Kinetic MHD for astrophysical plasmas

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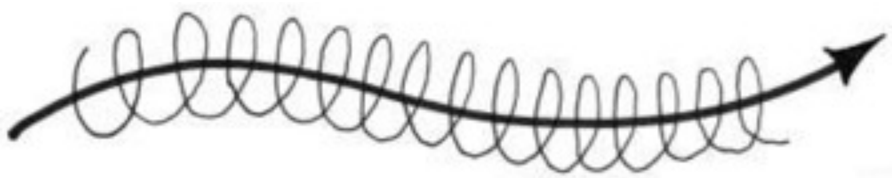
Weakly collisional ordering

There are a number of astrophysical plasmas where only part of the usual MHD ordering is satisfied. Yes,

$$\ell \ggg \lambda_{\mathrm{mfp}},\, \rho_{\mathrm{i,e}} \quad \text{and} \quad \omega^{-1} \ggg \nu_{\mathrm{i,e}}^{-1},\, \Omega_{\mathrm{i,e}}^{-1}$$

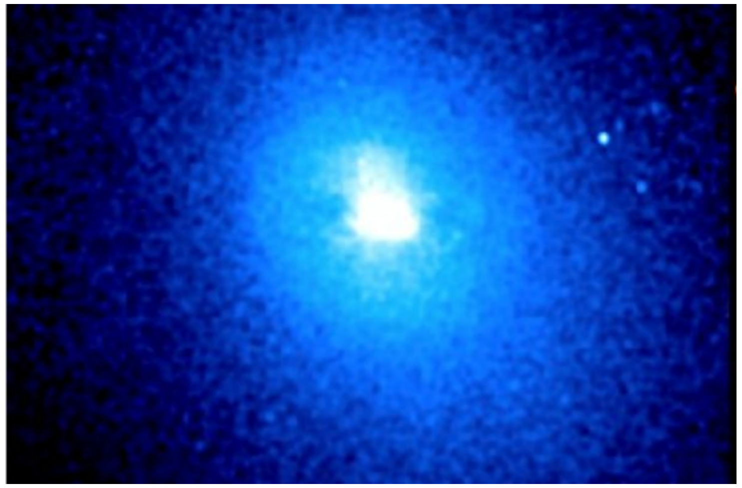
But, instead of $\rho_{\mathrm{i}} \gg \rho_{\mathrm{e}} \gg \lambda_{\mathrm{mfp}}$, these systems satisfy $\lambda_{\mathrm{mfp}} \gg \rho_{\mathrm{i}} \gg \rho_{\mathrm{e}}$.



Galactic centre at Bondi radius

$$\ell \sim 0.1 \text{ pc}$$
$$\lambda_{\mathrm{mfp}} \sim 0.01 \text{ pc}$$
$$\rho_{\mathrm{i}} \sim 1 \text{ ppc}$$

Intracluster medium of galaxy clusters



$$\ell \sim 100 \text{ kpc}$$
$$\lambda_{\mathrm{mfp}} \sim 1 \text{ kpc}$$
$$\rho_{\mathrm{i}} \sim 1 \text{ npc}$$

Goals of this lecture (series):

- To construct a formalism that can accurately describe the meso- and macroscale plasma physics at work in weakly collisional systems.

- To apply this formalism to study heat and (angular) momentum transport in some important astrophysical environments.

- To distract you from  .

Note! All the details of this lecture have been provided to you via my scanned lecture notes. Everything I present here (and more) is available in those notes.

Vlasov-Landau-Maxwell equations

The distribution function of species s (= i,e) satisfies

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \left[\frac{q_s}{m_s}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right) + \boldsymbol{g}\right]\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = \mathcal{C}[f_s]$$

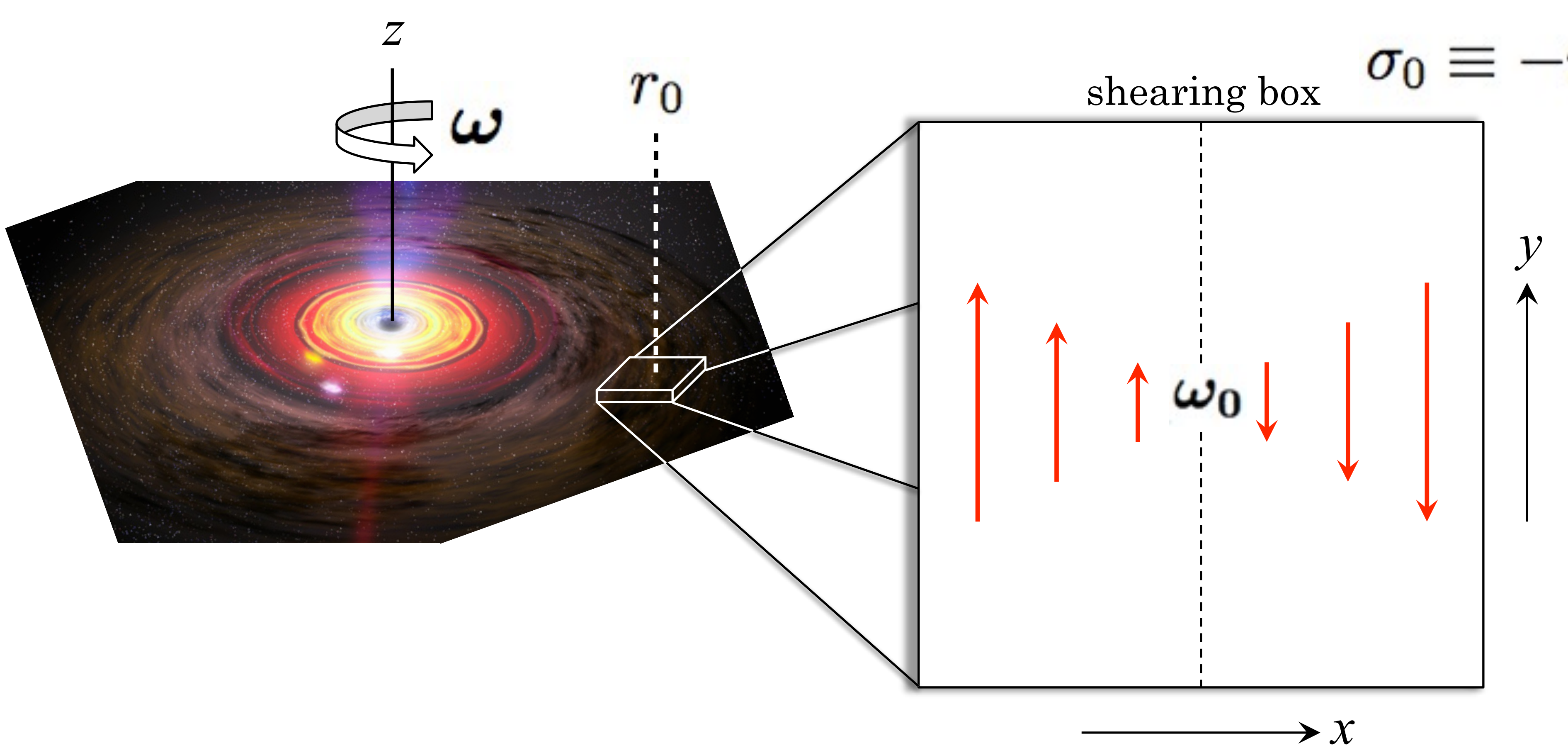
We'll use a simple Krook collision operator, $\mathcal{C}[f_s] = -\nu_s(f_s - F_{\mathrm{M,s}})$, which pushes f_s towards a Maxwellian at a rate ν_s. To this we append:

