Turbulence in Laboratory
& Astrophysical Plasmas

Many thanks to several
co-workers and collaborators

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Kinetic turbulence in laboratory and astrophysical plasmas

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Les Houches School on “The Future of Plasma Astrophysics”
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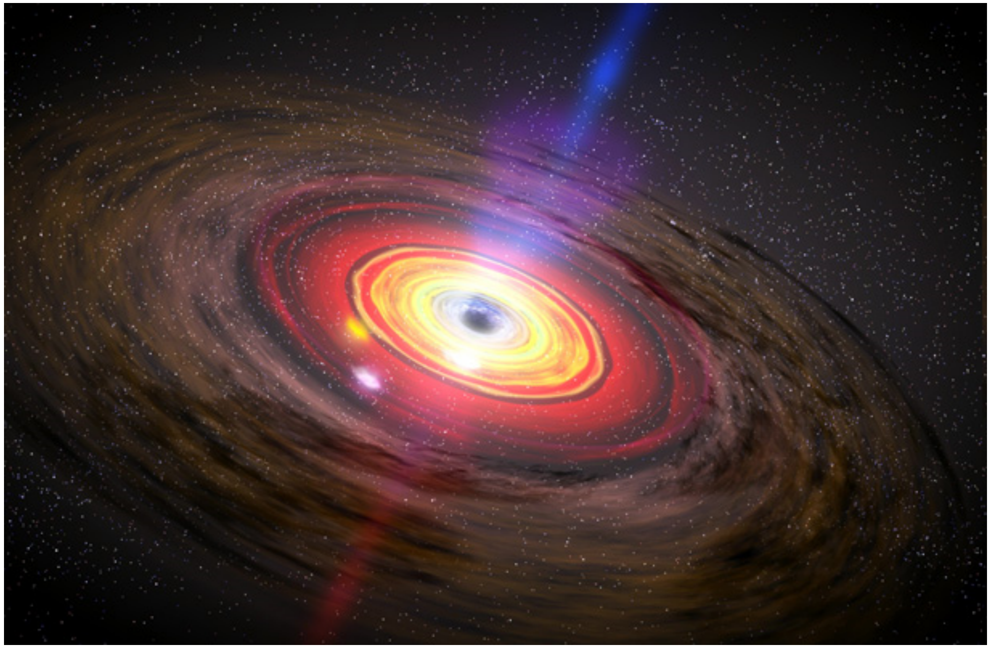
Kinetic turbulence in astrophysics

Key question:	How is the turbulent energy transformed into heat (and observable radiation) at small scales?

MHD is not able to address this question, a kinetic approach is required



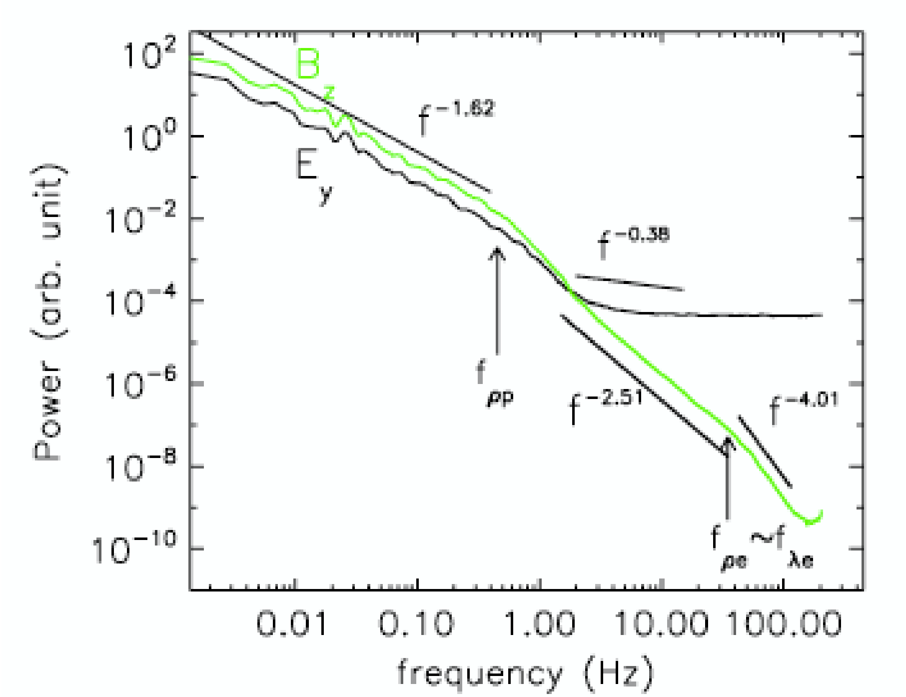
Black hole accretion disks

Applies also to **solar corona, interstellar medium, galaxy clusters**

The tail of the MHD cascade is kinetic: The solar wind dissipation range

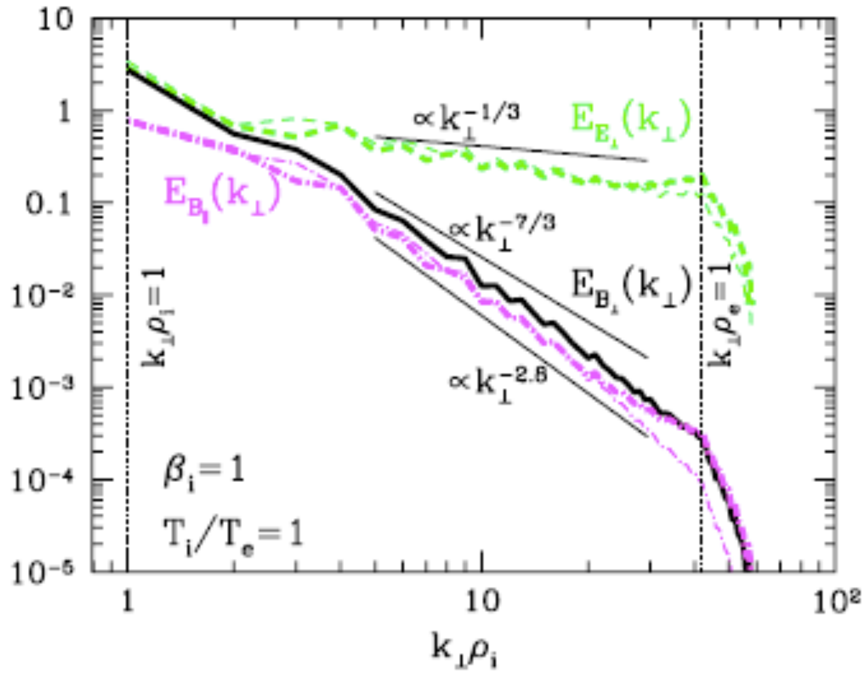
See talks by A. Schekochihin and F. Sahraoui

