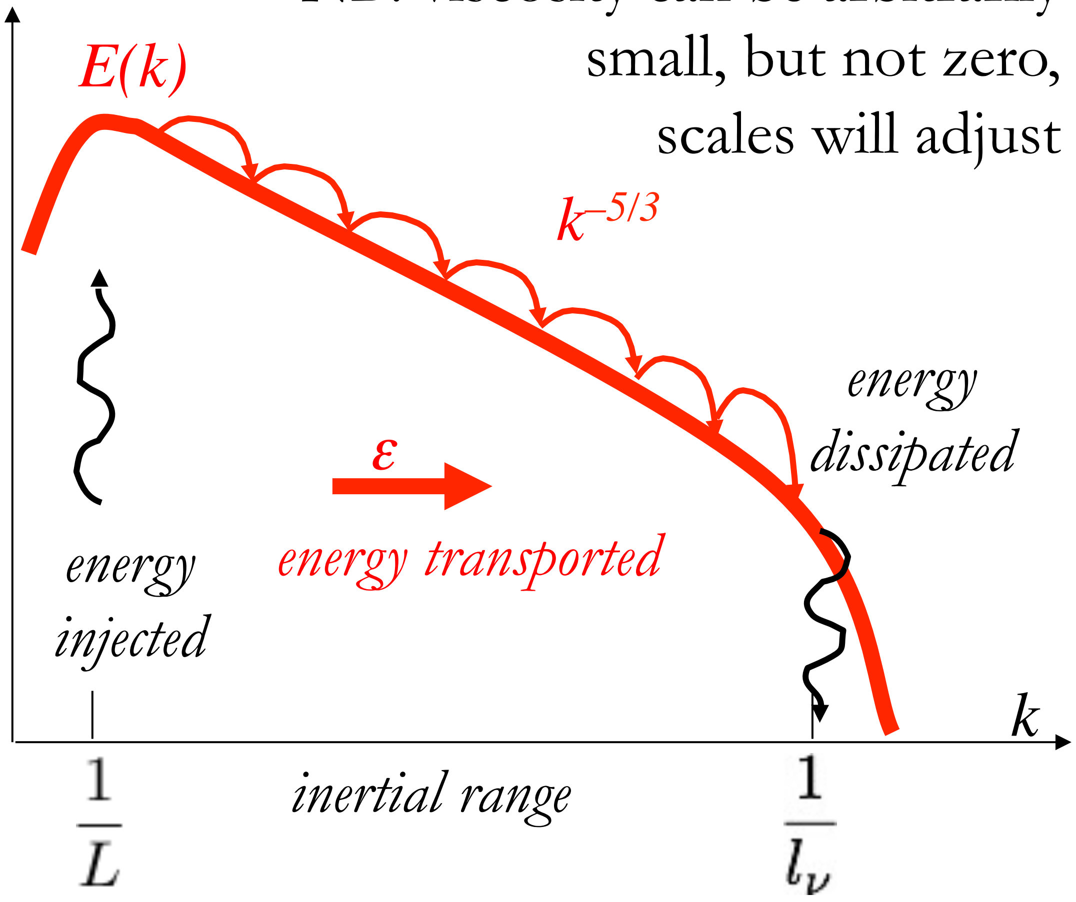
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\frac{u^2}{2} = \varepsilon - \nu\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}|\boldsymbol{\nabla}\boldsymbol{u}|^2}$$

$$l_\nu \sim (\nu^3/\varepsilon)^{1/4} \sim L\mathrm{Re}^{-3/4}$$

NB: viscosity can be arbitrarily small, but not zero, scales will adjust

$$\varepsilon = (1/V)\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{u}\cdot\boldsymbol{f}$$

If cascade is local, intermediate scales fill up ⇒K41, spectra, and all that (see Boldyrev's lecture)



Plasma Turbulence *Ab Initio*

$$\boxed{\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{c}}}$$

energy injection (model)

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\sum_s q_s n_s, \qquad n_s = \int \mathrm{d}^3\boldsymbol{v}\, f_s,$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\left(\boldsymbol{j} + \boldsymbol{j}_{\mathrm{ext}}\right), \qquad \boldsymbol{j} = \sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, \boldsymbol{v} f_s,$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0.$$

PPCF **50**, 124024 (2008)

Plasma Turbulence *Ab Initio*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{m_s v^2}{2}\, f_s = \int \frac{\mathrm{d}^3\boldsymbol{r}}{V}\,\boldsymbol{E}\cdot\boldsymbol{j} = \varepsilon - \frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\,\frac{E^2+B^2}{8\pi}}$$

Work done

$\varepsilon = -(1/V)\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}_{\mathrm{ext}}$

energy injection (model)

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_s q_s n_s, \qquad n_s = \int \mathrm{d}^3\boldsymbol{v}\, f_s,$$

$$\nabla\times\boldsymbol{B} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\left(\boldsymbol{j}+\boldsymbol{j}_{\mathrm{ext}}\right), \qquad \boldsymbol{j} = \sum_s q_s\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v} f_s,$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\nabla\times\boldsymbol{E}, \qquad \nabla\cdot\boldsymbol{B} = 0.$$

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Work done

$$\varepsilon = -(1/V)\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}_{\mathrm{ext}}$$

energy injection (model)

Entropy produced:

$$\frac{\mathrm{d}S_s}{\mathrm{d}t} \equiv \frac{d}{dt}\left[-\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \mathrm{d}^3\boldsymbol{v}\, f_s\ln f_s\right] = -\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \mathrm{d}^3\boldsymbol{v}\,\ln f_s\left(\frac{\partial f_s}{\partial t}\right)_{\mathrm{c}} \geq 0$$

Boltzmann 1872

$$f_s = F_{0s} + \delta f_s \qquad F_{0s} = n_{0s}(\pi v_{\mathrm{th}s}^2)^{-3/2}\exp(-v^2/v_{\mathrm{th}s}^2)$$

$$v_{\mathrm{th}s} = (2T_{0s}/m_s)^{1/2}$$

Then

$$f_s\ln f_s = (F_{0s}+\delta f_s)\ln\left[F_{0s}\left(1+\frac{\delta f_s}{F_{0s}}\right)\right]$$

$$\approx (F_{0s}+\delta f_s)\ln F_{0s} + \delta f_s - \frac{\delta f_s^2}{2F_{0s}}$$

(to second order)

PPCF **50**, 124024 (2008)

Plasma Turbulence *Ab Initio*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int\mathrm{d}^3\boldsymbol{v}\,\frac{m_s v^2}{2}\,f_s=\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\,\boldsymbol{E}\cdot\boldsymbol{j}=\varepsilon-\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\,\frac{E^2+B^2}{8\pi}$$

Work done

$$\varepsilon=-(1/V)\int\mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}_{\mathrm{ext}}$$

energy injection (model)

Entropy produced:

$$\boxed{\begin{aligned}T_{0s}\frac{\mathrm{d}S_s}{\mathrm{d}t}&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int\mathrm{d}^3\boldsymbol{v}\,\frac{m_s v^2}{2}\left(F_{0s}+\delta f_s\right)-\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int\mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s^2}{2F_{0s}}\right]\\&=-\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int\mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_{\mathrm{c}}-n_{0s}\nu_E^{ss'}\left(T_{0s}-T_{0s'}\right)\end{aligned}}$$

$$\begin{aligned}f_s\ln f_s&=\left(F_{0s}+\delta f_s\right)\ln\left[F_{0s}\left(1+\frac{\delta f_s}{F_{0s}}\right)\right]\\&\approx\left(F_{0s}+\delta f_s\right)\ln F_{0s}+\delta f_s-\frac{\delta f_s^2}{2F_{0s}}\end{aligned}$$

(to second order)

PPCF **50**, 124024 (2008)

Plasma Turbulence *Ab Initio*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{m_s v^2}{2}f_s = \int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\,\boldsymbol{E}\cdot\boldsymbol{j} = \varepsilon - \frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\,\frac{E^2+B^2}{8\pi}$$

Work done

$$\varepsilon = -(1/V)\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}_{\mathrm{ext}}$$

energy injection (model)

Entropy produced:

$$T_{0s}\frac{\mathrm{d}S_s}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \mathrm{d}^3\boldsymbol{v}\,\frac{m_s v^2}{2}\left(F_{0s}+\delta f_s\right) - \int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s^2}{2F_{0s}}\right]$$

$$= -\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial \delta f_s}{\partial t}\right)_{\mathrm{c}} - n_{0s}\nu_E^{ss'}\left(T_{0s}-T_{0s'}\right)$$

PPCF **50**, 124024 (2008)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s^2}{2F_{0s}} + \frac{E^2+B^2}{8\pi}\right]$$

$$= \varepsilon + \int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial \delta f_s}{\partial t}\right)_{\mathrm{c}}$$

Plasma Turbulence *Ab Initio*

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Work done

$\varepsilon = -(1/V)\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}_{\rm ext}$

energy injection (model)

Heating:

$$\frac{3}{2}n_{0s}\frac{\mathrm{d}T_{0s}}{\mathrm{d}t} = -\overline{\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_{\rm c}} - n_{0s}\nu_E^{ss'}(T_{0s}-T_{0s'})$$

Fluctuation energy budget:

PPCF **50**, 124024 (2008)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\underbrace{\frac{T_{0s}\delta f_s^2}{2F_{0s}}}_{-T\delta S}+\underbrace{\frac{E^2+B^2}{8\pi}}_{\text{energy}}\right]$$

$$=\underbrace{\varepsilon}_{\text{injection}}+\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\overset{\text{heating}}{\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_{\rm c}}$$

"Energy" in Plasma Turbulence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\underbrace{\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s^2}{2F_{0s}}}_{-T\delta S}+\underbrace{\frac{E^2+B^2}{8\pi}}_{\text{energy}}\right] = \underbrace{\varepsilon}_{\text{injection}}+\underbrace{\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_{\mathrm{c}}}_{\text{heating}}$$

Generalised energy = free energy of the particles + fields

Kruskal & Oberman 1958
Fowler 1968
Krommes & Hu 1994
Krommes 1999
Sugama et al. 1996
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PPCF **50**, 124024 (2008)

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Scott 2007
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NB: Landau damping is a redistribution between the e-m fluctuation energy and (negative) perturbed entropy (free energy). It was pointed out already by Landau 1946 that δf_s does not decay: "ballistic response" $\delta f_s \propto \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$

PPCF **50**, 124024 (2008)

Analogous to Fluid, But…

$$\boxed{\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\underbrace{\frac{T_{0s}\delta f_s^2}{2F_{0s}}}_{-T\delta S}+\underbrace{\frac{E^2+B^2}{8\pi}}_{\text{energy}}\right]\\ =\underbrace{\varepsilon}_{\text{injection}}+\underbrace{\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_{\mathrm{c}}}_{\text{heating}}\end{aligned}}$$

small scales in 6D phase space

$$\partial_t\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}=-\boldsymbol{\nabla}p+\nu\nabla^2\boldsymbol{u}+\boldsymbol{f},$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\frac{u^2}{2}=\varepsilon-\nu\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}|\boldsymbol{\nabla}\boldsymbol{u}|^2}$$

small scales in 3D physical space

$$\varepsilon=(1/V)\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{u}\cdot\boldsymbol{f}$$

PPCF **50**, 124024 (2008)

Analogous to Fluid, But…

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\left[\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\underset{-T\delta S}{\frac{T_{0s}\delta f_s^2}{2F_{0s}}}+\underset{\text{energy}}{\frac{E^2+B^2}{8\pi}}\right]$$

$$=\underset{\text{injection}}{\varepsilon}+\int\frac{\mathrm{d}^3\boldsymbol{r}}{V}\sum_s\int \mathrm{d}^3\boldsymbol{v}\,\underset{\text{heating}}{\frac{T_{0s}\delta f_s}{F_{0s}}\left(\frac{\partial\delta f_s}{\partial t}\right)_{\mathrm{c}}}$$

small scales in 6D
phase space

$$\boxed{\nu_{ii}v_{\mathrm{th}i}^2\left(\frac{\partial}{\partial v}\right)^2\sim\omega\ \Rightarrow\ \frac{\delta v}{v_{\mathrm{th}i}}\sim\left(\frac{\nu_{ii}}{\omega}\right)^{1/2}\sim\frac{1}{\sqrt{k_\parallel\lambda_{\mathrm{mfp}}}}\ll1}$$

Small scales in velocity space (**phase mixing**)

PPCF **50**, 124024 (2008)

Linear Phase Mixing

$$\frac{\partial \delta f}{\partial t} + v_\| \frac{\partial \delta f}{\partial z} + \text{stuff} = \left(\frac{\partial \delta f}{\partial t}\right)_c$$

Can think of this as shear in phase space

particle streaming along magnetic field

perpendicular nonlinearity, interaction with fields, etc. (to be dealt with later)

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"Ballistic response": $\delta f \propto e^{ik_{\parallel}v_{\parallel}t}$

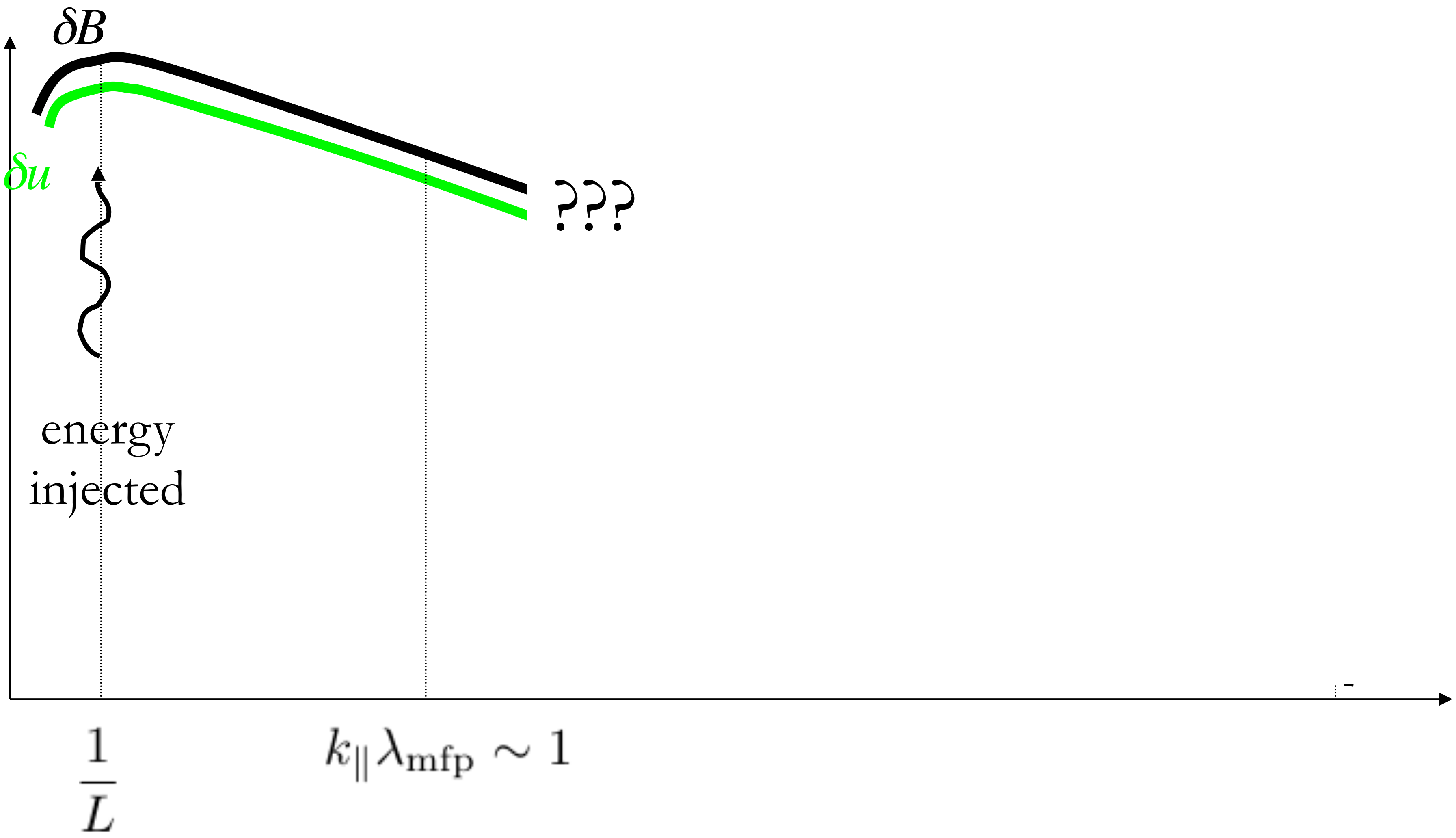
(always part of the Landau damping solution)

$$\frac{\partial \delta f}{\partial v_{\parallel}} = ik_{\parallel}t\delta f \quad \Rightarrow \quad \frac{\delta v_{\parallel}}{v_{\rm th}} \sim \frac{1}{k_{\parallel}v_{\rm th}t} \to 0 \quad \text{as} \quad t \to \infty$$

So small-scale structure forms and is eventually wiped out by collisions (but we will see that there is a much faster nonlinear mechanism for that: **turbulence**)

From MHD to Kinetic Scales

Let us now see what happens nonlinearly: the turbulent cascade starts as MHD turbulence and gets to collisionless scales, then what?