Quasi-neutrality: $\sum_s q_s n_s \equiv \sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, f_s = 0$

Ampère's law: $\boldsymbol{j} = \sum_s q_s n_s \boldsymbol{u}_s \equiv \sum_s q_s \int \mathrm{d}^3\boldsymbol{v}\, \boldsymbol{v} f_s = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}$

Faraday's law: $\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E}$ with $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$

Vlasov-Landau in a shearing sheet



$$\sigma_0 \equiv -r_0 \left.\frac{\mathrm{d}\omega}{\mathrm{d}r}\right|_{r=r_0}$$

$$\frac{\partial f_s}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_s + \left[\frac{q_s}{m_s}\left(\boldsymbol{E}' + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right) - 2\boldsymbol{\omega}_0\times\boldsymbol{v} - 2\omega_0\sigma_0 x\hat{\boldsymbol{x}} + \boldsymbol{g}_{\mathrm{eff}}\right]\cdot\frac{\partial f_s}{\partial \boldsymbol{v}} = \mathcal{C}[f_s]$$

electric field in rotating frame Coriolis shear $\boldsymbol{g}_{\mathrm{eff}} \equiv \boldsymbol{g} - \boldsymbol{\omega}_0\times(\boldsymbol{\omega}_0\times\boldsymbol{r})$

Peculiar velocities

It will be convenient to write this equation in terms of peculiar velocities:

$$\boldsymbol{v} \to \boldsymbol{v}' \equiv \boldsymbol{v} + \sigma_0 x \hat{\boldsymbol{y}} - \boldsymbol{u}_s(t, \boldsymbol{r})$$

Defining the convective derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u}_s \cdot \boldsymbol{\nabla} - \sigma_0 x \frac{\partial}{\partial y} \quad ,$$

our kinetic equation becomes

$$\frac{Df_s}{Dt} + \boldsymbol{v}' \cdot \boldsymbol{\nabla} f_s + \left[\boldsymbol{v}' \times \left(\frac{q_s \boldsymbol{B}}{m_s c} + 2\boldsymbol{\omega}_0 \right) + v'_x \sigma_0 \hat{\boldsymbol{y}} - \boldsymbol{v}' \cdot \boldsymbol{\nabla} \boldsymbol{u}_s + \boldsymbol{a}_s \right] \cdot \frac{\partial f_s}{\partial \boldsymbol{v}'} = C[f_s]$$

The acceleration $\boldsymbol{a}_s$ contains all the $\boldsymbol{v}'$-independent terms:

$$\boldsymbol{a}_s = \frac{q_s}{m_s} \left(\boldsymbol{E}'' + \frac{\boldsymbol{u}_s}{c} \times \boldsymbol{B} \right) - 2\boldsymbol{\omega}_0 \times \boldsymbol{u}_s + \boldsymbol{g}_{\text{eff}} - \frac{D\boldsymbol{u}_s}{Dt} + u_{s,x} \sigma_0 \hat{\boldsymbol{y}}$$

Moments of the kinetic equation

We follow standard procedure and take moments of the kinetic equation:

continuity eqn: $\int \mathrm{d}^3\boldsymbol{v}' \rightarrow \frac{Dn_s}{Dt} + n_s \boldsymbol{\nabla}\cdot\boldsymbol{u}_s = 0$

momentum eqn: $\int \mathrm{d}^3\boldsymbol{v}'\,\boldsymbol{v}' \rightarrow 0 = m_s n_s \boldsymbol{a}_s - \boldsymbol{\nabla}\cdot\mathbf{P}_s + R_s$

where $\mathbf{P}_s \equiv \int \mathrm{d}^3\boldsymbol{v}'\, m_s \boldsymbol{v}'\boldsymbol{v}' f_s$ is the pressure tensor

$R_s \equiv \int \mathrm{d}^3\boldsymbol{v}'\, m_s \boldsymbol{v}'\, \mathcal{C}[f_s]$ is the frictional force

Adding the momentum eqns of the ion and electrons gives (with $m_{\mathrm{e}} \ll m_{\mathrm{i}}$)

$$\frac{D\boldsymbol{u}_{\mathrm{i}}}{Dt} = -\frac{\boldsymbol{\nabla}\cdot\mathbf{P}}{m_{\mathrm{i}}n_{\mathrm{i}}} + \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi m_{\mathrm{i}}n_{\mathrm{i}}} + \boldsymbol{g}_{\mathrm{eff}} - 2\boldsymbol{\omega}_0\times\boldsymbol{u}_{\mathrm{i}} + u_{\mathrm{i},x}\sigma_0\hat{\boldsymbol{y}}$$