Sahraoui *et al*., PRL 2009



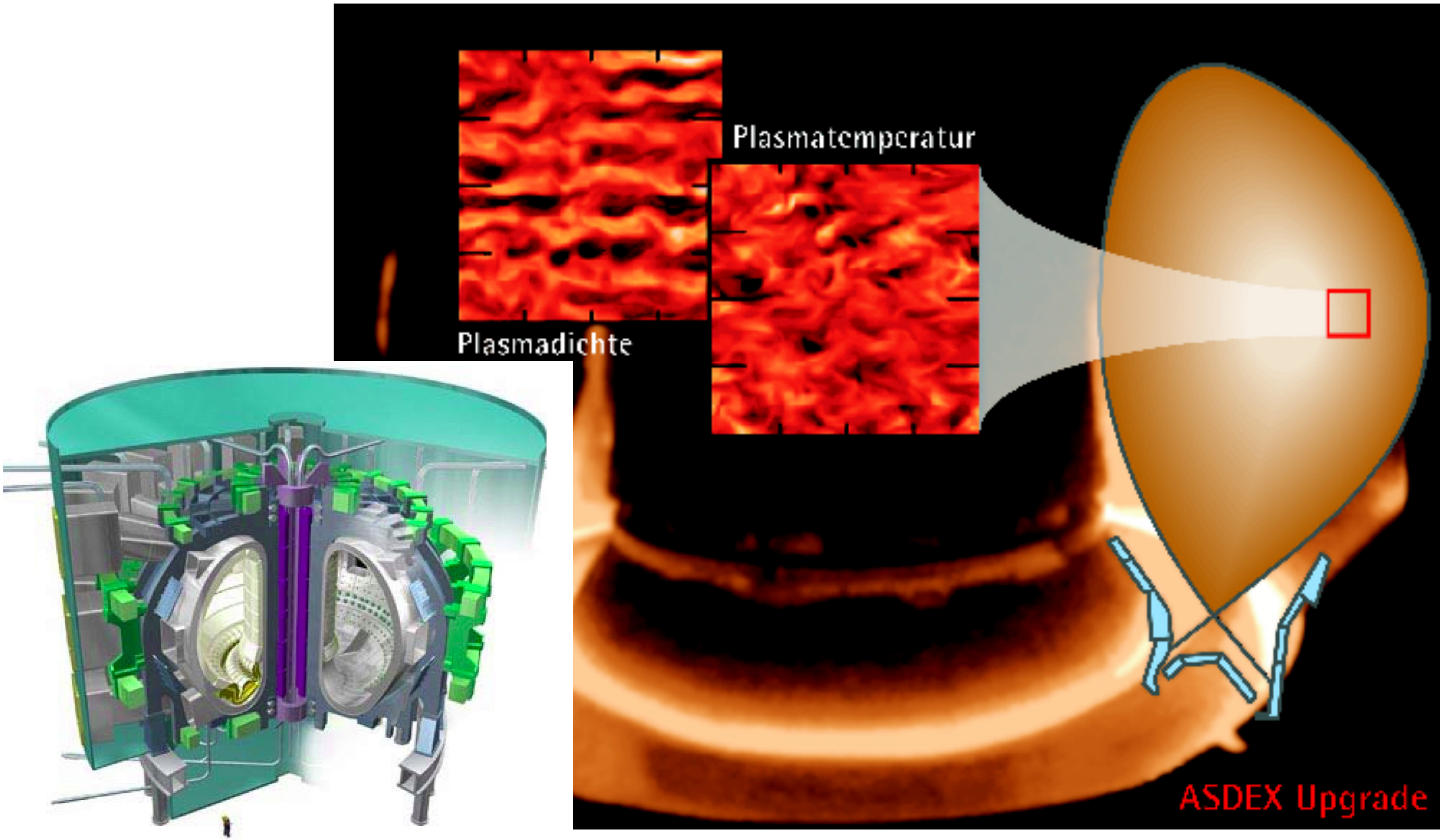
Cluster spacecraft measurements

Howes *et al*., PRL 2011



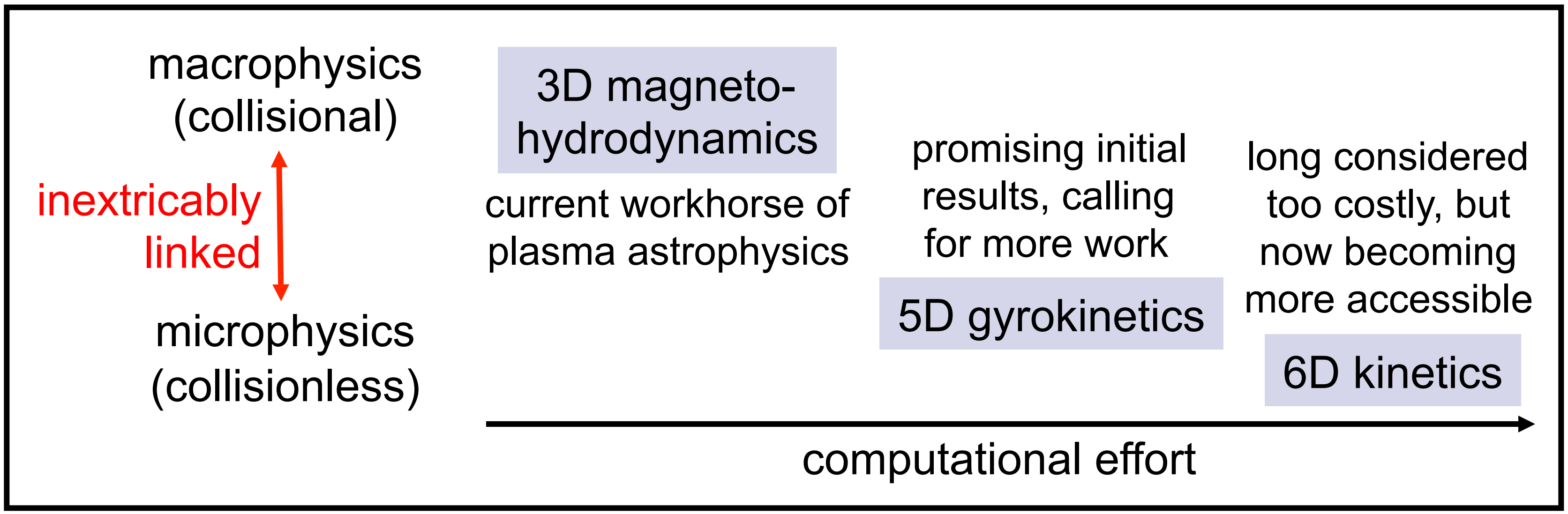
Gyrokinetic simulations below ρ_i

Kinetic turbulence in the laboratory



Many other toroidal and linear devices exist!