MHD Cascade Is Anisotropic

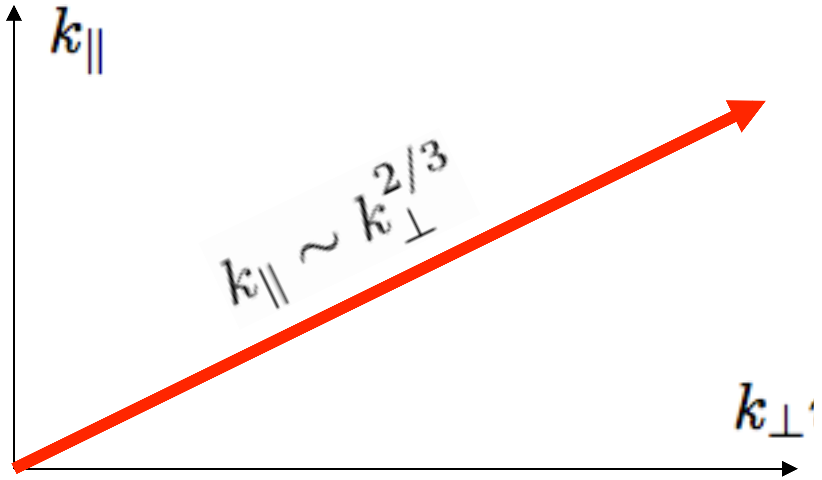
- **Strong anisotropy:** $\boxed{\frac{k_\parallel}{k_\perp} \ll 1}$

In magnetised plasma,
confirmed by numerics (MHD)
and observations (solar wind, ISM)

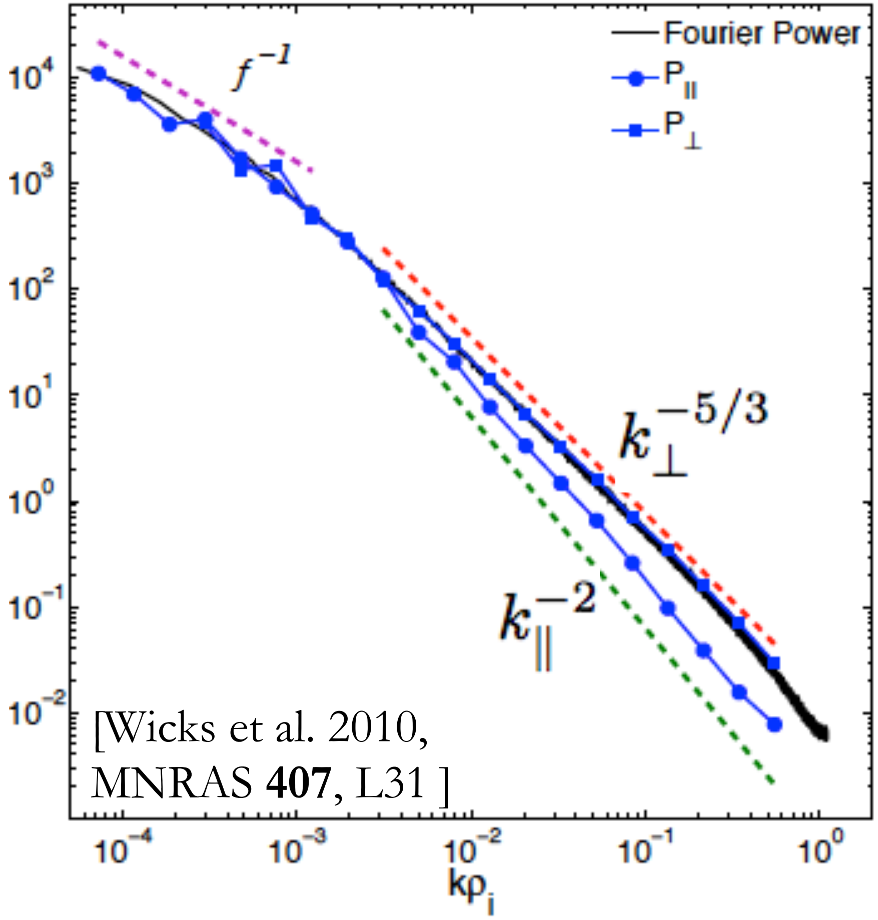
- **Strong nonlinearity:** $\boxed{\omega \sim k_\parallel v_A \sim k_\perp u_\perp}$

$\omega_{\text{linear}} \sim \omega_{\text{nonlinear}}$
Critical balance as a physical principle
(see Boldyrev's lectures)

- (2+1)D route through phase space:



SOLAR WIND



[Wicks et al. 2010, MNRAS **407**, L31]

Turbulence Reaches Larmor Scales and Beyond

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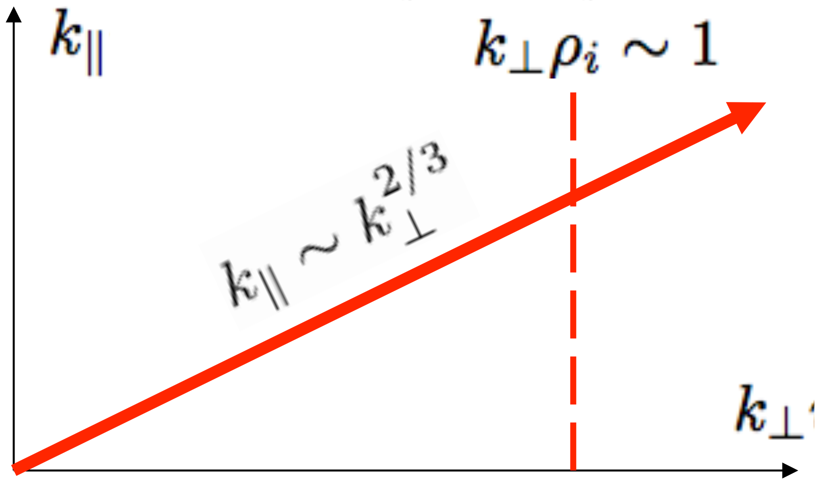
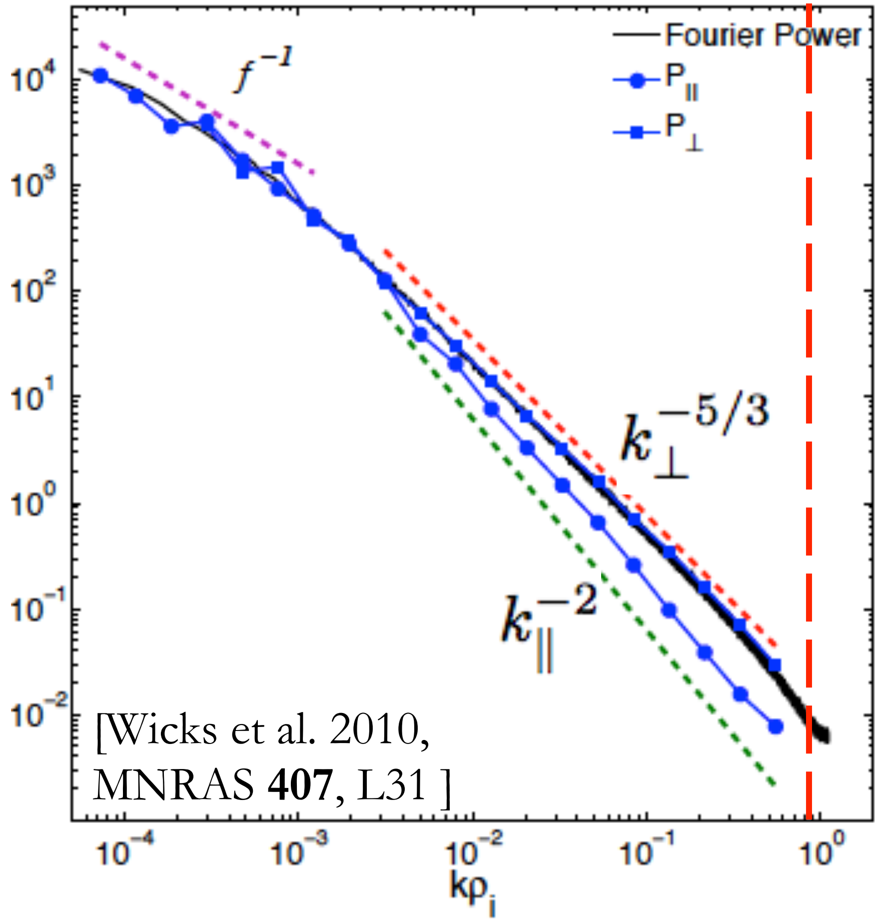


SOLAR WIND



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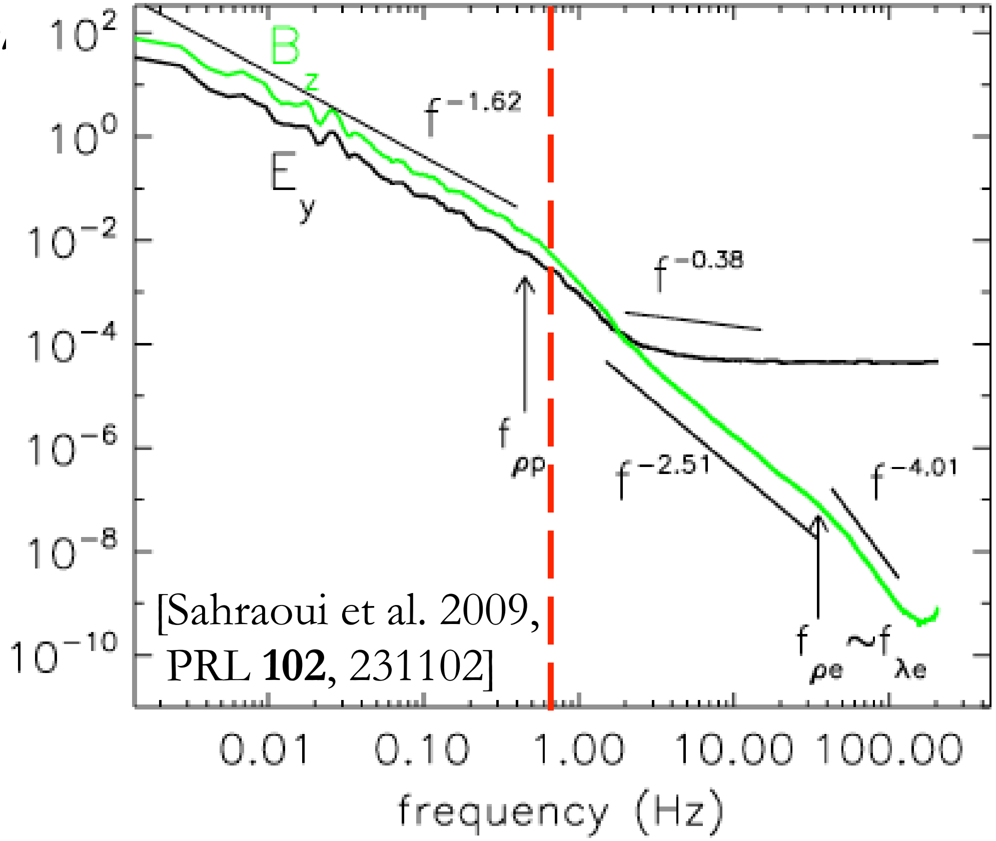
- **Strong nonlinearity:** $\boxed{\omega \sim k_\parallel v_A \sim k_\perp u_\perp}$

$\omega_{\text{linear}} \sim \omega_{\text{nonlinear}}$
Critical balance as a physical princip
(see Boldyrev's lectures)

- (2+1)D route through phase space
- Stuff happens at $k_\perp \rho_i \sim 1$

SOLAR WIND



[Sahraoui et al. 2009, PRL **102**, 231102]

Turbulence Reaches Larmor Scales and Beyond

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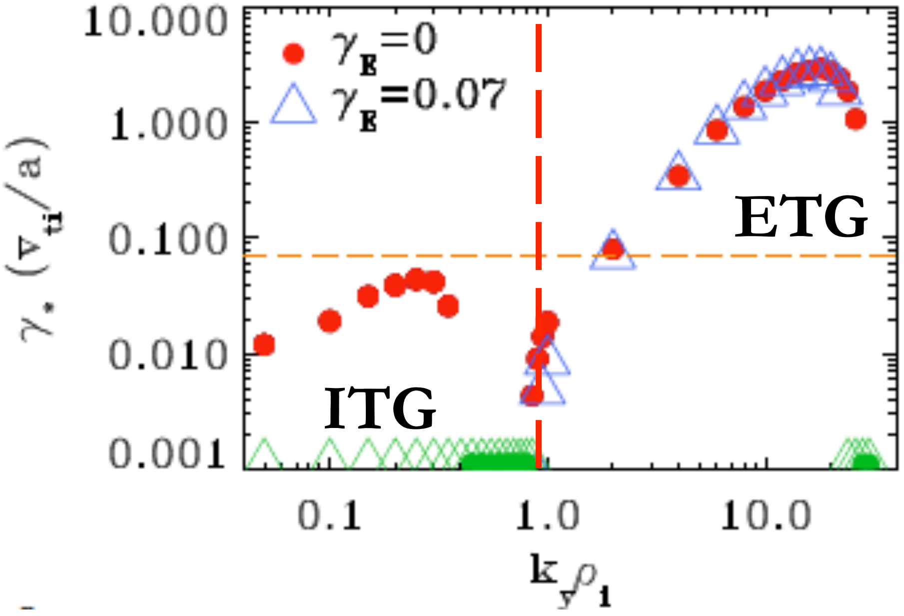
TOKAMAK DNS



[Roach et al. 2009, PPCF **51**, 124020]

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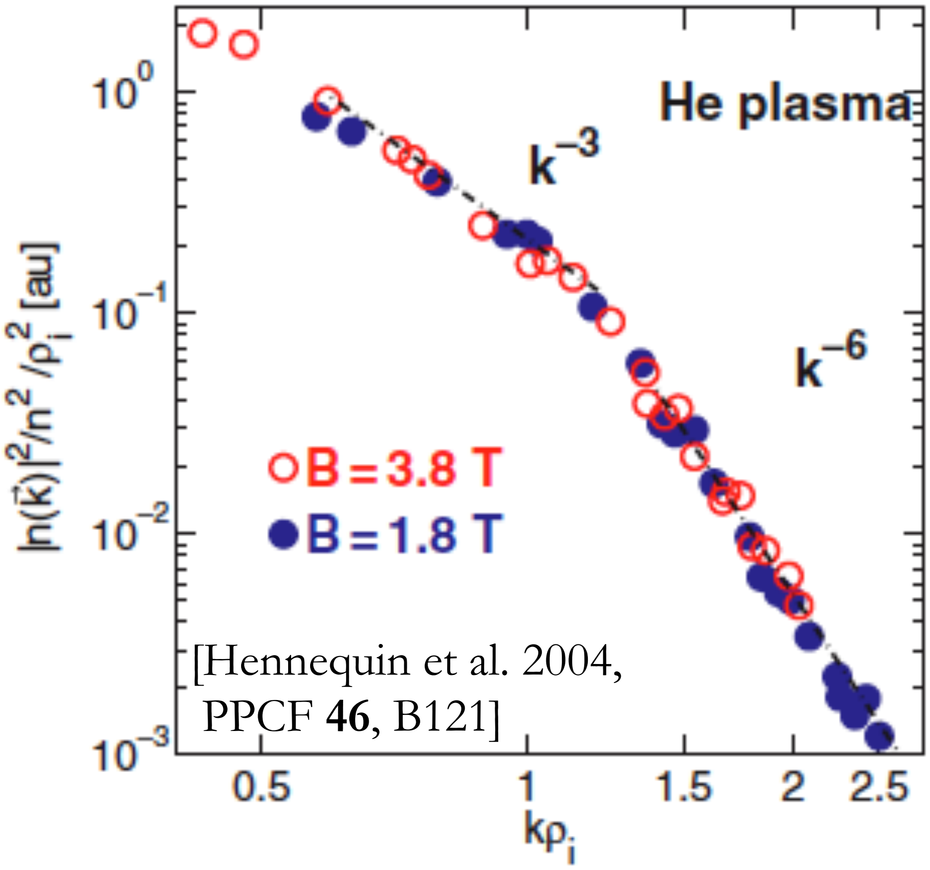
TORE SUPRA



[Hennequin et al. 2004, PPCF **46**, B121]

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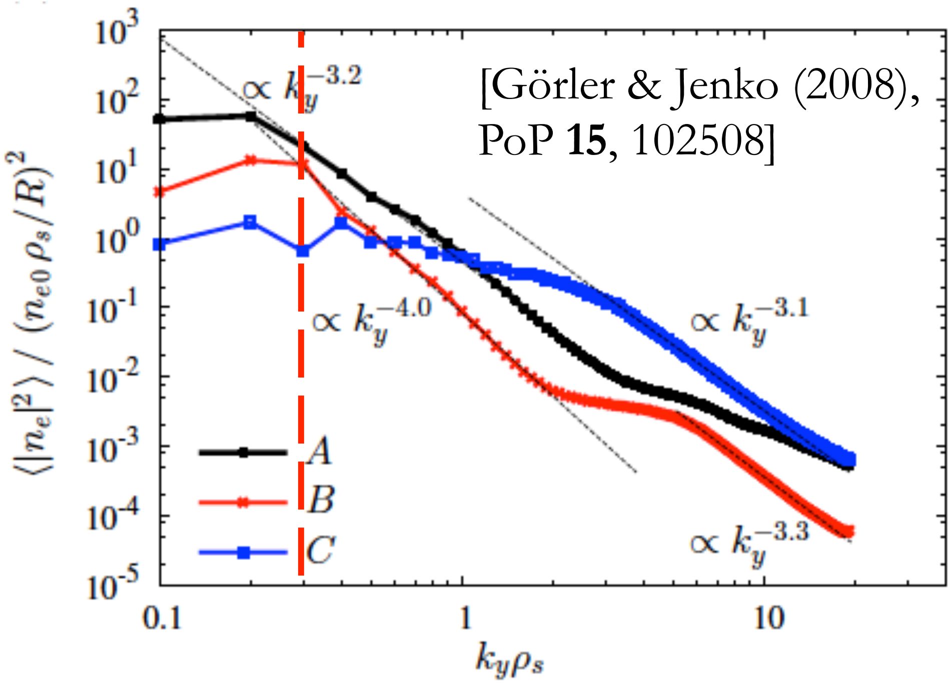

[Görler & Jenko (2008), PoP **15**, 102508]

Critical Balance as an Ordering Assumption

- **Strong anisotropy:** $\epsilon \sim \boxed{\frac{k_{\|}}{k_{\perp}} \ll 1}$ *(this is the small parameter!)*
- **Strong nonlinearity:** $\boxed{\omega \sim k_{\|} v_A \sim k_{\perp} u_{\perp}}$
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Gyrokinetics

- **Strong anisotropy:** $\epsilon \sim \frac{k_\parallel}{k_\perp} \ll 1$ *(this is the small parameter!)*

- **Strong nonlinearity:** $\omega \sim k_\parallel v_A \sim k_\perp u_\perp$
  (critical balance as an ordering assumption)

- **Finite Larmor radius:** $k_\perp \rho_i \sim 1$

$$\boxed{\frac{\omega}{\Omega_i} \sim \frac{k_\parallel v_A}{\Omega_i} \sim \frac{k_\perp \rho_i}{\sqrt{\beta_i}}\,\epsilon}$$