Ohm’s and Faraday’s laws

Electron momentum equation + quasi-neutrality gives:

$$\boldsymbol{E}'' = -\frac{\boldsymbol{u}_\mathrm{i}}{c}\times\boldsymbol{B} \boxed{\begin{aligned} &+\frac{1}{cen_\mathrm{e}}\boldsymbol{j}\times\boldsymbol{B} + \frac{1}{en_\mathrm{e}}R_\mathrm{e} - \frac{1}{en_\mathrm{e}}\boldsymbol{\nabla}\cdot\mathbf{P}_\mathrm{e} \\ &- \frac{m_\mathrm{e}}{e}\left[\frac{D\boldsymbol{u}_\mathrm{e}}{Dt} + 2\boldsymbol{\omega}_0\times\boldsymbol{u}_\mathrm{e} - \boldsymbol{g}_\mathrm{eff} - u_{\mathrm{e},x}\sigma_0\hat{\boldsymbol{y}}\right] \end{aligned}}$$

Induction equation is shearing-rotating frame is:

$$\left(\frac{\partial}{\partial t} - \sigma_0 x\frac{\partial}{\partial y}\right)\boldsymbol{B} = -c\boldsymbol{\nabla}\times\boldsymbol{E}'' - \sigma_0 B_x\hat{\boldsymbol{y}}$$

notes prove that these are negligible in considered ordering

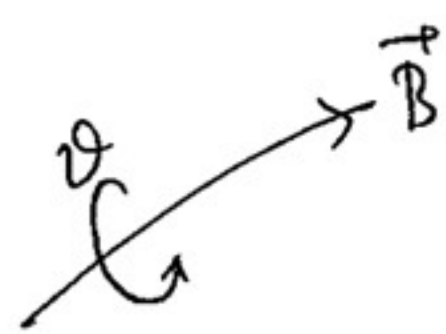
The pressure tensor

$$\mathbf{P}_s \equiv \int \mathrm{d}^3\boldsymbol{v}'\, m_s \boldsymbol{v}'\boldsymbol{v}'\, f_s$$

You may be suspecting that our (x,y,z) coordinate system is not the best for evaluating the components of this tensor — and you'd be right.

But there *is* a coordinate system where $\mathbf{P}_s$ is nice and diagonal.

To find it, we must show that the distribution function is "gyrotropic", i.e. independent of gyroangle:



The pressure tensor

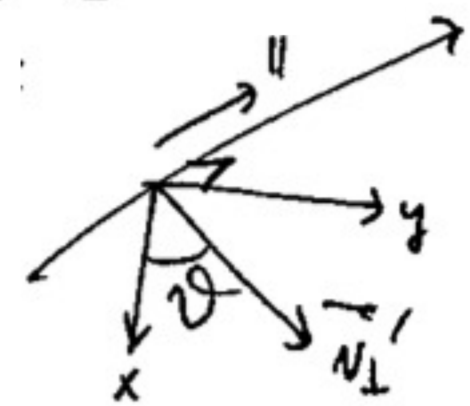
Now we come to our first major assumption: $\omega, \nu \ll \Omega$ and $k, \lambda_{\rm mfp}^{-1} \ll \rho^{-1}$
Recall our kinetic equation:

$$\frac{Df_s}{Dt} + \boldsymbol{v}'\cdot\boldsymbol{\nabla} f_s + \left[\boldsymbol{v}'\times\left(\frac{q_s\boldsymbol{B}}{m_s c} + 2\boldsymbol{\omega}_0\right) + v_x'\sigma_0\hat{\boldsymbol{y}} - \boldsymbol{v}'\cdot\boldsymbol{\nabla}\boldsymbol{u}_s + \boldsymbol{a}_s\right]\cdot\frac{\partial f_s}{\partial \boldsymbol{v}'} = C[f_s]$$

this is the largest term

$$\rightarrow 0 = \left(\boldsymbol{v}'\times\frac{q_s\boldsymbol{B}}{m_s c}\right)\cdot\frac{\partial f_s}{\partial\boldsymbol{v}'} \quad \text{to leading order}$$

$$= \frac{q_s B}{m_s c}\left(v_\perp'\sin\vartheta\frac{\partial f_s}{\partial v_x'} - v_\perp'\cos\vartheta\frac{\partial f_s}{\partial v_y'}\right) = -\Omega_s\left(\frac{\partial f_s}{\partial\vartheta}\right)_{v_\perp', v_\parallel'}$$



$\rightarrow f_s$ is gyrotropic $\rightarrow f_s = f_s(t, \boldsymbol{r}, v_\perp', v_\parallel')$

The pressure tensor

$$\mathbf{P}_s = \begin{pmatrix} p_{\perp,s} & 0 & 0 \\ 0 & p_{\perp,s} & 0 \\ 0 & 0 & p_{||,s} \end{pmatrix} = (\mathbf{I} - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}})\, p_{\perp,s} + \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\, p_{||,s}$$

$$\frac{D\boldsymbol{u}_{\rm i}}{Dt} = -\frac{1}{m_{\rm i} n_{\rm i}} \boldsymbol{\nabla}\cdot \left[\mathbf{I} \left(p_\perp + \frac{B^2}{8\pi} \right) - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} \left(p_\perp - p_{||} + \frac{B^2}{4\pi} \right) \right] + \boldsymbol{g}_{\rm eff} - 2\boldsymbol{\omega}_0 \times \boldsymbol{u}_{\rm i} + u_{{\rm i},x}\sigma_0 \hat{\boldsymbol{y}}$$

p in MHD

pressure anisotropy (augments tension)
absent in collisional MHD

causes all kinds of nasty stuff
that will break our ordering
(next talk by Pierre-Louis Sulem)

Why would $p_\perp \neq p_{||}$?