Promising advances in kinetic simulation

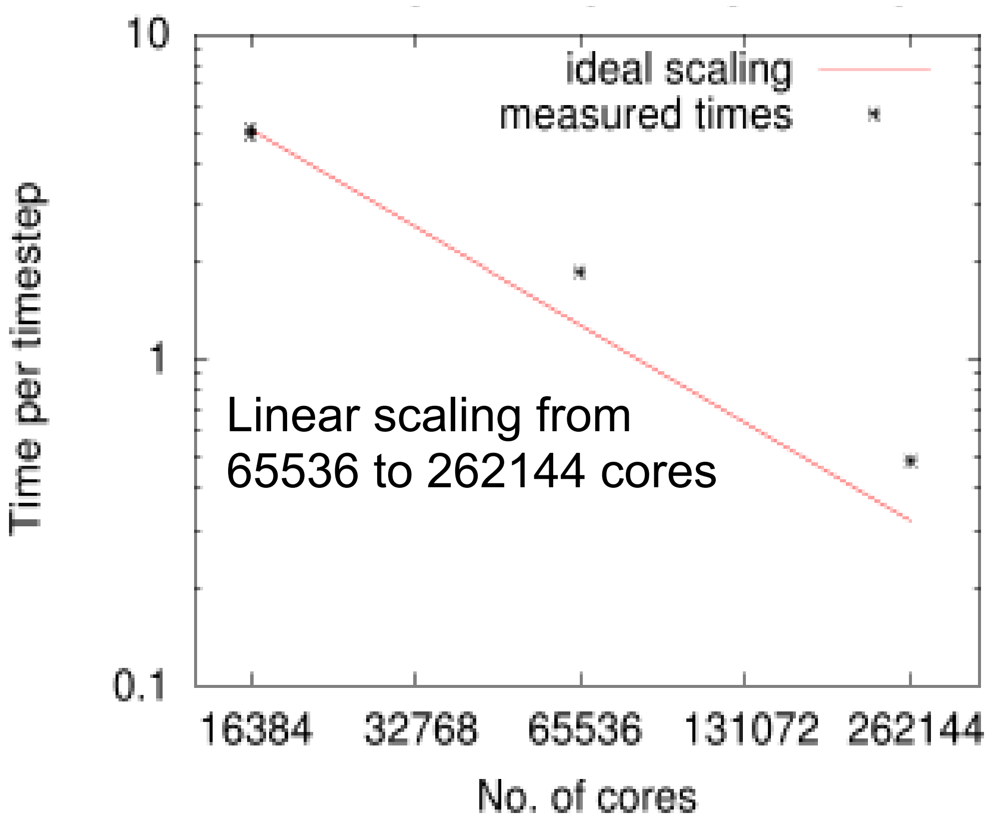


Thanks to the continuous advance in supercomputing power, „serious“ kinetic turbulence simulations have become feasible

GENE code: Parallel implementation

- code automatically adapts to chosen hardware & grid size (à la FFTW)
- efficient usage of petascale platforms

Strong scaling on BG/P



Close collaborations with experts in applied math and computer science

More info:
http://gene.rzg.mpg.de

Gyrokinetic theory in a nutshell

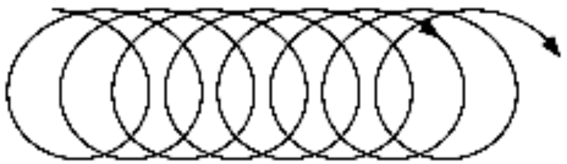
What is gyrokinetic theory?

Dilute and/or hot plasmas are almost collisionless.

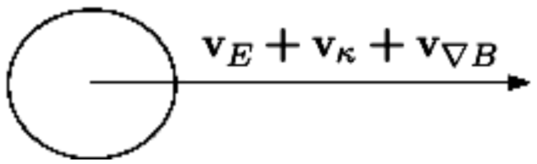
Thus, if kinetic effects (finite Larmor radius, Landau damping, magnetic trapping etc.) play a role, MHD is not applicable, and one has to use a kinetic description!

Vlasov-Maxwell equations $$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{x},\mathbf{v},t) = 0$$

Removing the fast gyromotion leads to a dramatic speed-up $\omega \ll \Omega$



Charged rings as quasiparticles; gyrocenter coordinates; keep kinetic effects



Details may be found in: Brizard & Hahm, Rev. Mod. Phys. **79**, 421 (2007)

The gyrokinetic ordering

- The gyrokinetic model is a Vlasov-Maxwell on which the GK ordering is imposed:

⇒ Slow time variation as compared to the gyro-motion time scale:

$$\omega/\Omega_i \sim \epsilon_g \ll 1$$

⇒ Spatial equilibrium scale much larger than the Larmor radius:

$$\rho/L_n \sim \rho/L_T \equiv \epsilon_g \ll 1$$

⇒ Strong anisotropy, i.e. only perpendicular gradients of the fluctuating quantities can be large ($k_\perp\rho \sim 1$, $k_\parallel\rho \sim \epsilon_g$):

$$k_\parallel/k_\perp \sim \epsilon_g \ll 1$$

⇒ Small amplitude perturbations, i.e. energy of perturbation much smaller than the thermal energy:

$$e\phi/T_e \sim \epsilon_g \ll 1$$

A Lagrangian approach

If the Lagrangian of a dynamical system is known...

Example: charged particle motion, in non canonical coordinates $(\vec{x}, \vec{v})$:

$$\begin{aligned} L &= \left(\frac{e}{c}\vec{A}(\vec{x},t) + m\vec{v}\right) \cdot \dot{\vec{x}} - H(\vec{x},\vec{v}) \\ H &= \frac{m}{2}v^2 + e\phi(\vec{x},t) \end{aligned}$$

with $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\phi - \partial_t \vec{A}/c$.