Low frequency

- **Weak collisions:** $\frac{\omega}{\nu_{ii}} \sim \frac{k_\parallel \lambda_{\rm mfp}}{\sqrt{\beta_i}} \sim 1$

GK ORDERING: $f_s = F_{0s} + \delta f_s$

$$\boxed{\frac{\delta f_s}{F_{0s}} \sim \frac{\delta B}{B_0} \sim \frac{e\varphi}{T_e} \sim \frac{k_\parallel}{k_\perp} \sim \frac{\rho_i}{\lambda_{\rm mfp}} \sim \frac{\omega}{\Omega_i} \sim \epsilon}$$

[Howes et al. 2006, ApJ **651**, 590
& refs. therein]

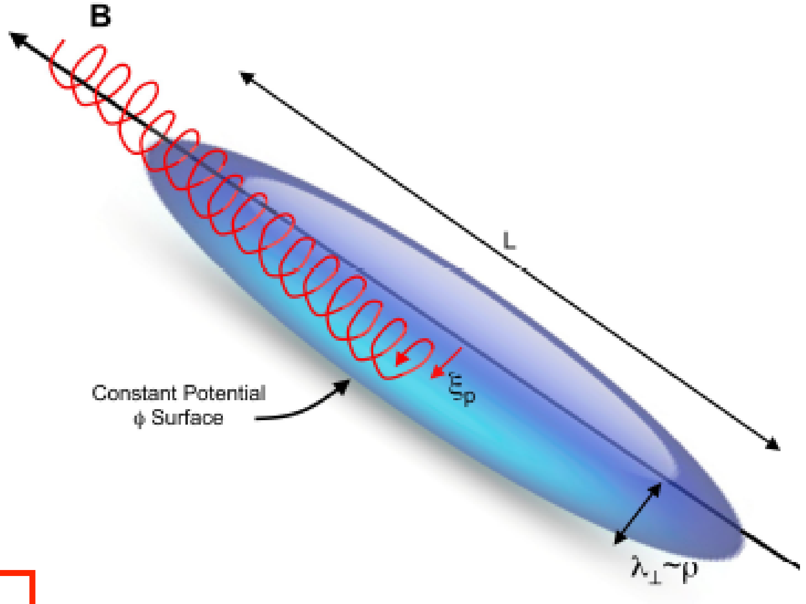
Gyrokinetics: Kinetics of Larmor Rings

Particle dynamics can be averaged over the Larmor orbits and everything reduces to kinetics of Larmor rings centered at

$$\mathbf{R}_s = \mathbf{r} + \frac{\mathbf{v}\times\hat{\mathbf{z}}}{\Omega_s}$$

Catto transformation

and interacting with the electromagnetic fluctuations.



$$\boxed{\delta f_s = -q_s\varphi F_{0s}/T_{0s} + h_s(t, \boldsymbol{R}_s, v_\perp, v_\parallel)}$$

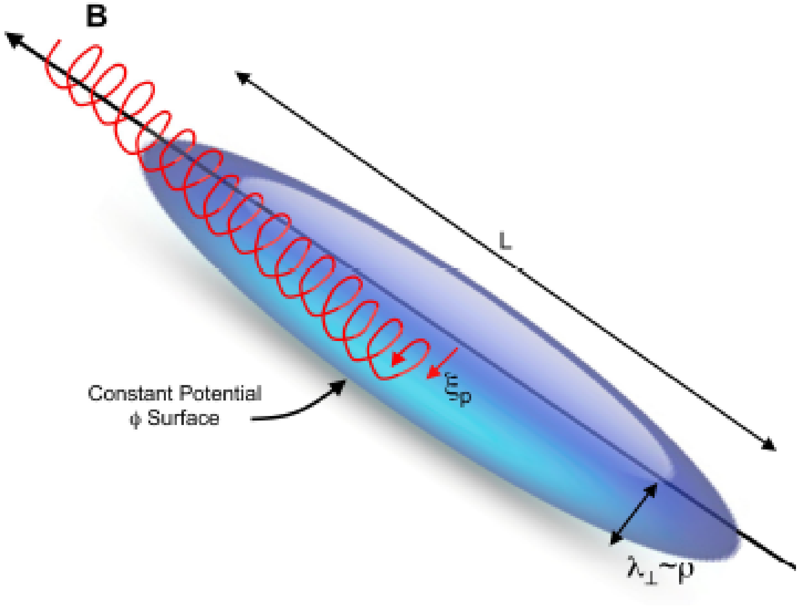
Boltzmann bit

distribution of rings

only two velocity variables,
i.e., 6D → 5D

[Howes et al. 2006, ApJ **651**, 590
& refs. therein]

Gyrokinetics: Kinetics of Larmor Rings



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$$\chi = \varphi - \boldsymbol{v} \cdot \boldsymbol{A}/c, \ \boldsymbol{B} = B_0 \hat{z} + \delta \boldsymbol{B}, \ \delta \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

+ Maxwell's equations (quasineutrality and Ampère's law)

$$\langle \chi(t, \mathbf{r}, \mathbf{v}) \rangle_{\mathbf{R}_s} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \chi\left(t, \mathbf{R}_s - \frac{\mathbf{v} \times \hat{\mathbf{z}}}{\Omega_s}, \mathbf{v}\right)$$

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Gyrokinetics: Kinetics of Larmor Rings

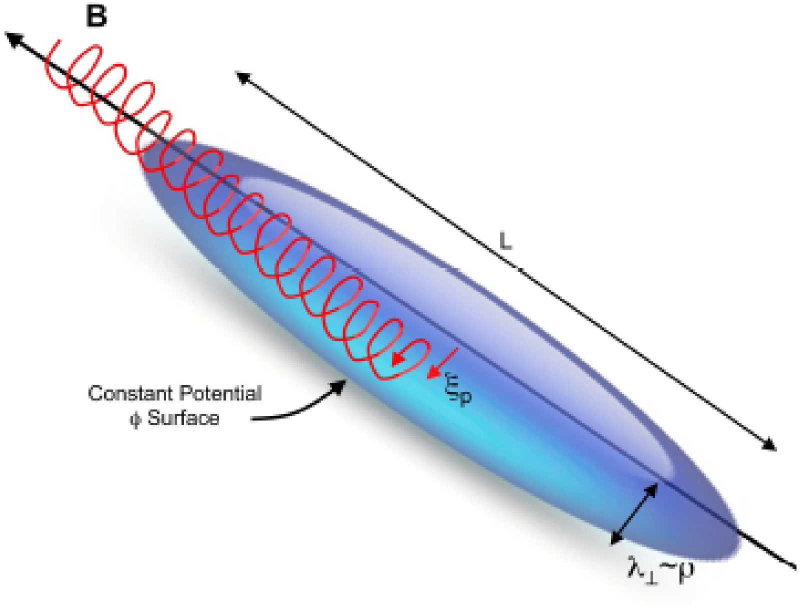
Averaged gyrocentre drifts:

- **E**×**B**$_0$ drift
- **∇**B drift
- motion along perturbed fieldline

$$\left\langle \frac{d\mathbf{R}_s}{dt} \right\rangle_{\mathbf{R}_s} \cdot \frac{\partial h_s}{\partial \mathbf{R}_s}$$

Averaged wave-ring interaction

$$-\left\langle \frac{d\mathcal{E}_s}{dt}\frac{\partial f_s}{\partial \mathcal{E}_s} \right\rangle_{\mathbf{R}_s}$$



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GK Phase Mixing (Entropy Cascade)

linear phase mixing

(slow; ask me later!)

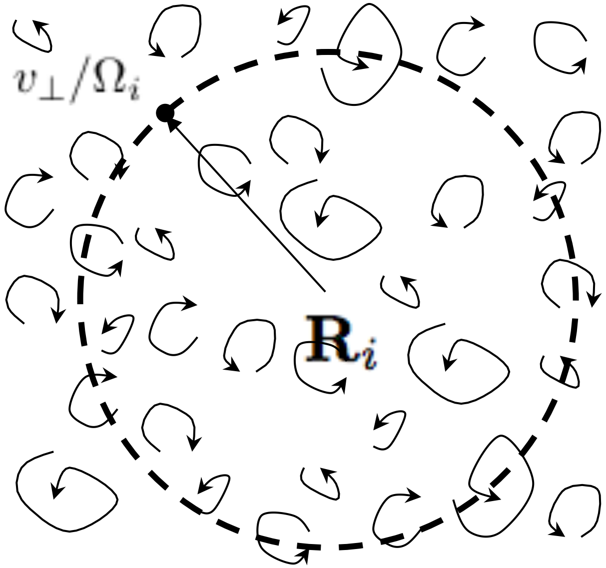
nonlinearity $\langle \mathbf{v}_E \rangle_{\mathbf{R}_i} \cdot \nabla h_i$

$$\frac{\partial h_i}{\partial t} + v_\parallel \frac{\partial h_i}{\partial z} + \frac{c}{B_0}\{\langle\varphi\rangle_{\mathbf{R}_i}, h_i\} - \left(\frac{\partial h_i}{\partial t}\right)_{\mathrm{c}} = \frac{\partial}{\partial t}\frac{Ze\langle\varphi\rangle_{\mathbf{R}_i}}{T_{0i}} F_{0i}$$

electrostatic for simplicity: $\chi = \varphi$

collisions

- **Gyroveraged fluctuations mix *h_i* via this term, so *h_i* developes small (perpendicular) scales in the gyrocenter space:** $\boxed{k_\perp \rho_i \gg 1}$



PPCF **50**, 124024 (2008)

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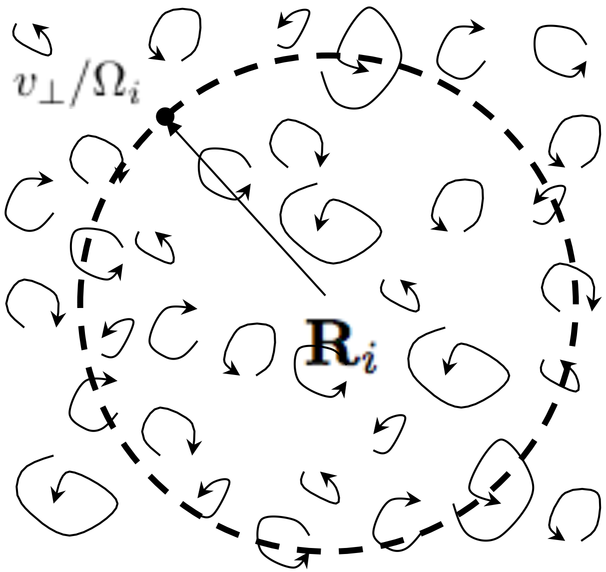
- **In this limit, free energy conservation is**

$$\frac{d}{dt}\int d^3\mathbf{R}_i d^3\mathbf{v}\,\frac{h_i^2}{2F_{0i}} = \int d^3\mathbf{R}_i d^3\mathbf{v}\,\frac{h_i}{F_{0i}}\left(\frac{\partial h_i}{\partial t}\right)_c \le 0$$

This is (minus) the entropy of the perturbed distribution; it is damped only by collisions (Boltzmann!), so h_i must be phase mixed to small scales in velocity space. HOW?



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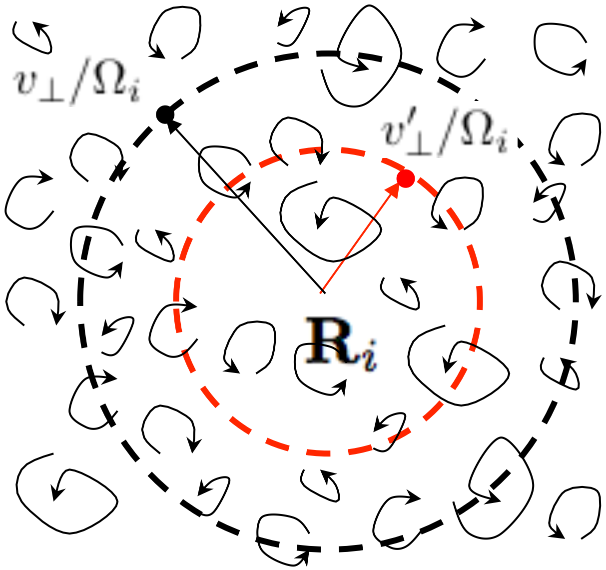
$$\frac{v_\perp}{\Omega_i} - \frac{v'_\perp}{\Omega_i} \sim \frac{1}{k_\perp} \Rightarrow \boxed{\frac{\delta v_\perp}{v_{\mathrm{th}i}} \sim \frac{1}{k_\perp \rho_i}}$$

[This perpendicular nonlinear phase-mixing mechanism was anticipated by Dorland & Hammett 1993]



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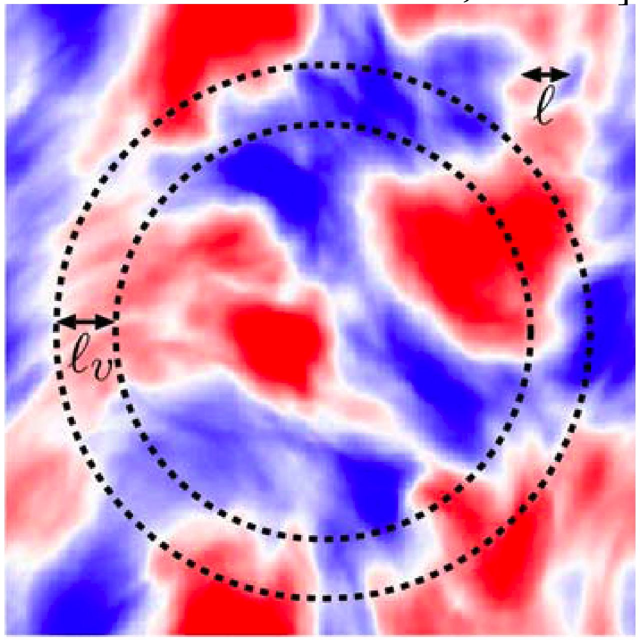

PPCF **50**, 124024 (2008)

Entropy Cascade

- **The cascade is now in phase space, involving both position and velocity, because entropy must get to small scales in velocity**
- **G. Plunk** has developed a spectral formalism to quantify perpendicular velocity-space structure via Hankel transforms:

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$$E(k, p) = p\langle|\hat{h}_i(\mathbf{k}, p)|^2\rangle$$

- **T. Tatsuno** found the cascade along the *(k, p)* diagonal in his 2D GK DNS

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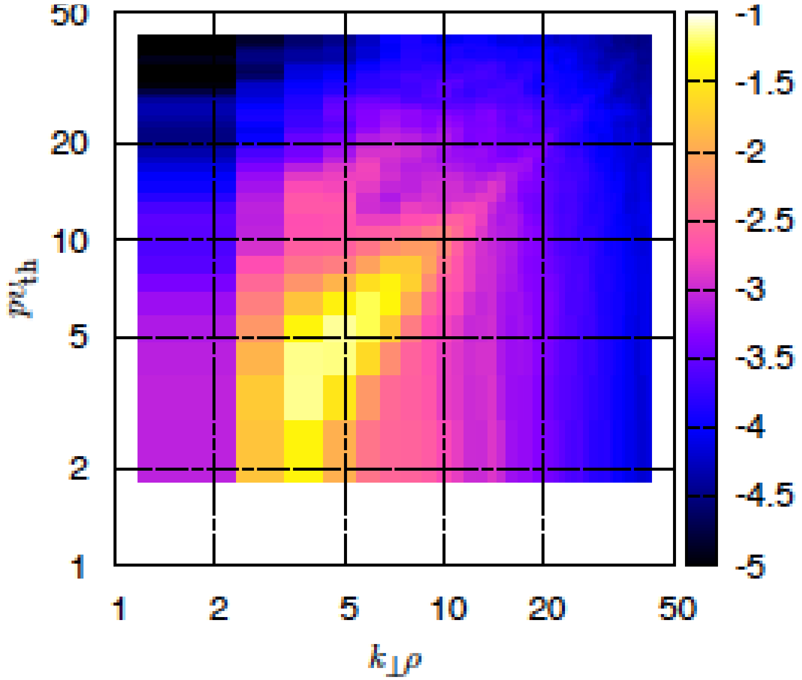

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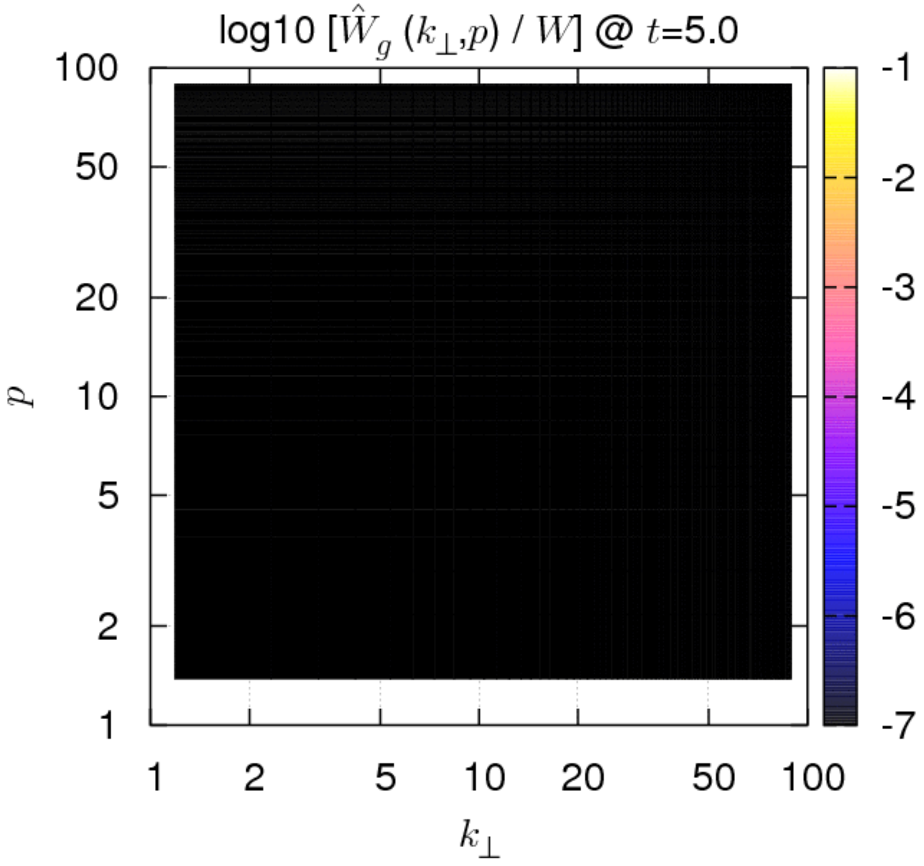