1. Magnetic moment: conservation of angular momentum

$$\mu \equiv \frac{mv_\perp'^2}{2B} \sim \text{const} \qquad mv_\perp' \rho = \frac{mv_\perp'^2}{\Omega} \propto \frac{mv_\perp'^2}{B}$$

sum over particles: $\left(\int \mathrm{d}^3 \boldsymbol{v}' \mu f = \frac{p_\perp}{B} \right) \times \text{volume} \rightarrow \frac{p_\perp}{nB} \sim \text{const}$

2. Longitudinal invariant: conservation of linear momentum

$$J \equiv \oint mv_{||}' \mathrm{d}\ell \sim \text{const}$$

sum over particles: $\left(\int \mathrm{d}^3 \boldsymbol{v}' J f = \frac{p_{||} B^2}{n^2} \right) \times \text{volume} \rightarrow \frac{p_{||} B^2}{n^3} \sim \text{const}$

CGL (or "double adiabatic") equations

Chew, Goldberger & Low (1956)

$$\frac{D}{Dt}\left(\frac{p_\perp}{nB}\right) = 0 \qquad \frac{D}{Dt}\left(\frac{p_{||}B^2}{n^3}\right) = 0$$

takes the place of the usual "single adiabatic" equation:

$$\frac{D}{Dt}\left(\frac{p}{n^{5/3}}\right) = 0$$

But much remains to be desired — e.g. collisions, heat fluxes, etc...

Let's derive equations for parallel and perpendicular pressure rigorously.

Gyroaveraged kinetic equation in $v'_\perp$ and $v'_{||}$

$$\frac{Df_s}{Dt} + v'_{||}\,\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} f_s + \frac{v'^2_\perp}{2}(\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}})\frac{\partial f_s}{\partial v'_{||}} + a_{||,s}\left(\frac{v'_{||}}{v'}\frac{\partial f_s}{\partial v'} + \frac{\partial f_s}{\partial v'_{||}}\right)$$
$$+\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_s - b_x b_y \sigma_0\right)\left[\left(\frac{v'^2_\perp}{2v'} - \frac{v'^2_{||}}{v'}\right)\frac{\partial f_s}{\partial v'} - v'_{||}\frac{\partial f_s}{\partial v'_{||}}\right] = \langle C[f_s]\rangle_\vartheta$$

Gyroaveraged kinetic equation in $v'_\perp$ and $v'_{||}$

$$\boxed{\frac{Df_s}{Dt}} + v'_{||}\,\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla} f_s + \frac{v'^2_\perp}{2}(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})\frac{\partial f_s}{\partial v'_{||}} + a_{||,s}\left(\frac{v'_{||}}{v'}\frac{\partial f_s}{\partial v'} + \frac{\partial f_s}{\partial v'_{||}}\right)$$
$$+\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_s - b_x b_y \sigma_0\right)\left[\left(\frac{v'^2_\perp}{2v'} - \frac{v'^2_{||}}{v'}\right)\frac{\partial f_s}{\partial v'} - v'_{||}\frac{\partial f_s}{\partial v'_{||}}\right] = \langle C[f_s]\rangle_\vartheta$$

temporal change in distribution function in frame co-moving with fluid

Gyroaveraged kinetic equation in $v'_\perp$ and $v'_{||}$

$$\frac{Df_s}{Dt} + \boxed{v'_{||}\,\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} f_s} + \frac{v'^2_\perp}{2}(\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}})\frac{\partial f_s}{\partial v'_{||}} + a_{||,s}\left(\frac{v'_{||}}{v'}\frac{\partial f_s}{\partial v'} + \frac{\partial f_s}{\partial v'_{||}}\right)$$
$$+ \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_s - b_x b_y \sigma_0\right)\left[\left(\frac{v'^2_\perp}{2v'} - \frac{v'^2_{||}}{v'}\right)\frac{\partial f_s}{\partial v'} - v'_{||}\frac{\partial f_s}{\partial v'_{||}}\right] = \langle C[f_s]\rangle_\vartheta$$

parallel advection of distribution function

Gyroaveraged kinetic equation in $v'_\perp$ and $v'_{||}$

$$\frac{Df_s}{Dt} + v'_{||}\,\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla} f_s + \boxed{\frac{v'^2_\perp}{2}(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})\frac{\partial f_s}{\partial v'_{||}}} + a_{||,s}\left(\frac{v'_{||}}{v'}\frac{\partial f_s}{\partial v'} + \frac{\partial f_s}{\partial v'_{||}}\right)$$
$$+\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_s - b_x b_y \sigma_0\right)\left[\left(\frac{v'^2_\perp}{2v'} - \frac{v'^2_{||}}{v'}\right)\frac{\partial f_s}{\partial v'} - v'_{||}\frac{\partial f_s}{\partial v'_{||}}\right] = \langle C[f_s]\rangle_\vartheta$$

magnetic mirror force

Gyroaveraged kinetic equation in $v'_\perp$ and $v'_{||}$

$$\frac{Df_s}{Dt} + v'_{||}\,\hat{\boldsymbol{b}}\cdot\boldsymbol{\nabla} f_s + \frac{v'^2_\perp}{2}(\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}})\frac{\partial f_s}{\partial v'_{||}} + \boxed{a_{||,s}\left(\frac{v'_{||}}{v'}\frac{\partial f_s}{\partial v'} + \frac{\partial f_s}{\partial v'_{||}}\right)}$$
$$+ \left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_s - b_x b_y \sigma_0\right)\left[\left(\frac{v'^2_\perp}{2v'} - \frac{v'^2_{||}}{v'}\right)\frac{\partial f_s}{\partial v'} - v'_{||}\frac{\partial f_s}{\partial v'_{||}}\right] = \langle C[f_s]\rangle_\vartheta$$

parallel “fluid” force

Gyroaveraged kinetic equation in $v'_\perp$ and $v'_{||}$

$$\frac{Df_s}{Dt} + v'_{||}\,\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} f_s + \frac{v'^2_\perp}{2}(\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}})\frac{\partial f_s}{\partial v'_{||}} + a_{||,s}\left(\frac{v'_{||}}{v'}\frac{\partial f_s}{\partial v'} + \frac{\partial f_s}{\partial v'_{||}}\right)$$
$$+ \boxed{\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{:}\boldsymbol{\nabla}\boldsymbol{u}_s - b_x b_y \sigma_0\right)\left[\left(\frac{v'^2_\perp}{2v'} - \frac{v'^2_{||}}{v'}\right)\frac{\partial f_s}{\partial v'} - v'_{||}\frac{\partial f_s}{\partial v'_{||}}\right]} = \langle C[f_s]\rangle_\vartheta$$

inertial term
(particles responding to a changing magnetic field in the fluid frame)