...the equation of motion are given by the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{with } i = 1, \ldots, 6$$

Lagrange equation of motion for a charged particle:

$$\begin{aligned} \vec{v} \Rightarrow \quad & -\tfrac{\partial L}{\partial \vec{v}} = 0 && \Rightarrow \dot{\vec{x}} = \vec{v} \\ \vec{x} \Rightarrow \quad & \tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{\partial L}{\partial \dot{\vec{x}}} - \tfrac{\partial L}{\partial \vec{x}} = 0 && \Rightarrow \dot{\vec{v}} = \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \end{aligned}$$

Deriving the driftkinetic Lagrangian

GK ordering, but $k_\perp \rho \simeq 1 \Rightarrow k_\perp \rho \ll 1$
$\Rightarrow \vec{B}(\vec{x})$, static magnetic field.

- Single particle Lagrangian:

$$L = \left(\frac{e}{c}\vec{A}(\vec{x}) + m\vec{v}\right) \cdot \dot{\vec{x}} - \frac{m}{2}v^2 + e\phi(\vec{x},t)$$

- Change of coordinates:

particle coordinates $(\vec{x},\vec{v}) \Rightarrow$ guiding center coordinates $(\vec{R}, v_\|, \mu, \varphi)$

$$\begin{aligned}
\vec{x} &= \vec{R} + \vec{\rho} \equiv \vec{R} + \frac{v_\perp}{\Omega}\hat{a}(\vec{R},\varphi) \\
\mu &= v_\perp^2/2B(\vec{R}) \\
v_\| &= \vec{v}\cdot\vec{b} \\
\varphi &= \tan^{-1}\left(\frac{\vec{v}\cdot\vec{e}_1}{\vec{v}\cdot\vec{e}_2}\right)
\end{aligned}$$

$\vec{R}$ guiding center position; $\Omega \equiv eB/mc$ gyrofrequency.
$\hat{a} \equiv \cos(\varphi)\ \vec{e}_1 + \sin(\varphi)\ \vec{e}_2$
$\vec{e}_1(\vec{R},\varphi)$, $\vec{e}_2(\vec{R},\varphi)$ orthogonal unity vectors in the plane perpendicular to $\vec{b} \equiv \vec{B}/B$.

Driftkinetic equations

$$L_{DK} = \left(mv_\parallel \vec{b} + \frac{e}{c}\vec{A}(\vec{R})\right) \cdot \dot{\vec{R}} + \frac{\mu B}{\Omega}\dot{\varphi} - H_{DK}$$
$$H_{DK} = \frac{m}{2}v_\parallel^2 + \mu B + q\phi(\vec{R})$$

- Lagrange equations:

$$\dot{\vec{R}} = v_\parallel \vec{b} + \frac{B}{B_\parallel^*}\left(\vec{v}_{E\times B} + \vec{v}_{\nabla B} + \vec{v}_C\right)$$
$$\dot{v}_\parallel = \left(-\mu\nabla B + e\vec{E}\right) \cdot \frac{\dot{\vec{R}}}{mv_\parallel} \quad ; \quad \dot{\mu} = 0 \quad ; \quad \dot{\varphi} = \Omega$$

$\vec{v}_{E\times B} \equiv \frac{c}{B^2}\,\vec{E}\times\vec{B}$		$E \times B$ drift
$\vec{v}_{\nabla B} \equiv \frac{\mu}{m\Omega}\,\vec{b}\times\nabla B$		∇B drift
$\vec{v}_C \equiv \frac{v_\parallel^2}{\Omega}\,\vec{b}\times(\vec{b}\cdot\nabla)\vec{b}$		Curvature drift

with $\vec{B}^* \equiv \vec{B} + (mc/e)v_\parallel \nabla\times\vec{b} = B(1+\mathcal{O}(\rho_\parallel/L_B))$.

Introducing fluctuations

Eliminate explicit gyrophase dependence via near-identity (Lie) transforms to gyrocenter coordinates; the resulting Lagrangian 1-form reads:

$$\Gamma \;=\; \left(m v_{||}\mathbf{b}_0 + \frac{e}{c}\,\bar{A}_{1||}\,\mathbf{b}_0 + \frac{e}{c}\,\mathbf{A}_0\right)\cdot d\mathbf{X} + \frac{mc}{e}\,\mu\, d\theta \,- \\ - \left(\frac{m}{2}v_{||}^2 + \mu B_0 + \mu \bar{B}_{1||} + e\,\bar{\phi}_1\right)\,dt$$

$$\bar{\phi}_1 \equiv I_0(\lambda)\,\phi_1\,, \quad \bar{A}_{1||} \equiv I_0(\lambda)\,A_{1||}\,, \quad \bar{B}_{1||} \equiv I_1(\lambda)\,B_{1||}$$

Euler-Lagrange equations

$$\dot{\mathbf{X}} = v_{||}\,\mathbf{b} + \frac{B}{B_{||}^*}\left(\frac{v_{||}}{B}\bar{\mathbf{B}}_{1\perp} + \frac{c}{B^2}\bar{\mathbf{E}}_1 \times \mathbf{B} + \frac{\mu}{m\Omega}\mathbf{b} \times \nabla(B + \bar{B}_{1||}) + \frac{v_{||}^2}{\Omega}(\nabla \times \mathbf{b})_\perp\right)$$

$$\dot{v}_{||} = \frac{\dot{\mathbf{X}}}{m v_{||}} \cdot \left(e\bar{\mathbf{E}}_1 - \mu\nabla(B + \bar{B}_{1||})\right) \qquad \dot{\mu} = 0$$

$$f = f(\mathbf{X}, v_{||}, \mu; t)$$

$$\frac{\partial f}{\partial t} + \dot{\mathbf{X}} \cdot \frac{\partial f}{\partial \mathbf{X}} + \dot{v}_{||}\frac{\partial f}{\partial v_{||}} = 0$$

Gyroaveraged potentials

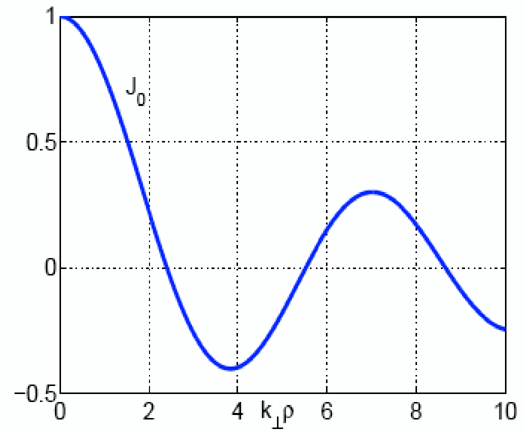
Drift-kinetic : $\vec{E}(\vec{R},t)$.