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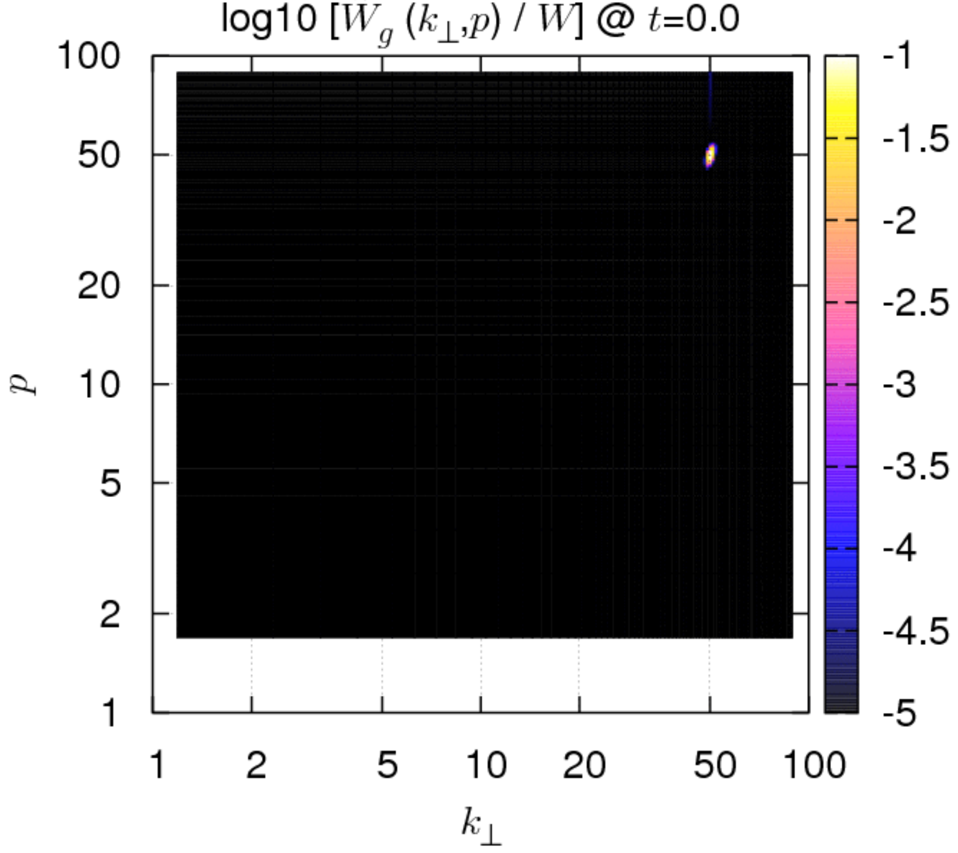

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PPCF **50**, 124024 (2008)

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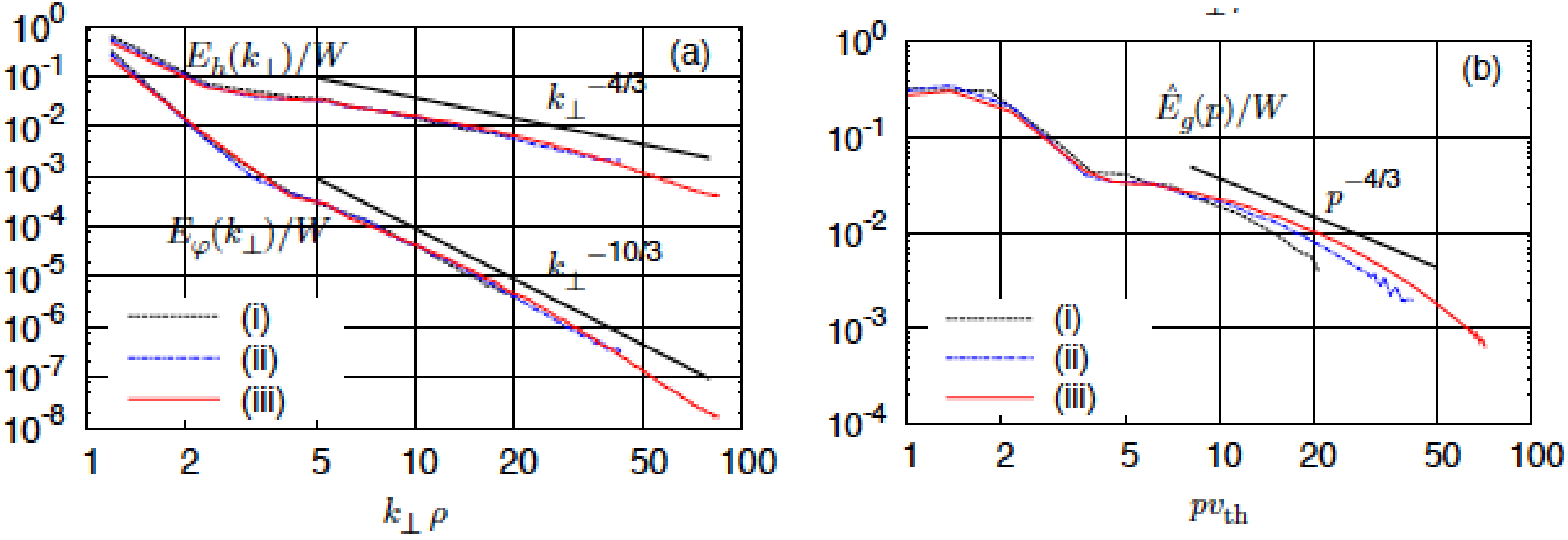

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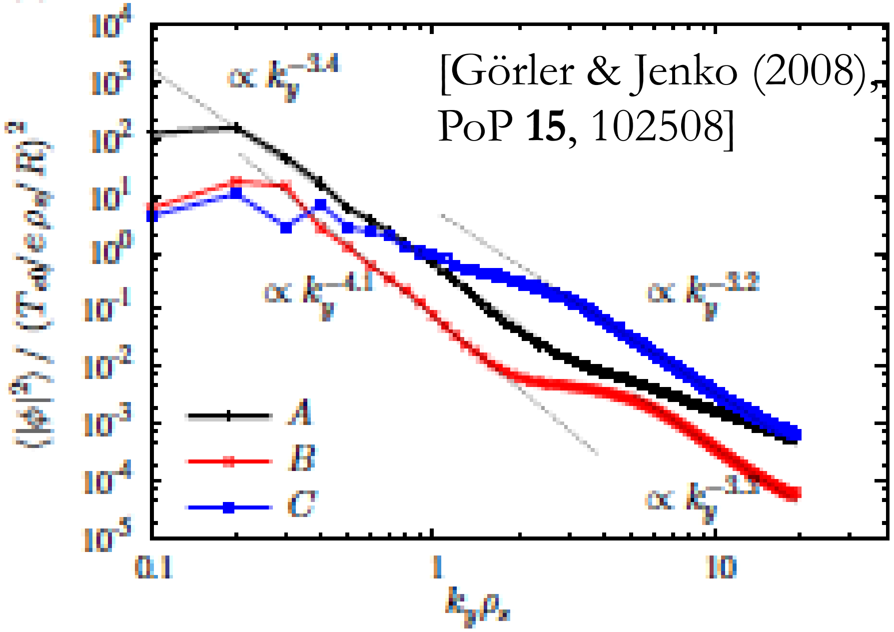
- It is attractive to think of this as **a universal theory of sub-Larmor turbulence** and, for example, attribute to it the sub-Larmor scalings seen in 3D GK DNS of tokamak turbulence by by Görler & Jenko (2008)



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- **Dissipation scale in phase space**

(cf. Kolmogorov scale vs. Re)

$$\frac{\delta v_{\perp c}}{v_{\mathrm{th}i}} \sim \frac{1}{k_{\perp c}\rho_i} \sim \mathbf{Do}^{-3/5}$$

Dorland Number

$$\mathrm{Do} = \frac{1}{\nu_{ii}\tau_{\rho_i}}$$

characteristic time at the ion gyroscale

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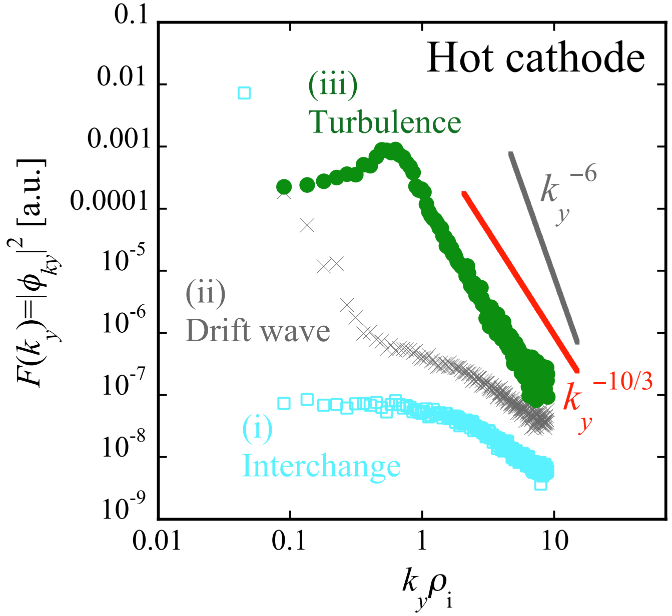
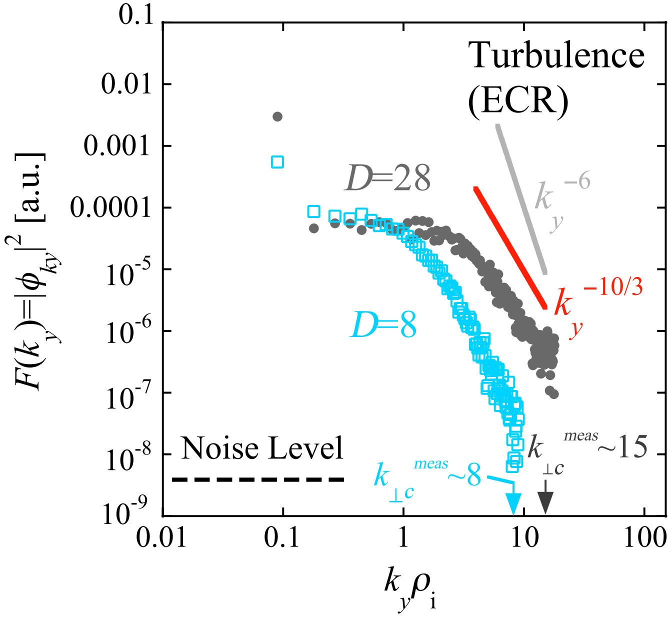
PPCF **50**, 124024 (2008)

- Just last week a PRL by **E. Kawamori** came out claiming a laboratory measurement that confirms the entropy cascade:

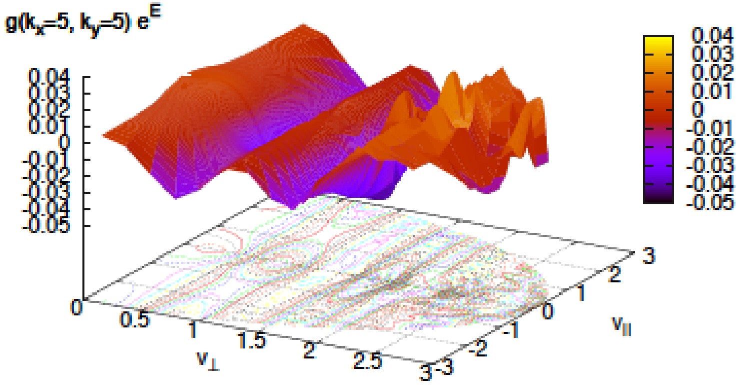


[Kawamori (2013), PRL **110**, 195001]



Linear vs. Nonlinear (GK) Phase Mixing

NONLINEAR



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Since cascade is nonlinear, mixing occurs in one turnover time (fast)

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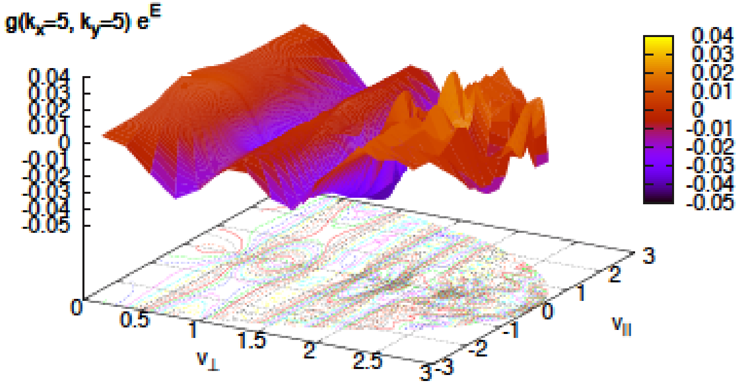

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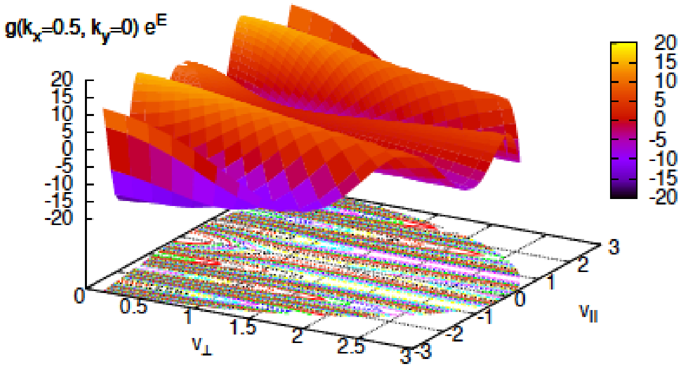


$$\frac{\partial h_i}{\partial t} + v_\parallel \frac{\partial h_i}{\partial z} + \cdots = 0$$

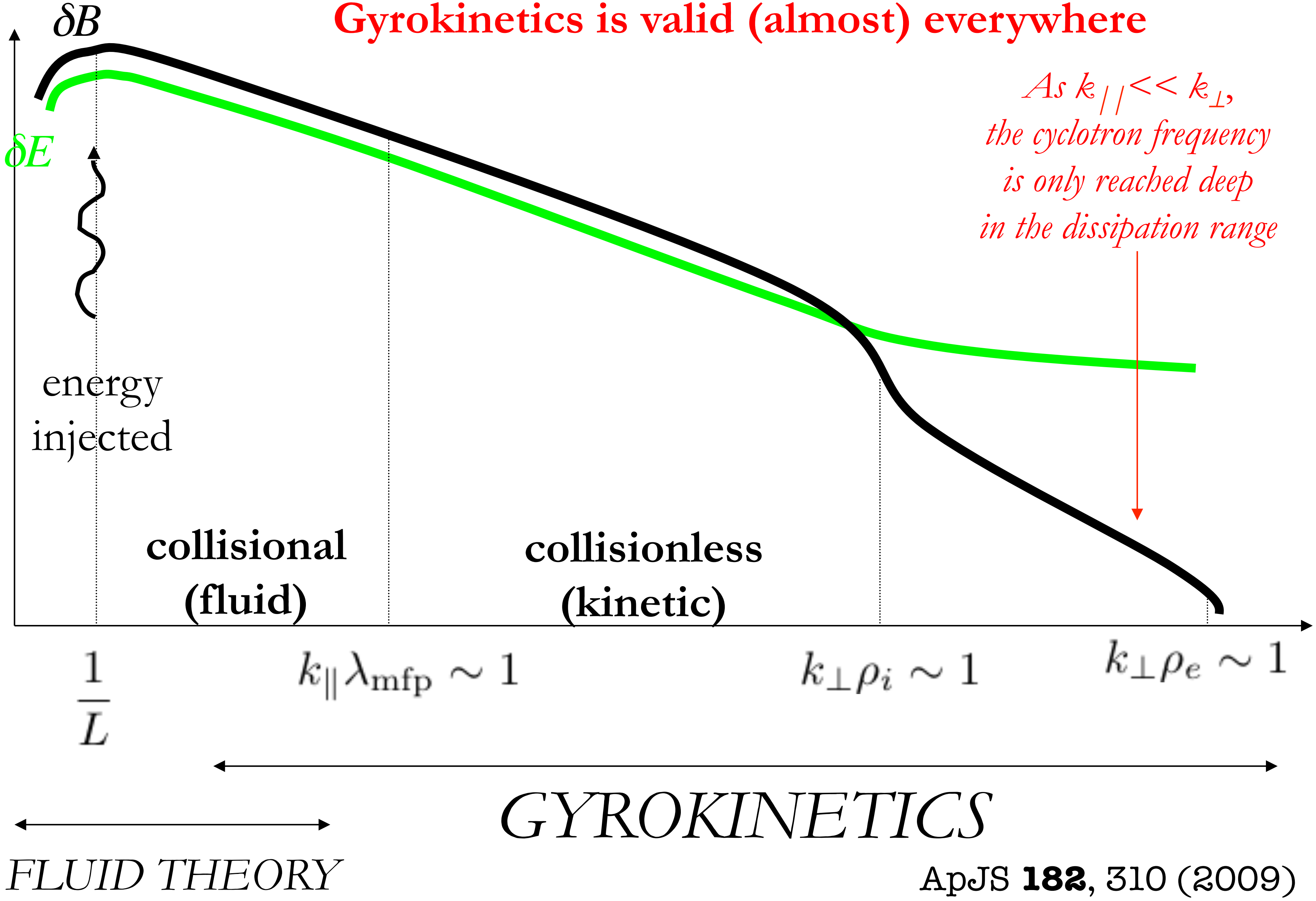
"ballistic response": $h_i \propto e^{ik_\parallel v_\parallel t}$

$$\boxed{\frac{\delta v_\parallel}{v_{\rm thi}} \sim \frac{1}{k_\parallel v_{\rm thi} t} \sim 1}$$

after one turnover time if "critical balance" holds, so linear phase mixing is slow

So How Do MHD and GK Tie Together?