Pressure equations

Can be written in a form that generalises the CGL result:

$$p_{\perp,s}\frac{D}{Dt}\ln\frac{p_{\perp,s}}{n_s B} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}_{\perp,s} - q_{\perp,s}\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}} - \frac{1}{3}\nu_s\left(p_{\perp,s} - p_{||,s}\right)$$

$$p_{||,s}\frac{D}{Dt}\ln\frac{p_{||,s}B^2}{n_s^3} = -\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{q}_{||,s} + 2q_{\perp,s}\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}} - \frac{2}{3}\nu_s\left(p_{||,s} - p_{\perp,s}\right)$$

adiabatic invariance

re-distribution of heat

collisional relaxation

Pressure equations

Add these with $p_s = \frac{2}{3} p_{\perp,s} + \frac{1}{3} p_{||,s}$:

$$\frac{3}{2} p_s \frac{D}{Dt} \ln \frac{p_s}{n_s^{5/3}} = (p_{\perp,s} - p_{||,s}) \frac{D}{Dt} \ln \frac{B}{n_s^{2/3}} - \boldsymbol{\nabla}\cdot\left(\boldsymbol{q}_{\perp,s} + \frac{1}{2}\boldsymbol{q}_{||,s}\right)$$

change in entropy

"viscous" heating

redistribution of heat

(more on these shortly)

Pressure equations

Subtract these:

$$\frac{1}{3}\left[\frac{D}{Dt}+\nu_s+\frac{D}{Dt}\ln\frac{B}{n_s^{7/3}}\right]\left(p_{\perp,s}-p_{||,s}\right)$$

$$=p_s\frac{D}{Dt}\ln\frac{B}{n_s^{2/3}}-\frac{1}{3}\boldsymbol{\nabla}\cdot\left(\boldsymbol{q}_{\perp,s}-\boldsymbol{q}_{||,s}\right)-q_{\perp,s}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}}$$

collisional relaxation

adiabatic invariance

redistribution of heat

pinching of field lines

Enter collisions...

So far, we have only assumed the plasma to be magnetised. No assumption has been made regarding the relative size of $k_\perp \rho_\mathrm{i},\, k_{||}\lambda_\mathrm{mfp},\, \omega/\nu_\mathrm{i}, m_\mathrm{e}/m_\mathrm{i}$.

Let’s do that:

$$\frac{\omega}{\nu_\mathrm{i}} \sim \frac{k_{||}\lambda_\mathrm{mfp}}{\sqrt{\beta}}, \quad k_\perp \rho_\mathrm{i} \ll k_{||}\lambda_\mathrm{mfp} \sim \sqrt{\frac{m_\mathrm{e}}{m_\mathrm{i}}} \ll 1$$

Then:

$$\frac{1}{3}\left[\cancel{\frac{D}{Dt}} + \overset{s=\mathrm{i}}{\nu_s} + \cancel{\frac{D}{Dt}\ln\frac{B}{n_s^{7/3}}}\right]\left(p_{\perp,s} - p_{||,s}\right)$$

$$= p_s \frac{D}{Dt}\ln\frac{B}{n_s^{2/3}} - \frac{1}{3}\boldsymbol{\nabla}\boldsymbol{\cdot}\left(\boldsymbol{q}_{\perp,s} - \boldsymbol{q}_{||,s}\right) - q_{\perp,s}\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}}$$

$$\rightarrow \boxed{\frac{p_\perp - p_{||}}{p_\mathrm{i}} = \frac{1}{\nu_\mathrm{i}}\frac{D}{Dt}\ln\frac{B^3}{n_\mathrm{i}^2} - \left[\frac{\boldsymbol{\nabla}\boldsymbol{\cdot}(\boldsymbol{q}_{\perp,\mathrm{i}} - \boldsymbol{q}_{||,\mathrm{i}}) + 3q_{\perp,\mathrm{i}}\boldsymbol{\nabla}\boldsymbol{\cdot}\hat{\boldsymbol{b}}}{p_\mathrm{i}\nu_\mathrm{i}}\right]}$$

Braginskii viscosity

$$\frac{p_\perp - p_{||}}{p_\mathrm{i}} = \frac{1}{\nu_\mathrm{i}}\boxed{\frac{D}{Dt}\ln\frac{B^3}{n_\mathrm{i}^2}} - \cancel{\left[\frac{\boldsymbol{\nabla}\cdot(\boldsymbol{q}_{\perp,\mathrm{i}} - \boldsymbol{q}_{||,\mathrm{i}}) + 3q_{\perp,\mathrm{i}}\boldsymbol{\nabla}\cdot\hat{\boldsymbol{b}}}{p_\mathrm{i}\nu_\mathrm{i}}\right]} \sim k_{||}\lambda_\mathrm{mfp}\frac{v_\mathrm{th}}{u}$$

$$= 3\,\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}:\boldsymbol{\nabla}\boldsymbol{u}_\mathrm{i} - 3b_x b_y \sigma_0 - \boldsymbol{\nabla}\cdot\boldsymbol{u}_\mathrm{i}$$ by induction and continuity equations

Then the ion momentum equation becomes

$$\frac{D\boldsymbol{u}_\mathrm{i}}{Dt} = -\frac{1}{m_\mathrm{i}n_\mathrm{i}}\boldsymbol{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi m_\mathrm{i}n_\mathrm{i}} + \boldsymbol{g}_\mathrm{eff} - 2\boldsymbol{\omega}_0\times\boldsymbol{u}_\mathrm{i} + u_{\mathrm{i},x}\sigma_0\hat{\boldsymbol{y}}$$