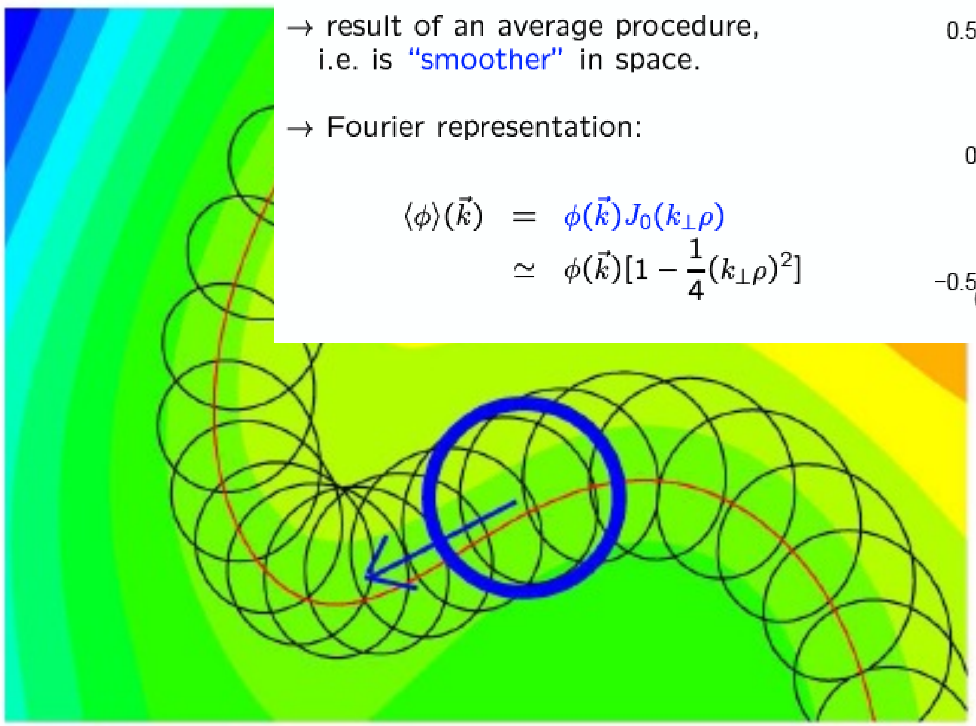
Gyro-kinetic : $\langle\vec{E}\rangle$

→ result of an average procedure, i.e. is "smoother" in space.

→ Fourier representation:

$$\langle\phi\rangle(\vec{k}) = \phi(\vec{k})J_0(k_\perp\rho) \simeq \phi(\vec{k})[1-\frac{1}{4}(k_\perp\rho)^2]$$





- Full Lorentz dynamics
- Gyrokinetic approx.: $\phi^{\text{eff}}(\vec{x},\rho) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \Phi(\vec{x}+\vec{\rho})$ $= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\vec{k}\, e^{i\vec{k}\vec{x}}\phi(\vec{k})J_0(|\vec{k}|\rho)$

Appropriate field equations

Reformulate Maxwell's equations in gyrocenter coordinates:

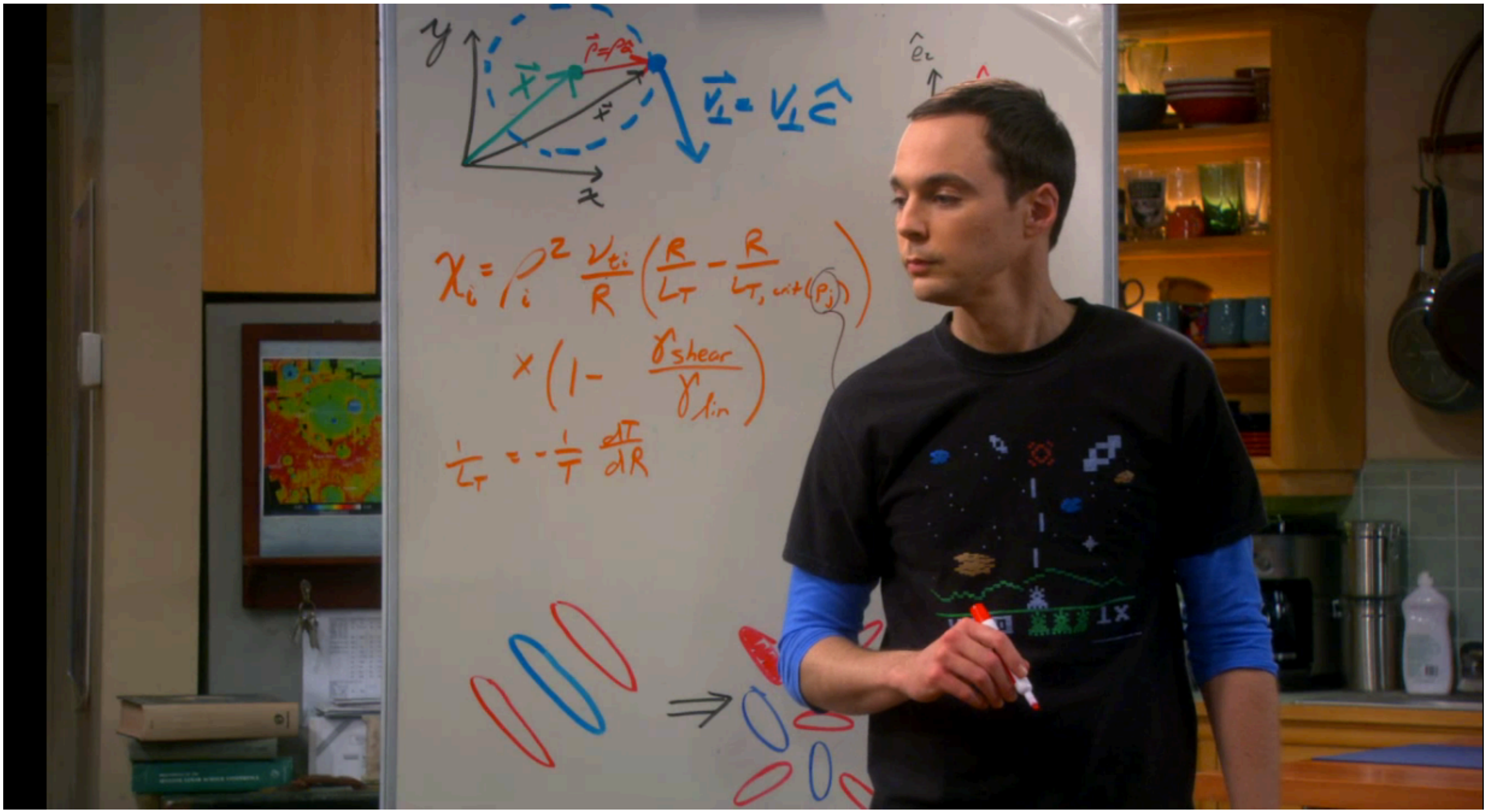
$$\nabla_\perp^2 \phi_1 = -4\pi \sum e n_1 \,, \quad \frac{n_1}{n_0} = \frac{\bar{n}_1}{n_0} - \left(1 - \|I_0^2\|\right) \frac{e\phi_1}{T} + \|x I_0 I_1\| \frac{B_{1\|}}{B} \,,$$