Gyrokinetics is valid (almost) everywhere



As $k_{||} << k_\perp$, the cyclotron frequency is only reached deep in the dissipation range

ApJS **182**, 310 (2009)

Gyrokinetics: Long-Wavelength Limit

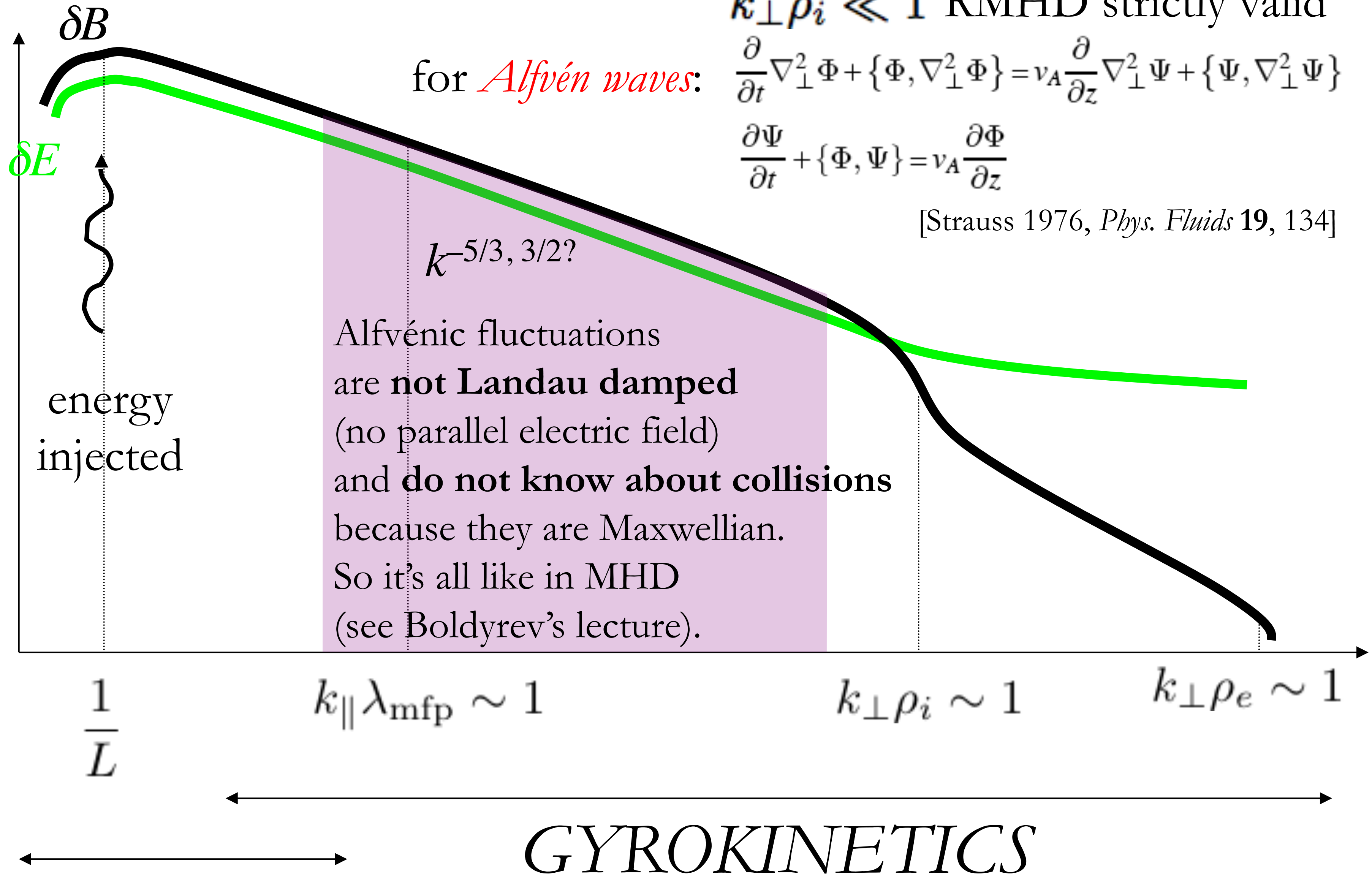
$k_\perp \rho_i \ll 1$ RMHD strictly valid

for *Alfvén waves*:

$$\frac{\partial}{\partial t}\nabla_\perp^2\Phi+\{\Phi,\nabla_\perp^2\Phi\}=v_A\frac{\partial}{\partial z}\nabla_\perp^2\Psi+\{\Psi,\nabla_\perp^2\Psi\}$$

$$\frac{\partial\Psi}{\partial t}+\{\Phi,\Psi\}=v_A\frac{\partial\Phi}{\partial z}$$

[Strauss 1976, *Phys. Fluids* **19**, 134]



Alfvénic fluctuations
are **not Landau damped**
(no parallel electric field)
and **do not know about collisions**
because they are Maxwellian.
So it's all like in MHD
(see Boldyrev's lecture).

GYROKINETICS

FLUID THEORY

ApJS **182**, 310 (2009)

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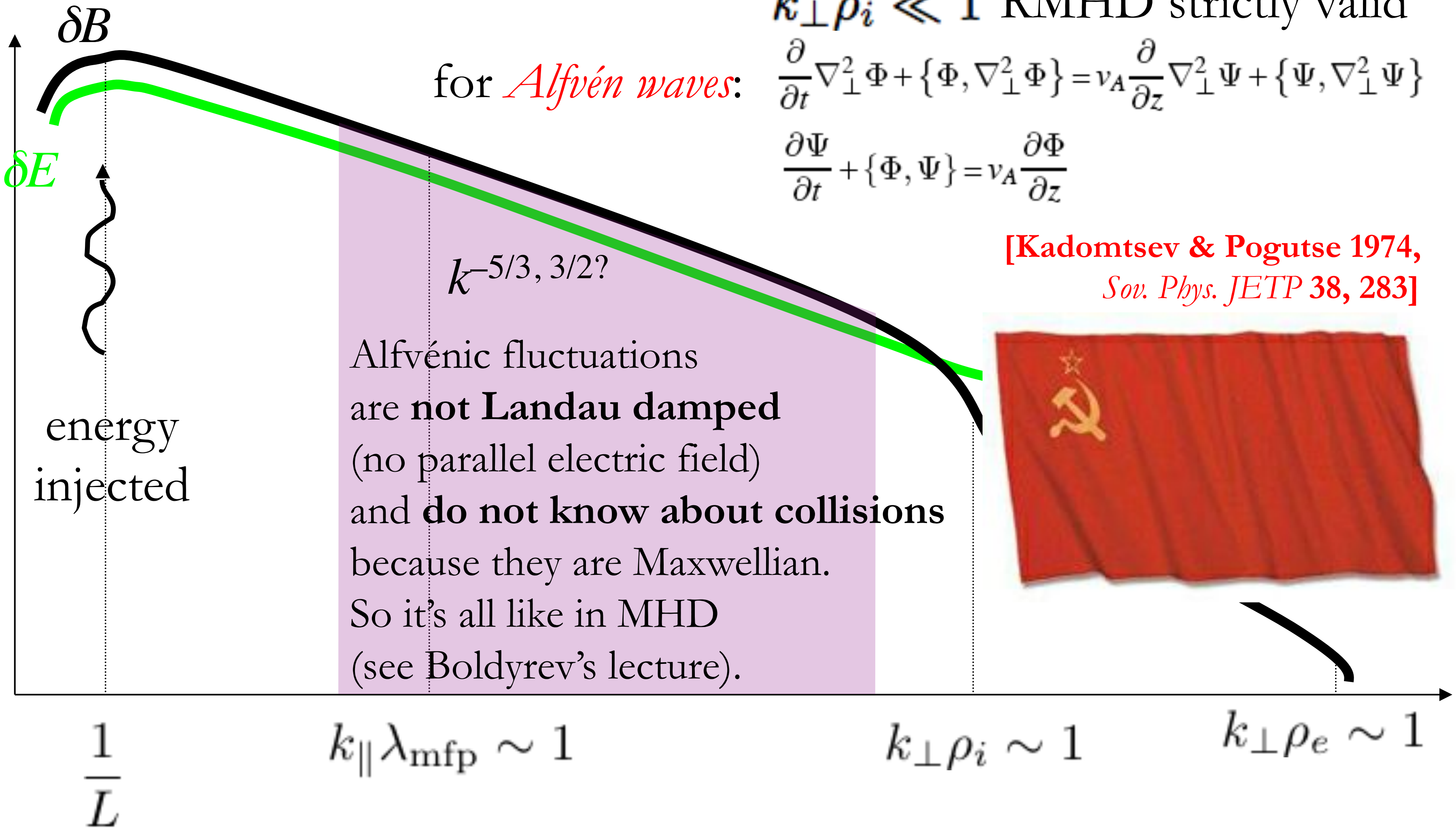
[Kadomtsev & Pogutse 1974, *Sov. Phys. JETP* **38, 283]**



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GYROKINETICS

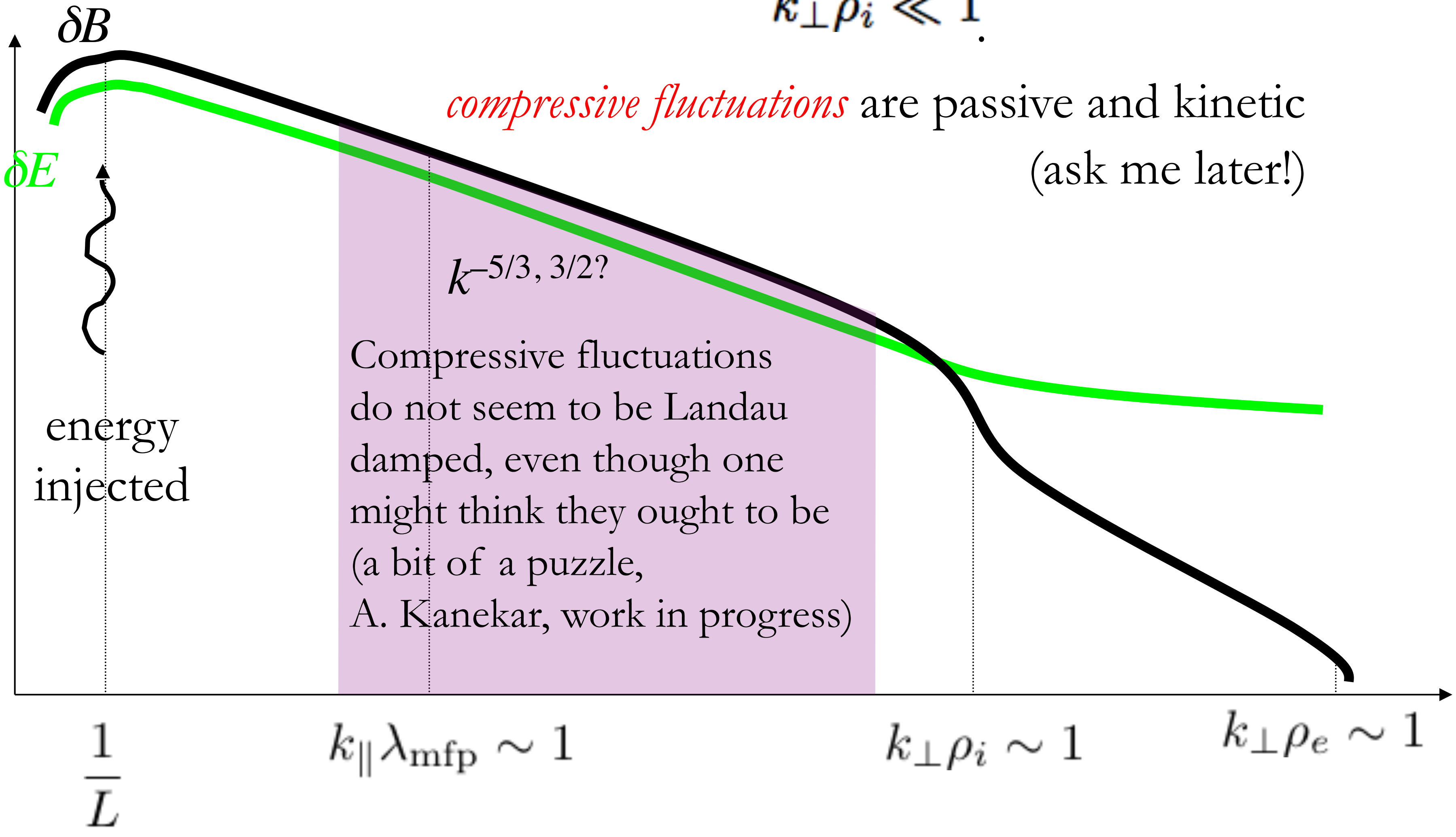
FLUID THEORY

ApJS **182**, 310 (2009)

Gyrokinetics: Long-Wavelength Limit

$k_\perp \rho_i \ll 1$.

compressive fluctuations are passive and kinetic
(ask me later!)



Compressive fluctuations
do not seem to be Landau
damped, even though one
might think they ought to be
(a bit of a puzzle,
A. Kanekar, work in progress)

GYROKINETICS

FLUID THEORY

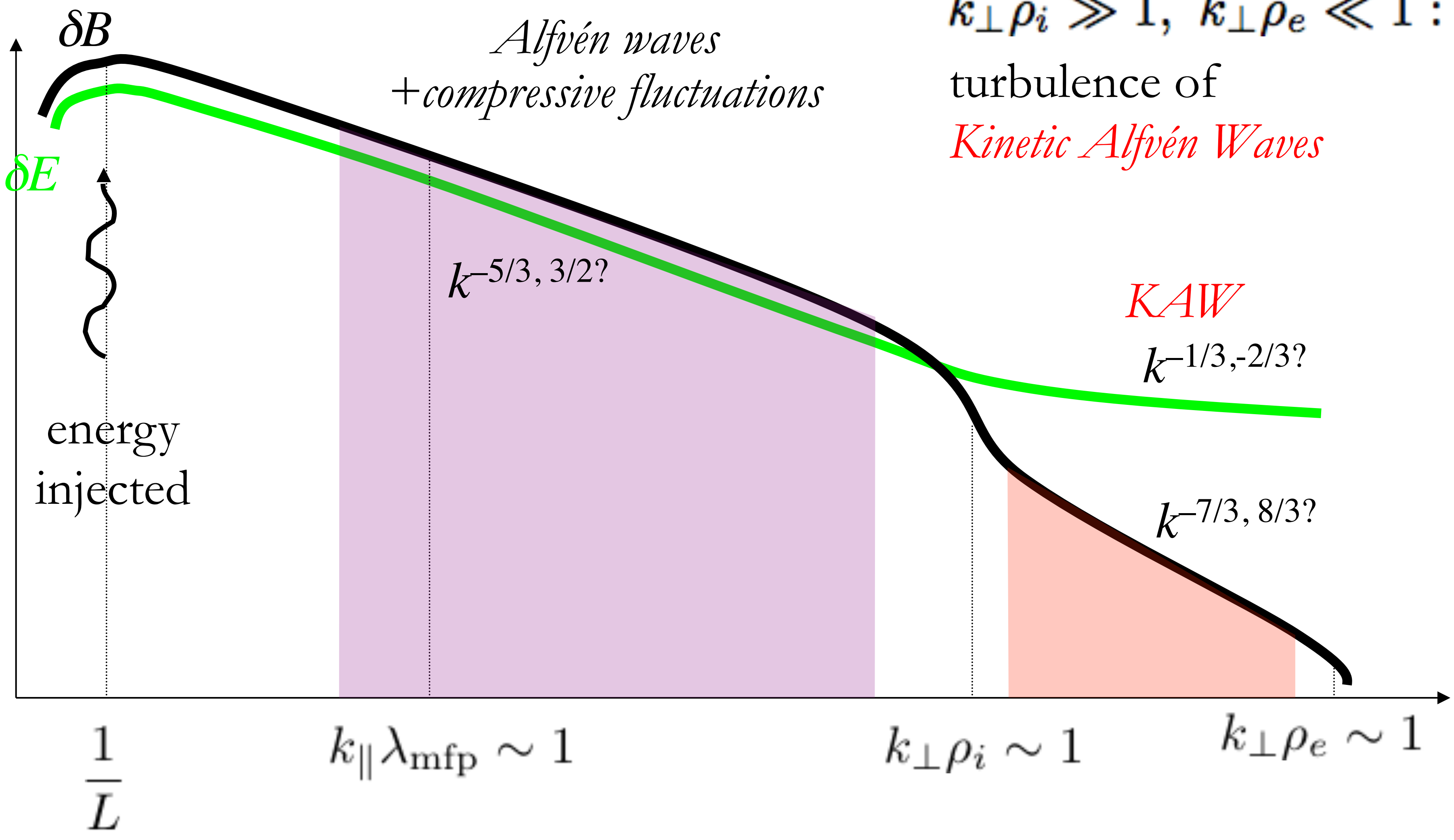
ApJS **182**, 310 (2009)