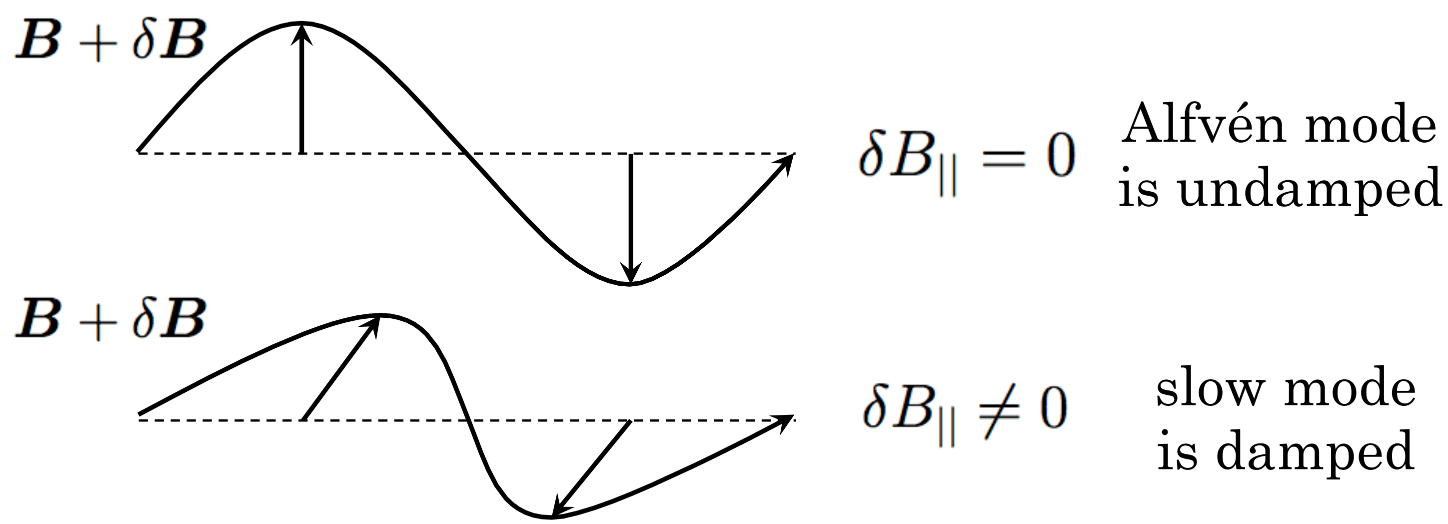
$$+\frac{1}{m_\mathrm{i}n_\mathrm{i}}\boldsymbol{\nabla}\cdot\left[\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{1}{3}\mathsf{I}\right)\frac{3}{2}\frac{p_\mathrm{i}}{\nu_\mathrm{i}}\left(\hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - \frac{1}{3}\mathsf{I}\right):\left(\boldsymbol{\nabla}\boldsymbol{u}_\mathrm{i} - \sigma_0\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\right)\right]$$

"Braginskii viscosity"

Braginskii viscosity — physical interpretation

1. Adiabatic invariance produces pressure anisotropy, which is viscously relaxed, thereby damping the motions that created the anisotropy.

   Linearly, this means that:



2. Since particles only sample velocity gradients along field lines, these are the only velocity gradients that can be viscously relaxed.

For sufficiently weak collisions, field lines become isotachs.

Heat flow

For derivation of equations governing perpendicular and parallel heat flow, see my lecture notes. For our weakly collisional ordering, the punchline is

$$q_{\perp} = -\frac{1}{2}\frac{n_{\mathrm{e}} v_{\mathrm{th,e}}^2}{\nu_{\mathrm{e}}}\,\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} T_{\mathrm{e}} \qquad \text{and} \qquad q_{||} = -\frac{3}{2}\frac{n_{\mathrm{e}} v_{\mathrm{th,e}}^2}{\nu_{\mathrm{e}}}\,\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} T_{\mathrm{e}}$$

with $v_{\mathrm{th,e}} \equiv \sqrt{\dfrac{2T_{\mathrm{e}}}{m_{\mathrm{e}}}}$.

For sufficiently weak collisions,
field lines become isotherms.

$$\frac{3}{2}p\,\frac{D}{Dt}\ln\frac{p}{n^{5/3}} = \underbrace{\frac{1}{3}p_{\mathrm{i}}\nu_{\mathrm{i}}\left(\frac{p_{\perp,\mathrm{i}} - p_{||,\mathrm{i}}}{p_{\mathrm{i}}}\right)^2}_{\text{viscous heating}} + \underbrace{\boldsymbol{\nabla\cdot}\left(\frac{5}{4}\frac{n_{\mathrm{e}} v_{\mathrm{th,e}}^2}{\nu_{\mathrm{e}}}\,\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot}\boldsymbol{\nabla} T_{\mathrm{e}}\right)}_{\text{redistribution of heat}}$$

Summary

- Kinetic MHD: fluid theory + closure scheme that respects the directionality of the magnetic field and weak collisions
  $\longrightarrow$ anisotropic pressure + anisotropic heat flow

- Braginskii MHD: these anisotropies manifest as a source of viscosity in momentum equation and viscous heating in entropy equation

$$\frac{Dn_\mathrm{i}}{Dt} = -n_\mathrm{i}\boldsymbol{\nabla\cdot u}_\mathrm{i}$$

$$\frac{D\boldsymbol{u}_\mathrm{i}}{Dt} = -\frac{1}{m_\mathrm{i}n_\mathrm{i}}\boldsymbol{\nabla\cdot}\left[\mathsf{I}\left(p_\perp + \frac{B^2}{8\pi}\right) - \hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\left(p_\perp - p_{||} + \frac{B^2}{4\pi}\right)\right] + \boldsymbol{g}_\mathrm{eff} - 2\boldsymbol{\omega}_0\boldsymbol{\times u}_\mathrm{i} + u_{\mathrm{i},x}\sigma_0\hat{\boldsymbol{y}}$$

$$\frac{3}{2}p\,\frac{D}{Dt}\ln\frac{p}{n^{5/3}} = (p_\perp - p_{||})\frac{D}{Dt}\ln\frac{B}{n^{2/3}} - \boldsymbol{\nabla\cdot}\left(\chi_{||}\hat{\boldsymbol{b}}\hat{\boldsymbol{b}}\boldsymbol{\cdot\nabla}T_\mathrm{e}\right)$$

$$\frac{D\boldsymbol{B}}{Dt} - \boldsymbol{B\cdot\nabla u}_\mathrm{i} = \boldsymbol{B\nabla\cdot u}_\mathrm{i}$$

Application: Magneto-viscous instability (MVI)

Fastest derivation of the MVI ever...