$$\nabla_\perp^2 A_{1\|} = -\frac{4\pi}{c} \sum \bar{J}_{1\|} \,,$$

$$\frac{B_{1\|}}{B} = -\sum \epsilon_\beta \left(\frac{\bar{p}_{1\perp}}{n_0 T} + \|x I_1 I_0\| \frac{e\phi_1}{T} + \|x^2 I_1^2\| \frac{B_{1\|}}{B} \right) \,,$$

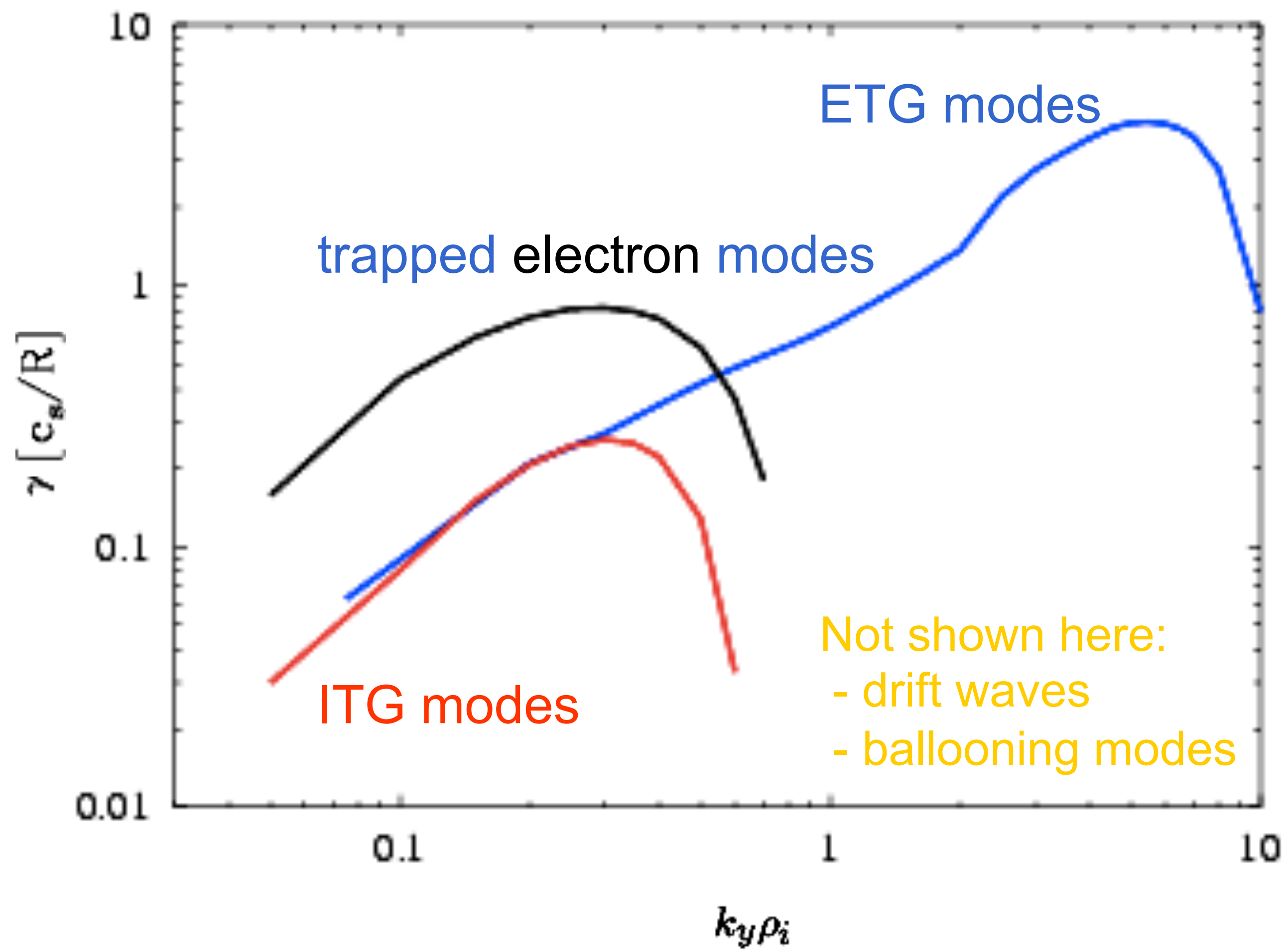
Nonlinear partial integro-differential equations in 5D...