Gyrokinetics: Short-Wavelength Limit

$k_\perp \rho_i \gg 1,\ k_\perp \rho_e \ll 1:$

turbulence of

Kinetic Alfvén Waves

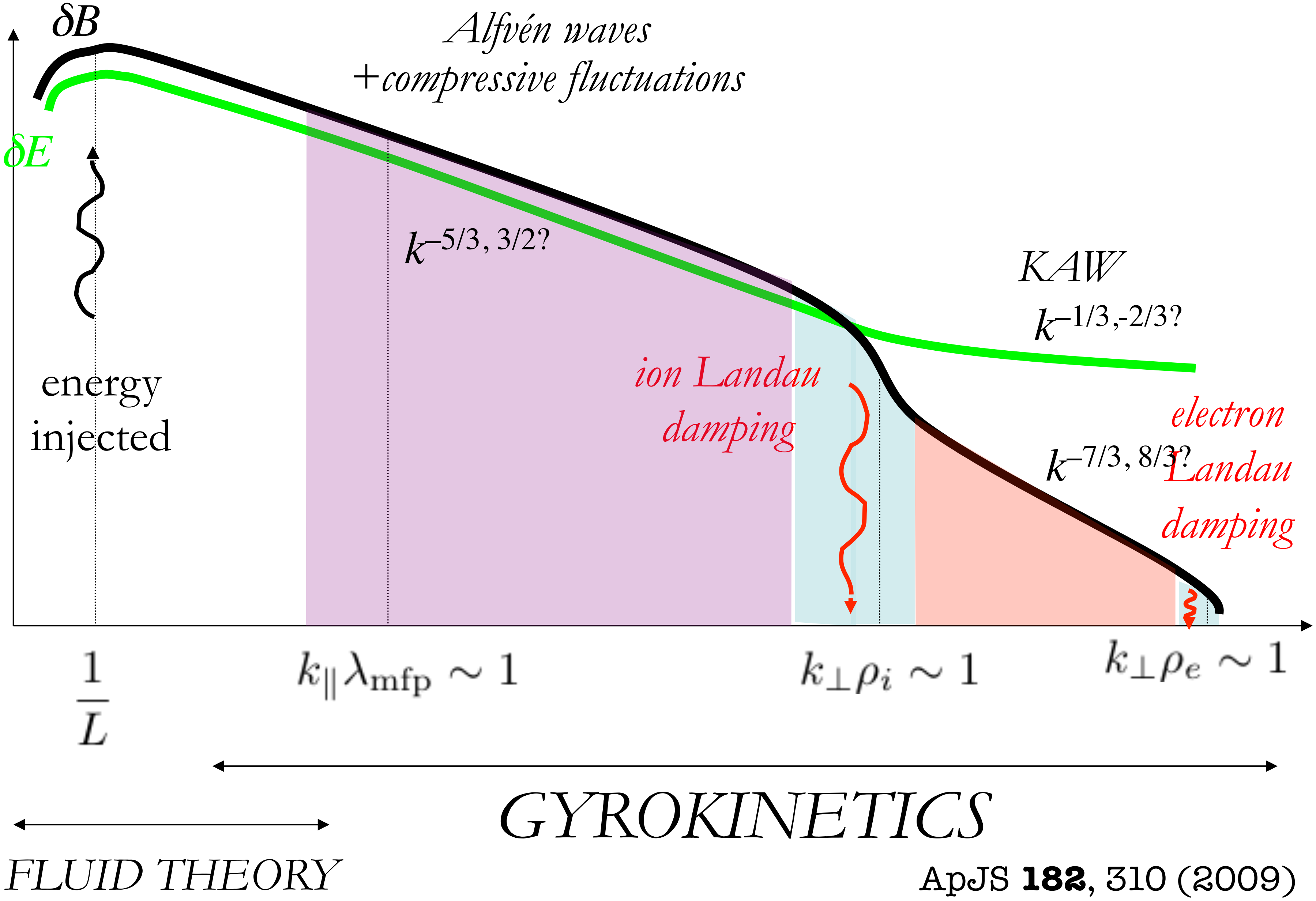


GYROKINETICS

FLUID THEORY

ApJS **182**, 310 (2009)

Gyrokinetics: Larmor-Scale Transition



ApJS **182**, 310 (2009)

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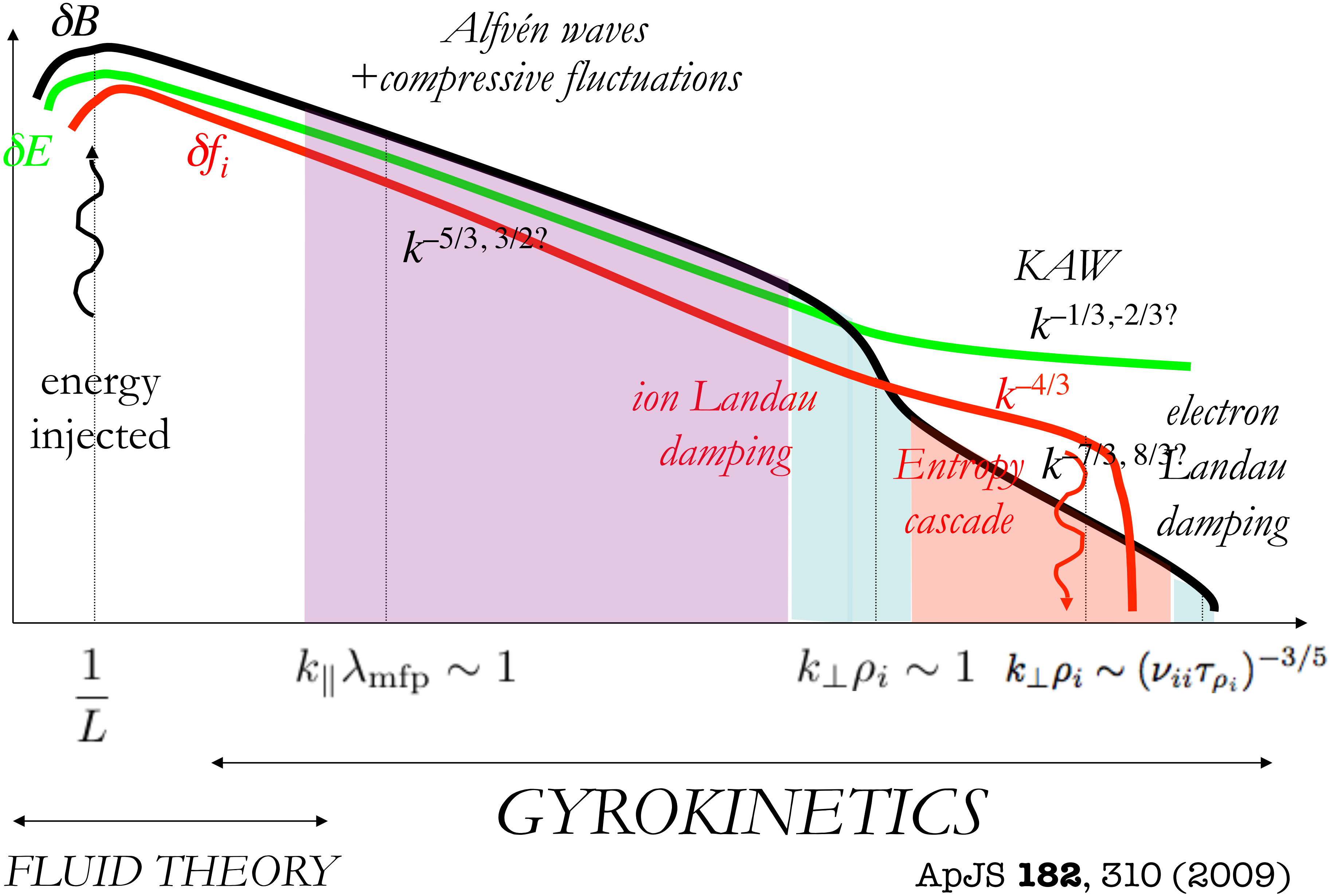
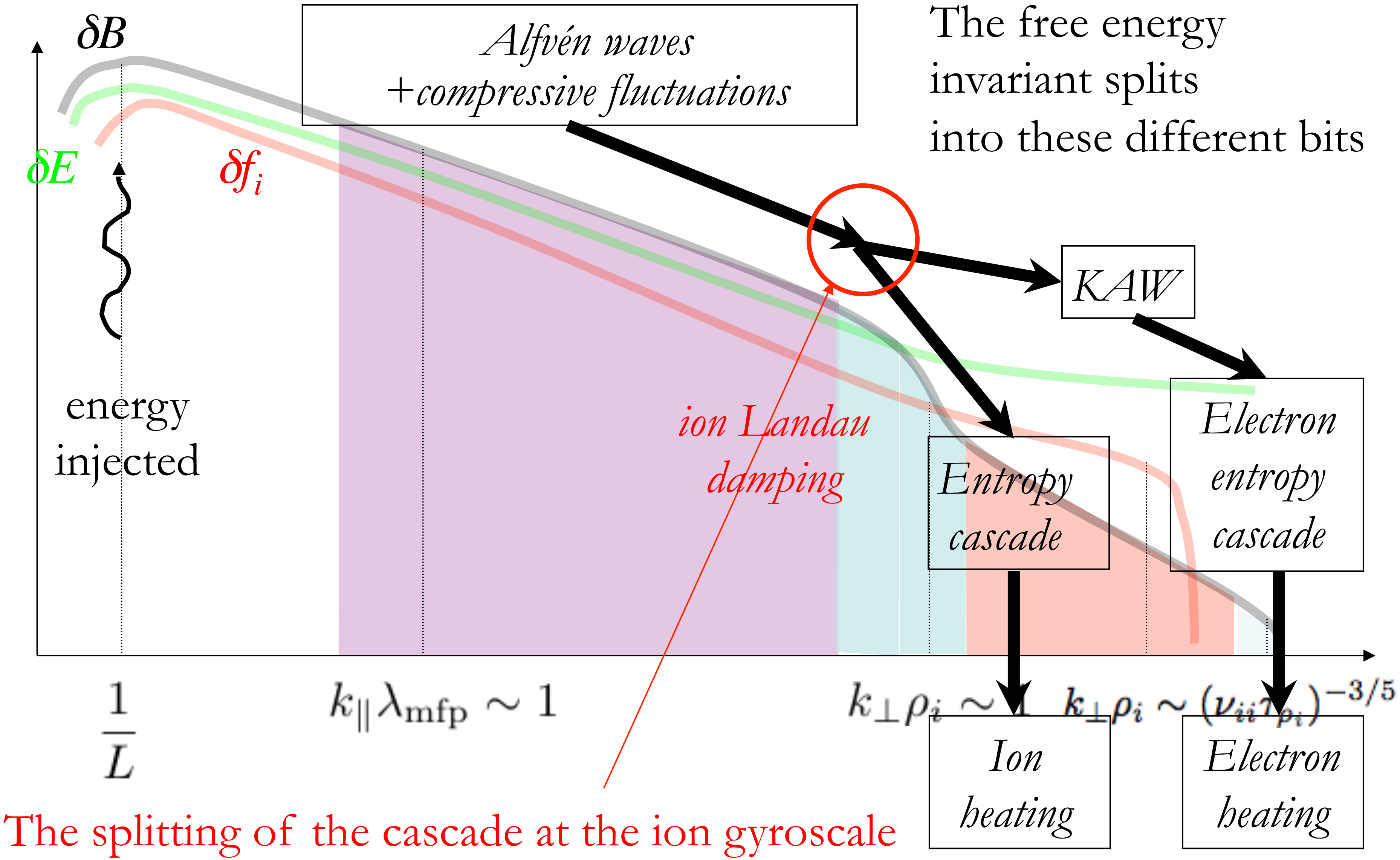


ApJS **182**, 310 (2009)

Free Energy Cascade

The free energy
invariant splits
into these different bits



Alfvén waves
+compressive fluctuations

KAW

Entropy cascade

Electron entropy cascade

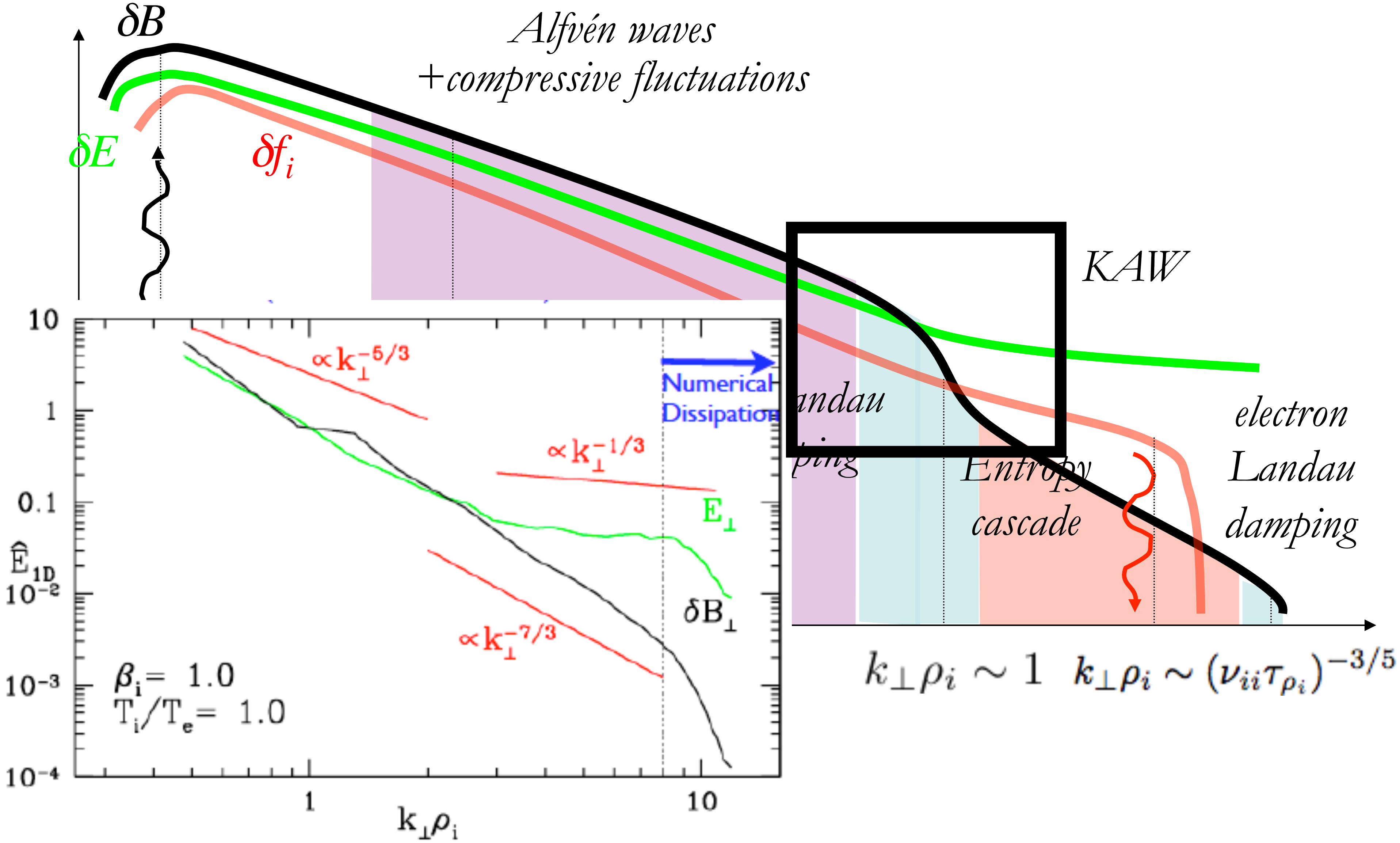
Ion heating

Electron heating

The splitting of the cascade at the ion gyroscale
determines relative heating of the species

ApJS **182**, 310 (2009)

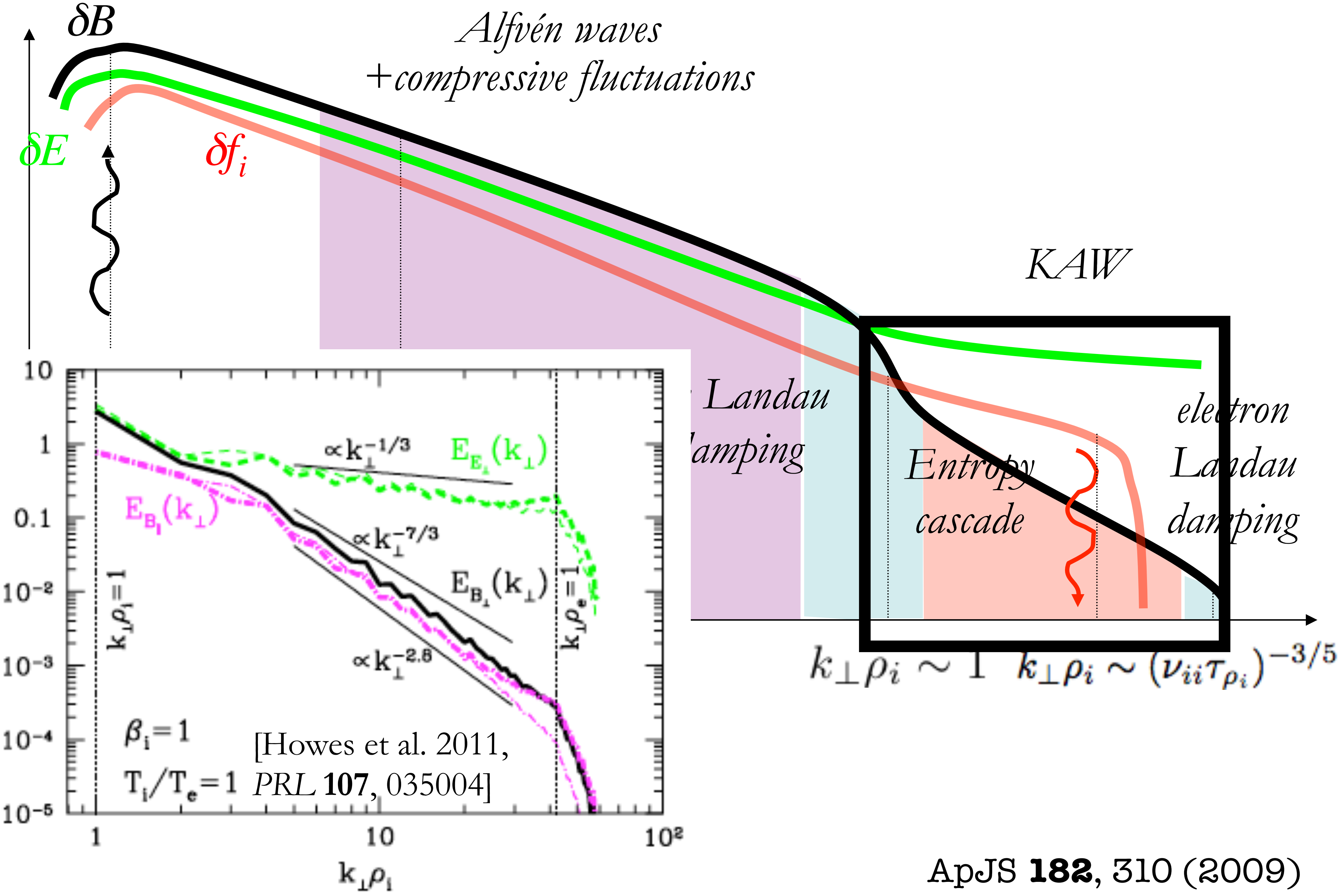
Larmor Transition: 3D GK DNS (by G. Howes)