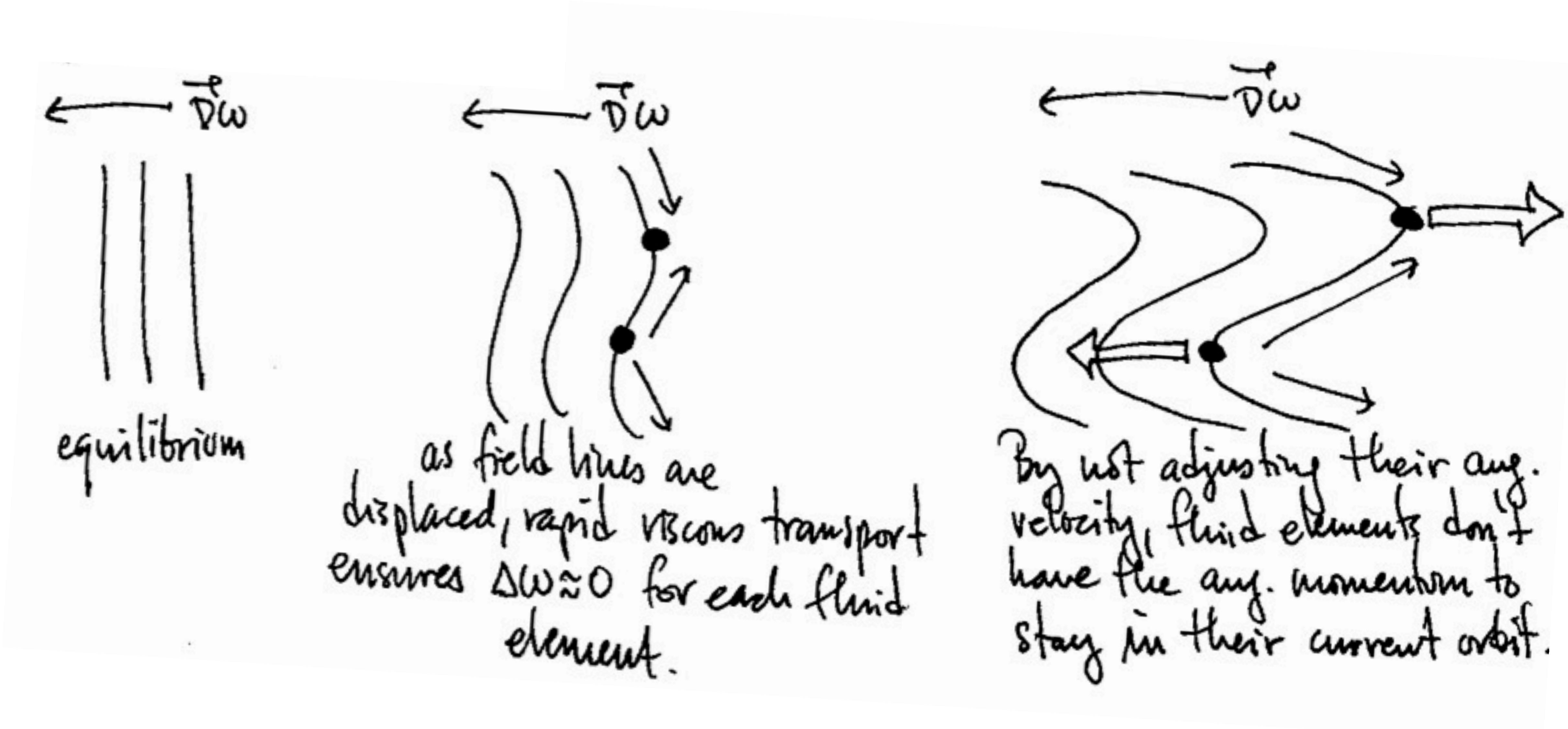
Recall, from Braginskii:

For sufficiently weak collisions, field lines become isotachs.

In an accretion disc, this means that magnetically coupled fluid elements satisfy $\Delta\omega \approx 0^-$ when displaced.

Since $\dfrac{\Delta L}{L_0} = 2\,\dfrac{\xi_x}{r_0} + \dfrac{\Delta\omega}{\omega_0} \approx 2\,\dfrac{\xi_x}{r_0}$, outwardly (inwardly) displaced fluid elements gain (lose) angular momentum. For $\sigma_0 > 0$, this is enough to guarantee instability.

MVI (Balbus 2004; Islam & Balbus 2005)



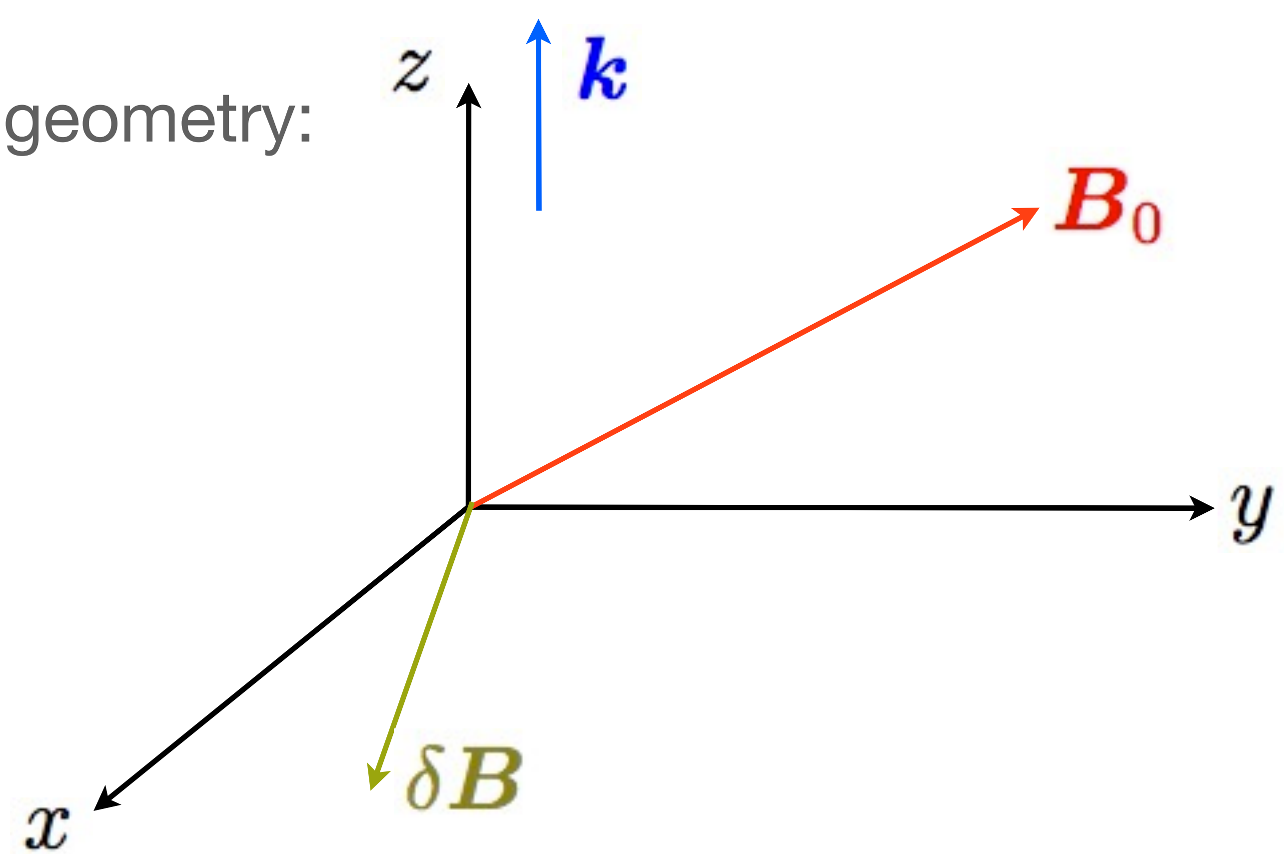
For this to work...