Sheldon also works on gyrokinetics…



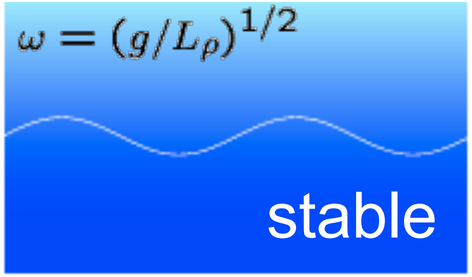
Gradient-driven linear microinstabilities

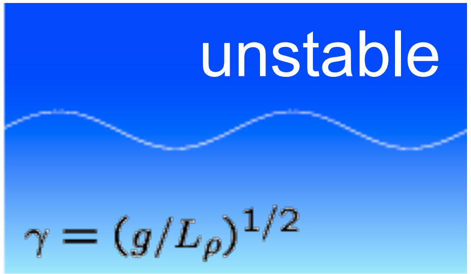
Some important microinstabilities



Gradient-driven microinstabilities

Perpendicular dynamics: de-/stabilization in out-/inboard regions



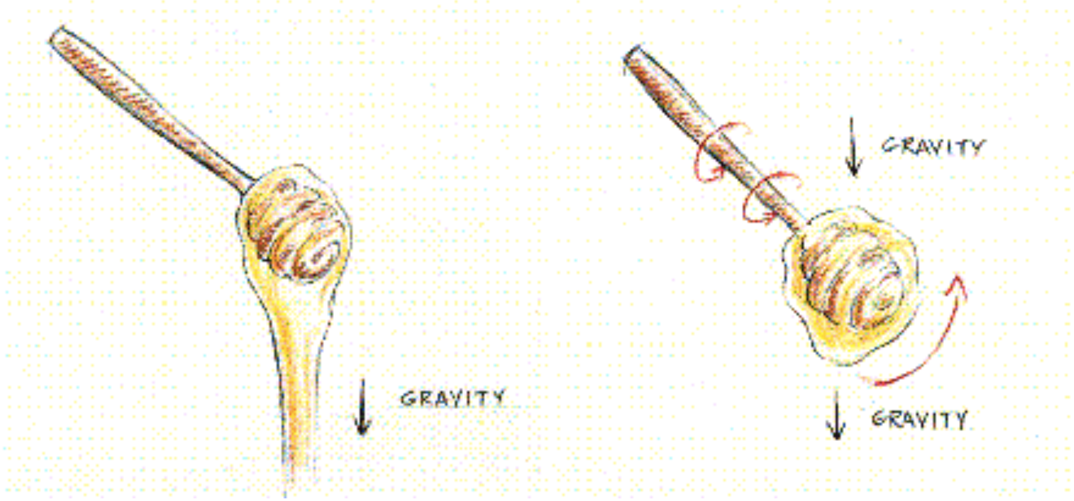


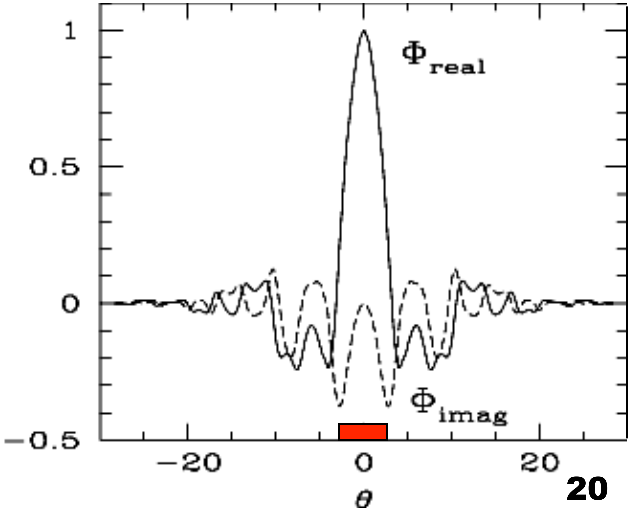
Rayleigh-Taylor instability

Analogy in a plasma:

$g_{\text{eff}} = v_t^2/R$

Parallel dynamics: localization in outboard regions

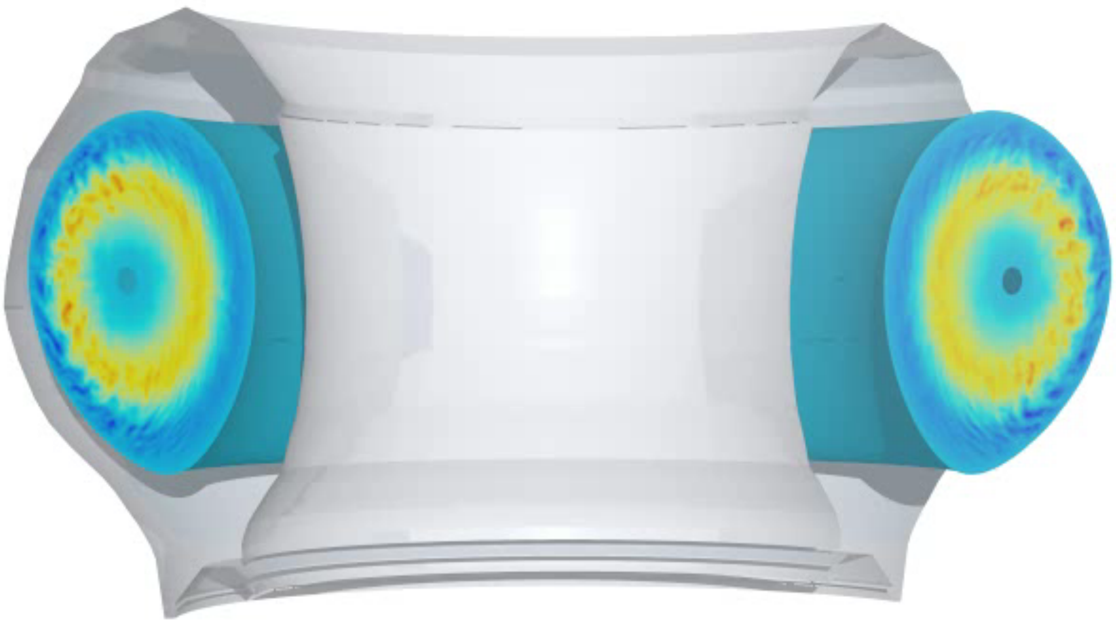




Nonlinear saturation via zonal flows

F. Jenko, Physics Letters A **351**, 417 (2006)

Self-organization: Zonal flow generation



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Charney-Hasegawa-Mima equation

Hasegawa & Mima, PRL 1977

In a certain limiting case, gyrokinetics leads to the CHM equation which is closely related to the 2D Navier-Stokes equation; also used in geophysics.