[Howes et al. 2008, *PRL* **100**, 065004]

ApJS **182**, 310 (2009)

Sub-Larmor Cascade: 3D GK DNS (by G. Howes)



[Howes et al. 2011, *PRL* **107**, 035004]

ApJS **182**, 310 (2009)

KAW Fluctuations

Start with GK, consider the scales such that $k_\perp \rho_i \gg 1,\ k_\perp \rho_e \ll 1$

This is not a very wide interval, but an important one:

$$\sqrt{\frac{m_i}{m_e}} \approx 42$$

(answer to the general question of life, Universe and everything)

ApJS **182**, 310 (2009)

KAW Fluctuations

Start with GK, consider the scales such that $k_\perp \rho_i \gg 1,\ k_\perp \rho_e \ll 1$

$$\boxed{\begin{aligned} \frac{\partial \Psi}{\partial t} &= v_A \left(1 + Z/\tau\right) \hat{\mathbf{b}} \cdot \nabla \Phi, \\ \frac{\partial \Phi}{\partial t} &= -\frac{v_A}{2 + \beta_i \left(1 + Z/\tau\right)} \hat{\mathbf{b}} \cdot \nabla \left(\rho_i^2 \nabla_\perp^2 \Psi\right) \end{aligned}}$$

$$\hat{\mathbf{b}} \cdot \nabla = \frac{\partial}{\partial z} + \frac{1}{v_A}\{\Psi, \ldots\}$$

This is the anisotropic version of EMHD [Kingsep *et al.* 1990, *Rev. Plasma Phys.* **16**, 243], which is derived (for $\beta_i >> 1$) by assuming magnetic field frozen into electron fluid and doing a RMHD-style anisotropic expansion:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi e n_{0e}} \boldsymbol{\nabla} \times \left[(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}\right]$$

$$\frac{\delta \mathbf{B}}{B_0} = \frac{1}{v_A} \hat{\mathbf{z}} \times \boldsymbol{\nabla}_\perp \Psi + \hat{\mathbf{z}} \frac{\delta B_\parallel}{B_0}$$

$$\frac{\delta n_e}{n_{0e}} = -\frac{Ze\phi}{T_{0i}} = -\frac{2}{\sqrt{\beta_i}} \frac{\Phi}{\rho_i v_A}$$ Boltzmann ions

$$\frac{\delta B_\parallel}{B_0} = \frac{\beta_i}{2}\left(1 + \frac{Z}{\tau}\right) \frac{Ze\phi}{T_{0i}} = \sqrt{\beta_i}\left(1 + \frac{Z}{\tau}\right) \frac{\Phi}{\rho_i v_A}$$ pressure balance

$$\frac{B_0 \delta B_\parallel}{4\pi} = -\delta p_i - \delta p_e = -T_{0i} \delta n_i - T_{0e} \delta n_e$$

$$u_{\parallel e} = \frac{c}{4\pi e n_{0e}} \nabla_\perp^2 A_\parallel = -\frac{\rho_i \nabla_\perp^2 \Psi}{\sqrt{\beta_i}}$$ no parallel ion current

[Ions more or less an immobile neutralising background]

ApJS **182**, 310 (2009)

Kinetic Alfvén Waves

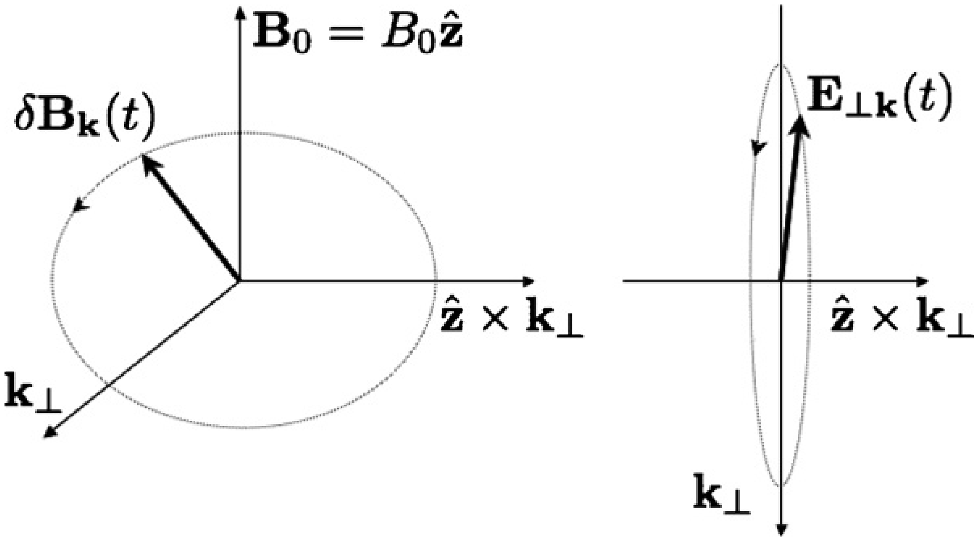
Start with GK, consider the scales such that $k_\perp \rho_i \gg 1,\ k_\perp \rho_e \ll 1$

$$\frac{\partial \Psi}{\partial t} = v_A \left(1+Z/\tau\right) \hat{\mathbf{b}}\cdot\nabla\Phi,$$

$$\frac{\partial \Phi}{\partial t} = -\frac{v_A}{2+\beta_i\left(1+Z/\tau\right)} \hat{\mathbf{b}}\cdot\nabla\left(\rho_i^2 \nabla_\perp^2 \Psi\right)$$

Linear wave solutions:

$$\boxed{\omega = \pm\sqrt{\frac{1+Z/\tau}{2+\beta_i\left(1+Z/\tau\right)}}\,k_\perp \rho_i k_\parallel v_A}$$



- Right-hand elliptically polarized
- $\delta E \sim k_\perp \phi \propto k_\perp \delta B$
- Landau-damped

ApJS **182**, 310 (2009)

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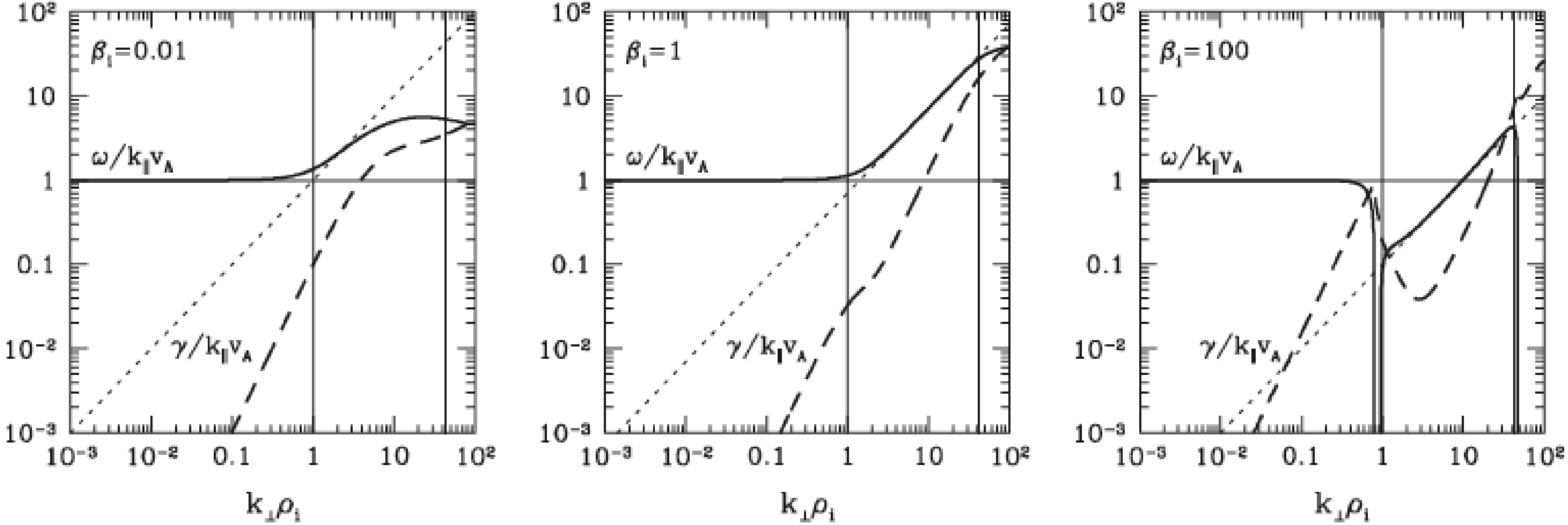

ApJS **182**, 310 (2009)

KAW Turbulence

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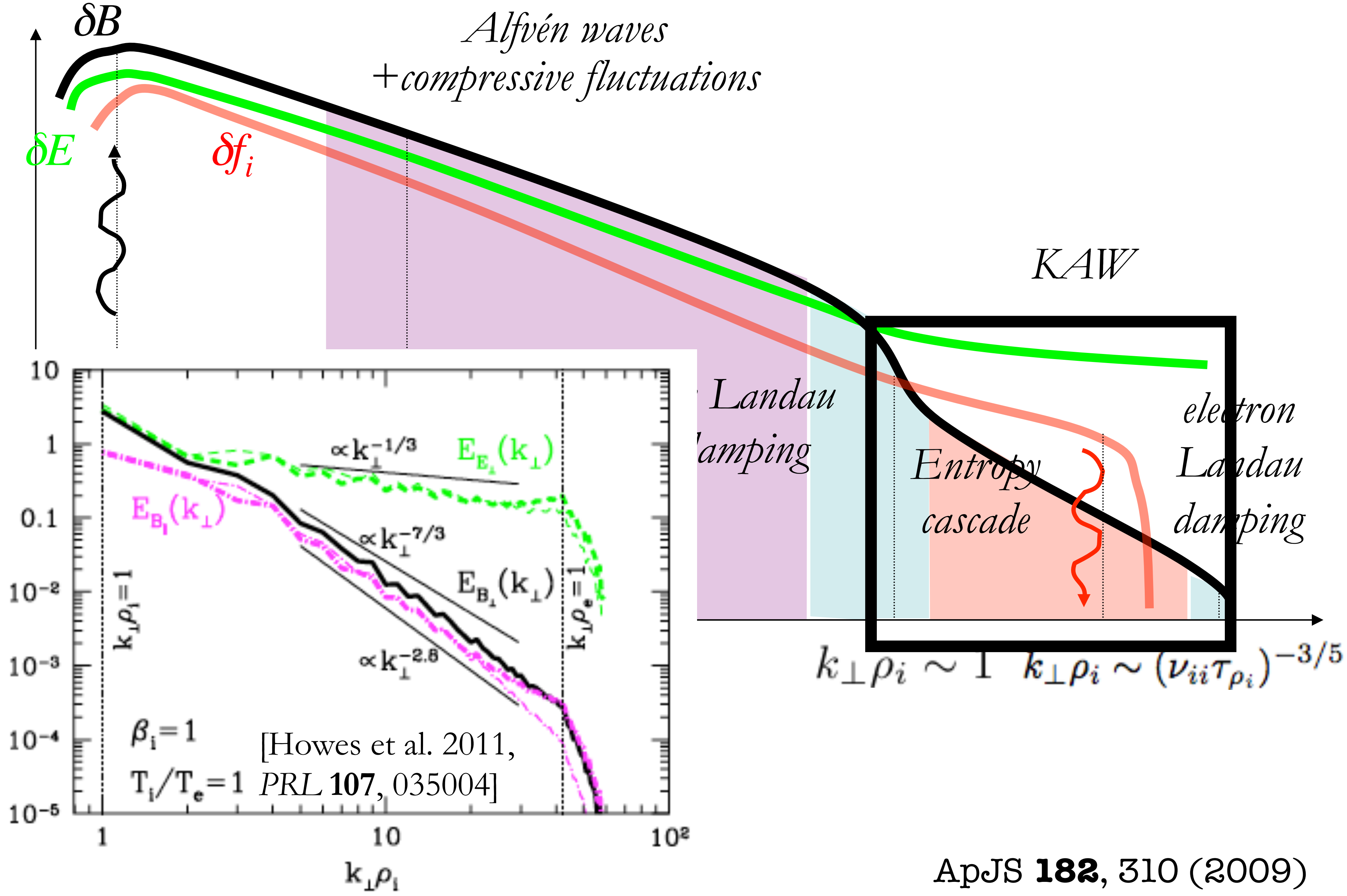
Linear wave solutions:

$$\omega = \pm\sqrt{\frac{1+Z/\tau}{2+\beta_i\left(1+Z/\tau\right)}}k_\perp\rho_i k_\parallel v_A$$

- There is a cascade of KAW,
  $\delta B_\parallel/B_0 \sim \Phi/\rho_i v_A \sim k_\perp\Psi/v_A \sim \delta B_\perp/B_0$
- Critical balance + constant flux argument à la K41/GS95 give
  $\boxed{k_\perp^{-7/3}}$ spectrum of magnetic field with anisotropy $k_\parallel \sim k_\perp^{1/3}$

  [Biskamp et al. 1996, *PRL* **76**, 1264; Cho & Lazarian 2004, *ApJ* **615**, L41]
- Electric field has $\boxed{k_\perp^{-1/3}}$ spectrum because $\delta E \sim k_\perp\phi \propto k_\perp\delta B$
- Recent modification of the theory by Boldyrev amends the spectrum to $\boxed{k_\perp^{-8/3}}$ by restricting cascade to 2D sheets [arXiv:1204.5809]
- NB: none of this takes into account Landau damping

Sub-Larmor Cascade: 3D GK DNS (by G. Howes)



[Howes et al. 2011, *PRL* **107**, 035004]

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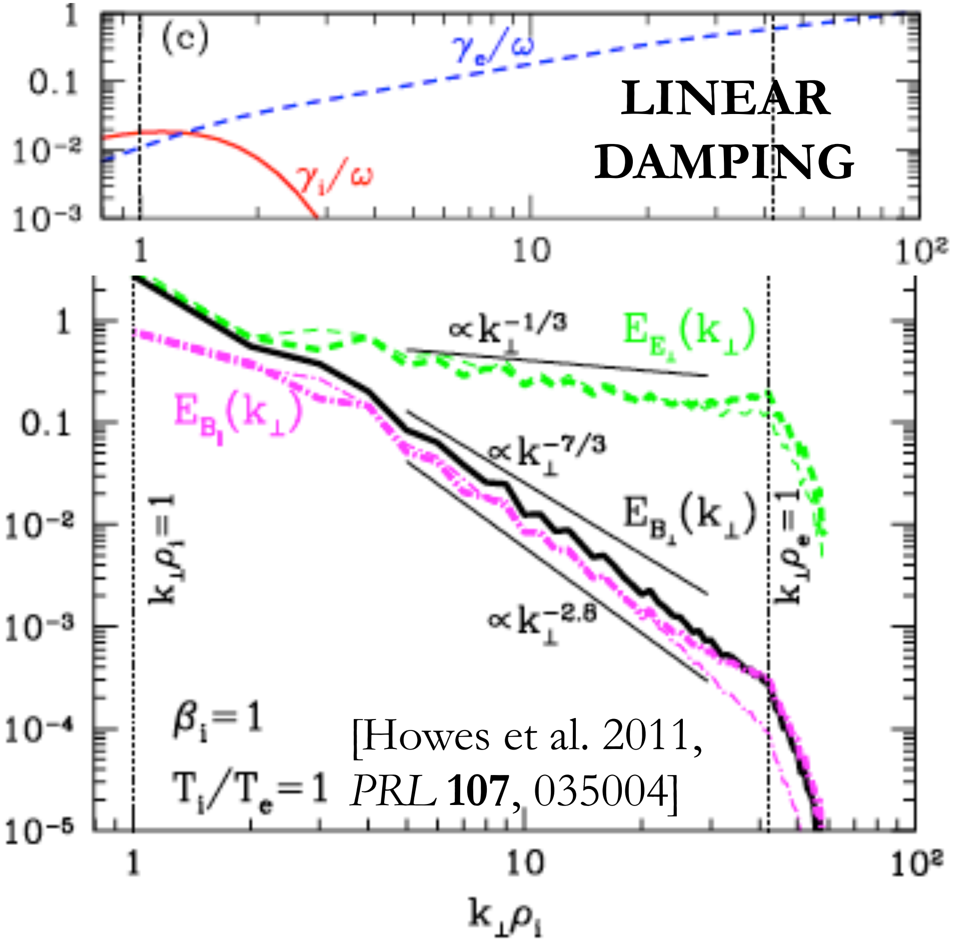
1. Power law spectra all the way to electron gyroscale despite electron Landau damping



LINEAR DAMPING

[Howes et al. 2011, *PRL* **107**, 035004]

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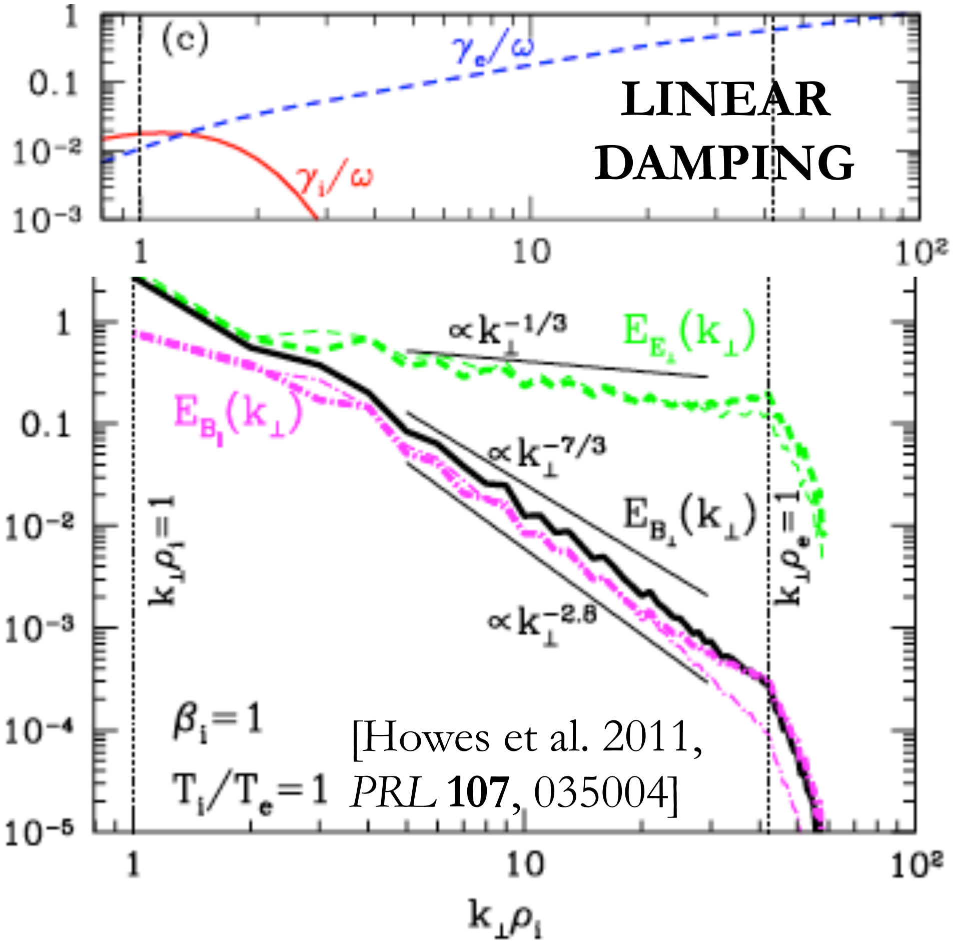