...there must be a pressure anisotropy generated:

$$p_\perp - p_{||} \sim \frac{1}{\nu_\mathrm{i}} \frac{D \ln B}{Dt} \rightarrow \frac{\gamma}{\nu_\mathrm{i}} \frac{\delta B_{||}}{B_0}$$

(recall that slow modes are damped, but not Alfvénic perturbations).

Simplest geometry:



Dispersion relation (see provided notes for derivation)

$$\widetilde{\gamma}^2 \left(\widetilde{\gamma}^2 + \gamma\, \omega_{\rm visc} \frac{k_\perp^2}{k^2} - 2\sigma_0\omega_0 \frac{k_z^2}{k^2} \right) = \gamma\, \omega_{\rm visc} \frac{k_z^2 b_y^2}{k^2} 2\sigma_0\omega_0 - 4\omega_0^2\gamma^2 \frac{k_z^2}{k^2}$$

slow mode

anisotropic viscous damping

free-energy gradient (shear)

epicyclic coupling

Alfvén wave

viscous coupling of slow & Alfvén modes

Dispersion relation (see provided notes for derivation)

$$\widetilde{\gamma}^2 \left(\widetilde{\gamma}^2 + \gamma\, \omega_{\rm visc} \frac{k_\perp^2}{k^2} - 2\sigma_0\omega_0\, \frac{k_z^2}{k^2} \right) = \gamma\, \omega_{\rm visc} \frac{k_z^2 b_y^2}{k^2}\, 2\sigma_0\omega_0 - 4\omega_0^2\gamma^2 \frac{k_z^2}{k^2}$$

$$\omega_{\rm visc} = 0 : \quad \widetilde{\gamma}^2 \left(\widetilde{\gamma}^2 - 2\sigma_0\omega_0\, \frac{k_z^2}{k^2} \right) = -4\omega_0^2\gamma^2 \frac{k_z^2}{k^2}$$ (MRI)

$$\gamma_{\rm max} = \frac{1}{2}\sigma_0 \qquad k_{||}^2 v_{\rm A}^2 \sim \omega_0^2 \qquad \frac{\delta B_y}{\delta B_x} = -1$$

$$\omega_{\rm visc} \gg \omega_0 : \quad \widetilde{\gamma}^2 \approx 2\sigma_0\omega_0 \frac{k_z^2 b_y^2}{k_\perp^2}$$ (MVI)

$$\gamma_{\rm max} \approx \sqrt{2\sigma_0\omega_0} \qquad k_{||}^2 v_{\rm A}^2 \sim \sqrt{\frac{\omega_0 \nu_{\rm i}}{\beta}} \qquad \frac{\delta B_y}{\delta B_x} \approx 0^-$$

Collisionless MRI (Quataert, Dorland & Hammett 2002)

This behaviour is not unique to the Braginskii closure. In fact, the details of the closure don't matter much (one can obtain this instability with CGL equations, or full kinetic equations, or kinetic MHD with Landau closure).

$$\ddot{\xi}_x - 2\omega_0 \dot{\xi}_y = (2\sigma_0\omega_0 - K_x)\,\xi_x$$

$$\ddot{\xi}_y + 2\omega_0 \dot{\xi}_x = -K_y\,\xi_y$$

$$\rightarrow \left(\gamma^2 + K_x - 2\sigma_0\omega_0\right)\left(\gamma^2 + K_y\right) = -4\omega_0^2\gamma^2$$

$$K_y \gg \omega_0 \gg K_x : \quad \gamma \approx \sqrt{2\sigma_0\omega_0}$$

K_y includes the perturbed pressure anisotropy

Angular momentum transport

$$T_{xy} = \rho v_x \delta v_y - \hat{b}_x \hat{b}_y \left(\frac{B^2}{4\pi} + p_\perp - p_{||} \right)$$

Reynolds stress

Maxwell stress

Pressure stress (in Braginskii, viscous stress)

$$\overline{T_{xy}} \approx \rho |\xi_x|^2 (\sigma_0 + 2\omega_0) \sqrt{2\sigma_0\omega_0}$$

$$= \rho |\xi_x|^2 \left| \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right|^{1/2} \left(\frac{\kappa^2}{2\Omega} \right)$$

Geoffroy notation

Current situation

- Sharma et al (2006):
  kinetic MHD with Landau fluid closure + anisotropy limiters
- Several groups currently pursuing collisionless MRI with PIC codes (myself included — I'd be happy to talk with you about it)
- How pressure anisotropy is limited (see next talk) affects turbulent heating (e.g. ion vs electron) and turbulent transport