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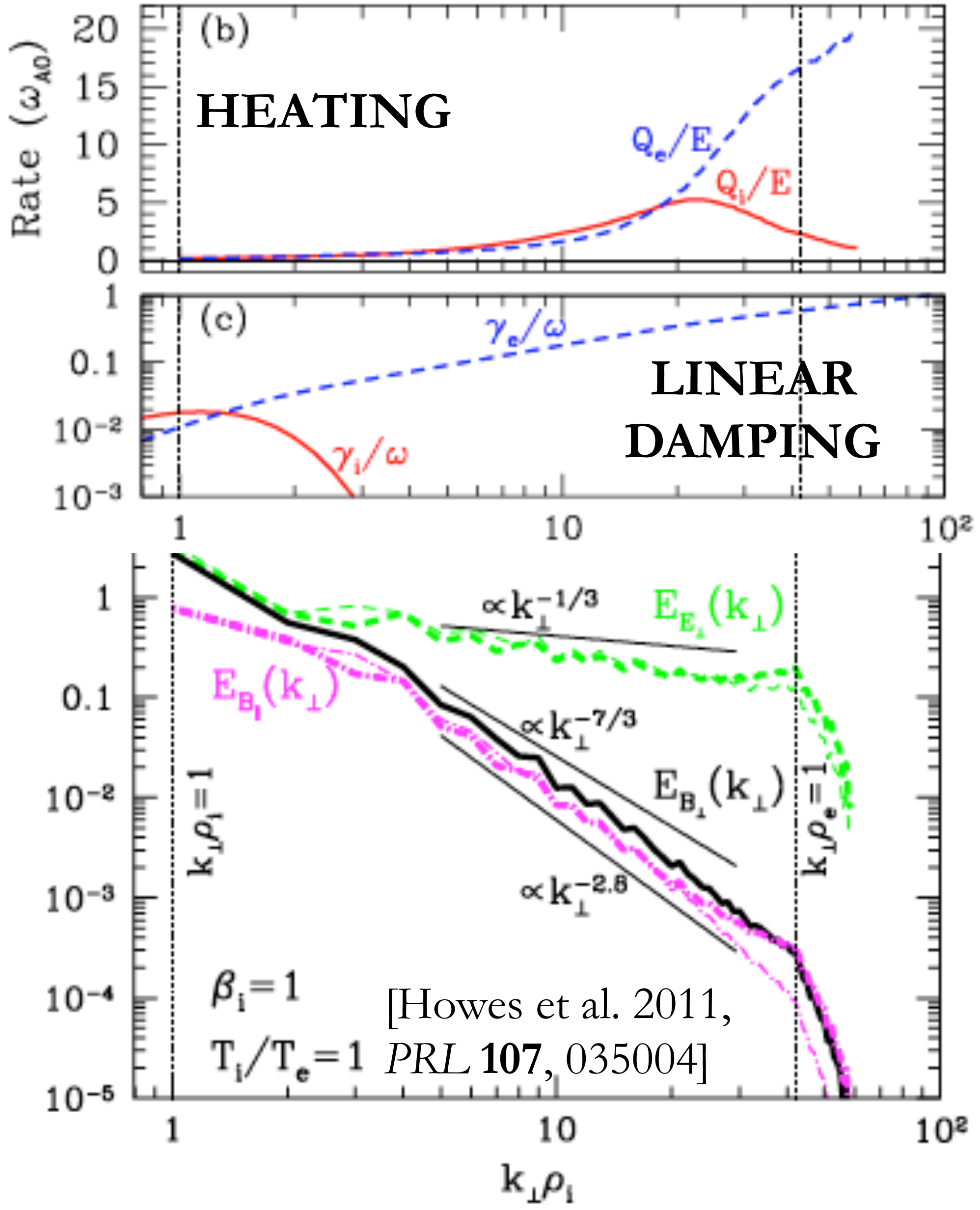

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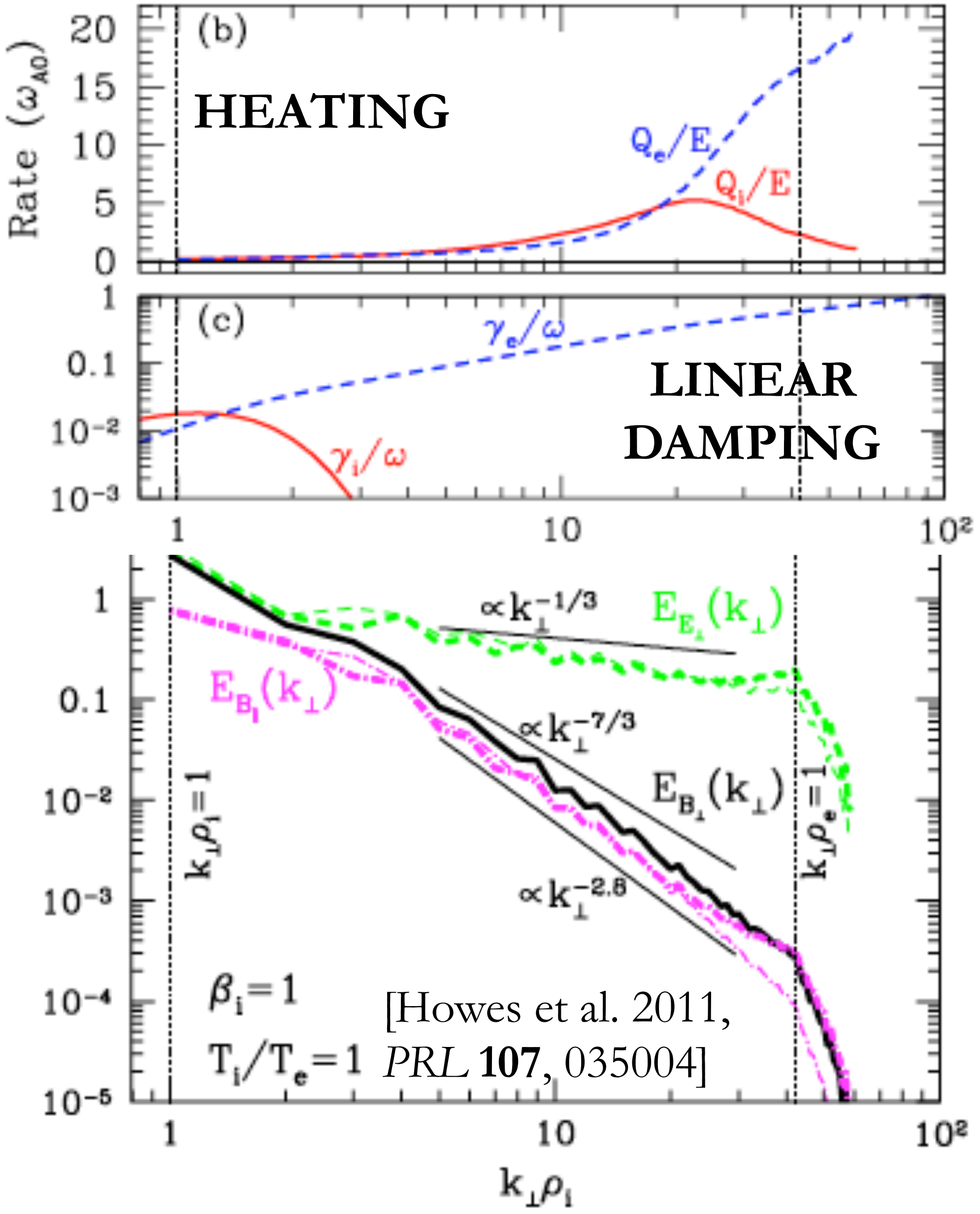

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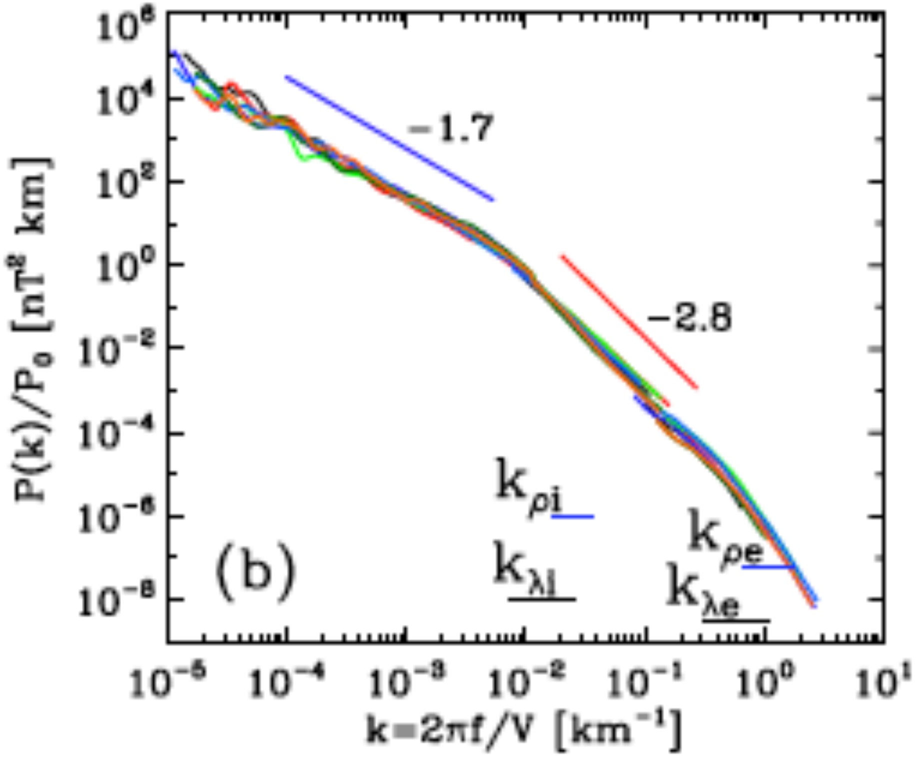
4. Magnetic spectrum ~ $\boxed{k_\perp^{-2.8}}$ which is steeper than KAW standard result –7/3 (perhaps closer to Boldyrev's –8/3) and in close agreement with solar wind data

Sub-Larmor Cascade: Solar Wind

[Alexandrova et al. 2009, *PRL* **103**, 165003]



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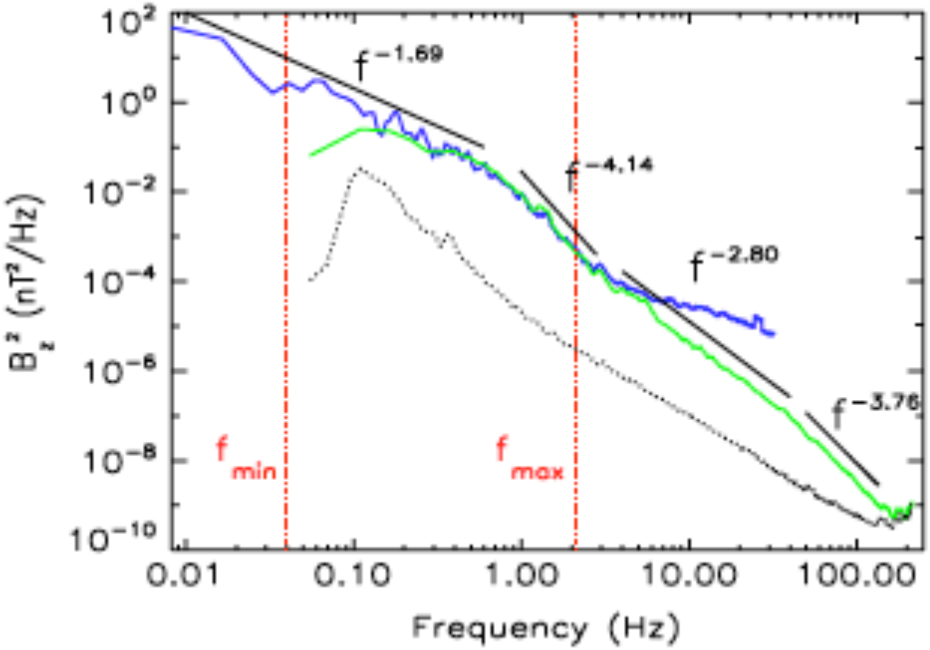
Sub-Larmor Cascade: Solar Wind

[Saharaoui et al. 2010, *PRL* **105**,131101]



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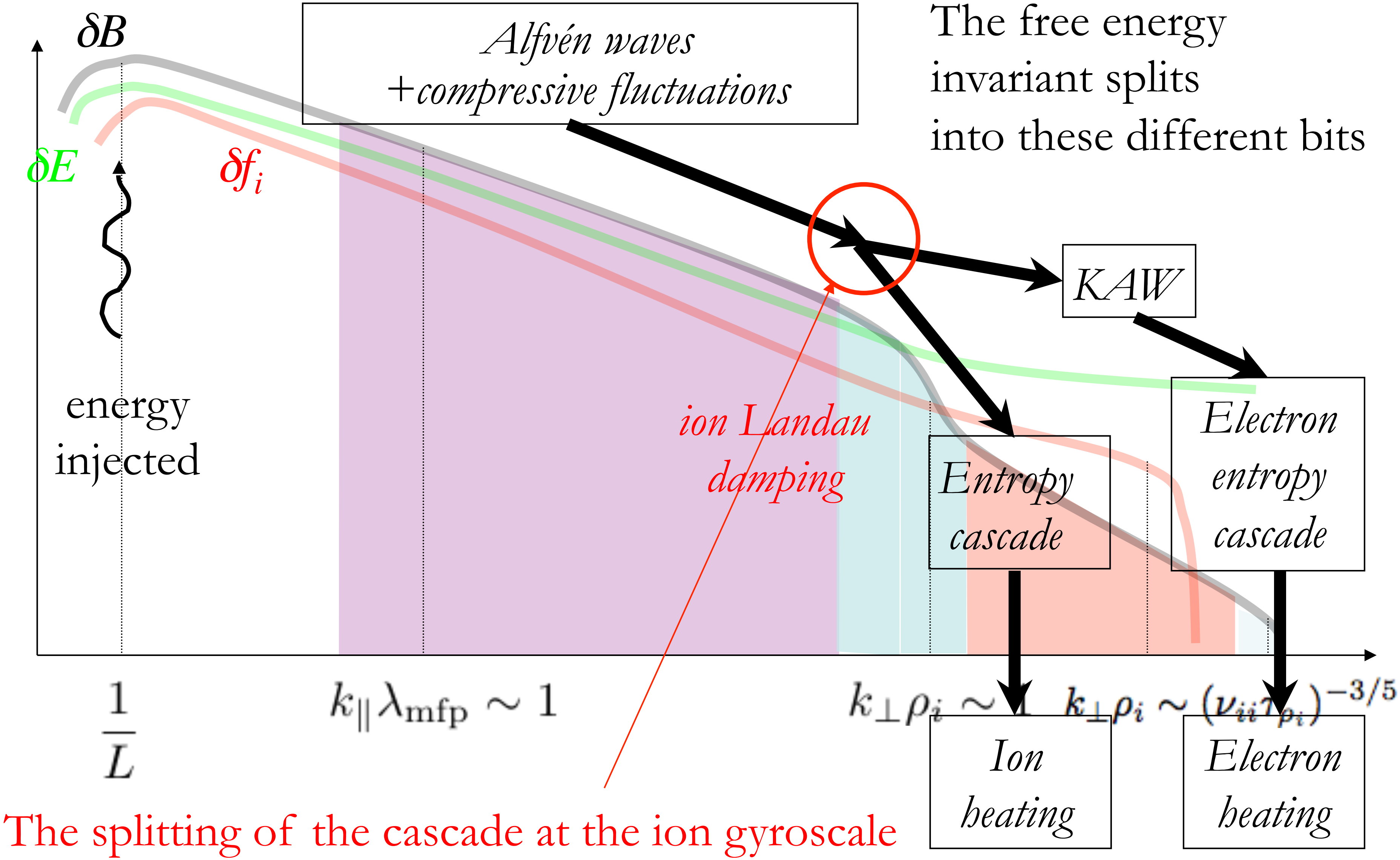
Conclusions

- **Kinetic turbulence is a free-energy cascade in phase space towards collisional scales**
- Gyrokinetics is a good approximation for magnetised turbulence
- In gyrokinetic turbulence, a fast nonlinear perpendicular phase-mixing mechanism allows small-scale structure to emerge simulataneously in physical and velocity space ("entropy cascade")
- We still need to understand how linear (| |) and nonlinear (⊥) phase mixing compete/coexist
- The free energy cascade splits into various channels:
  AW + compressive above ion gyroscale ("inertial range")
  KAW + entropy cascade below ion gyroscale ("dissipation range")
- The splitting at the ion gyroscale determines the relative heating of the two species ➔energy partition
- Structure of kinetic cascades (KAW turbulence, entropy cascade, compressive cascade) is interesting in its own right (and measurable!)

AAS et al. 2008, *PPCF* **50**, 124024 [arXiv:0806.1069]
AAS et al. 2009, *Astrophys. J. Suppl.* **182**, 310 [arXiv:0704.0044]

Free Energy Cascade



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