Standard CHME (ETG)

$$\frac{d}{dt}(\phi - \nabla^2 \phi - x) = 0$$

Modified CHME (ITG)

$$\frac{d}{dt}(\phi - \langle \phi \rangle - \nabla^2 \phi - x) = 0$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \frac{\partial \phi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial y}$$

4-mode analysis

ITG-type CHME in Fourier space

$$(1+k^2)\dot{\Phi}_{\mathbf{k}} + ik_y\Phi_{\mathbf{k}}$$

$$= \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} [\hat{\mathbf{z}}\cdot(\mathbf{k}_1 \times \mathbf{k}_2)(1+k_2^2)]\Phi_{\mathbf{k}_1}\Phi_{\mathbf{k}_2}$$

Reduction to just 4 modes (and their CC's)

streamer $(k_x, k_y) = (0, q)$

zonal flow $(k_x, k_y) = (p, 0)$

sidebands $(k_x, k_y) = (p, q)$

$(k_x, k_y) = (p, -q)$

Resulting amplitude equations

$$\dot{\phi}_q + i\Omega_q \phi_q = 0\,,$$

$$\dot{\phi}_0 = -\,qp(\phi_q \phi_- - \phi_q^* \phi_+)\,,$$

$$\dot{\phi}_+ + i\Omega_+ \phi_+ = \frac{qp(1+q^2-p^2)}{1+q^2+p^2}\phi_q \phi_0\,,$$

$$\dot{\phi}_- + i\Omega_- \phi_- = -\,\frac{qp(1+q^2-p^2)}{1+q^2+p^2}\phi_q^* \phi_0\,.$$

$$\Omega_+ = -\,\Omega_- = \frac{q}{1+q^2+p^2}\,, \qquad \Omega_q = \frac{q}{1+q^2}\,.$$

Zonal flow growth rate

If the streamer amplitude exceeds a certain threshold, the zonal flow becomes unstable.

Its growth rate is given by:

$$\gamma_0 = \sqrt{\frac{2q^2p^2(1+q^2-p^2)}{(1+q^2+p^2)}|\phi_q|^2 - \Delta\Omega^2}$$ ITG case

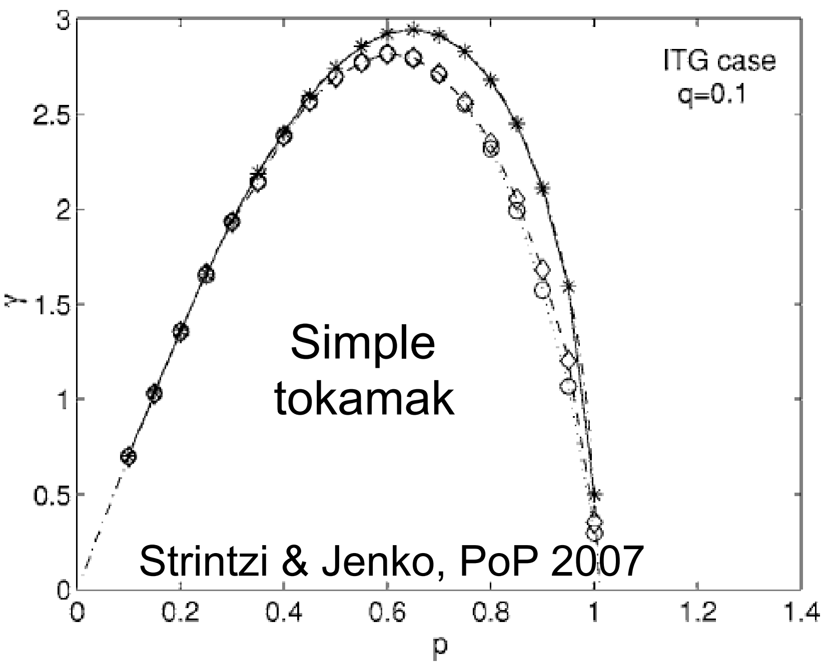
$$\gamma_0 = \sqrt{\frac{2q^2p^4(q^2-p^2)}{(1+p^2)(1+q^2+p^2)}|\phi_q|^2 - \Delta\Omega^2}$$ ETG case

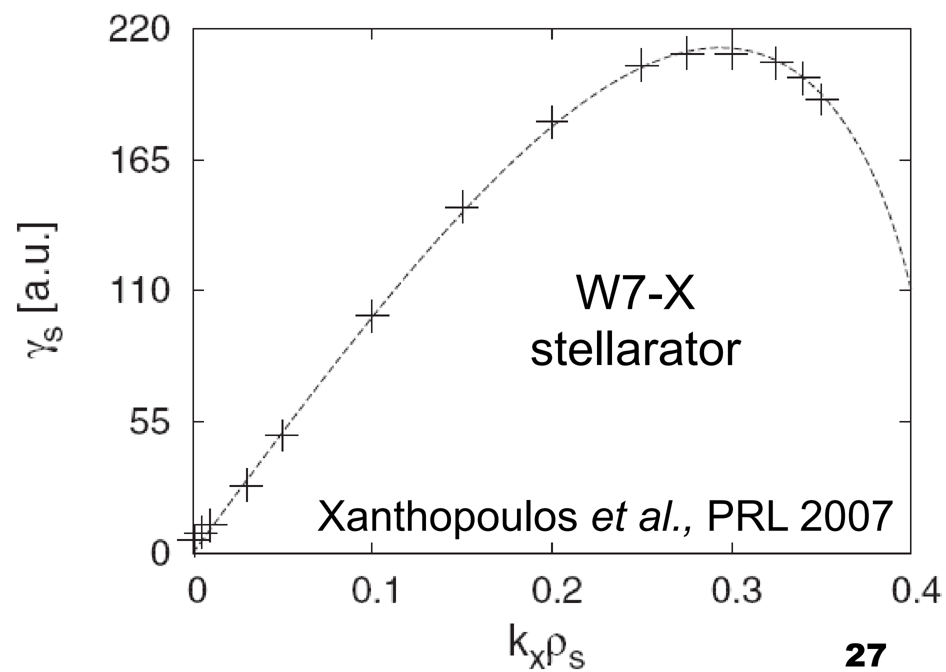
Strintzi & Jenko, PoP 2007

Secondary instabilities & ZF generation

- Large-amplitude streamers are Kelvin-Helmholtz unstable

  [Cowley at al. 1991; Dorland & Jenko PRL 2000]
- This secondary instability contains a zonal-flow component
- Near-equivalence to 4-mode and wave-kinetic approaches

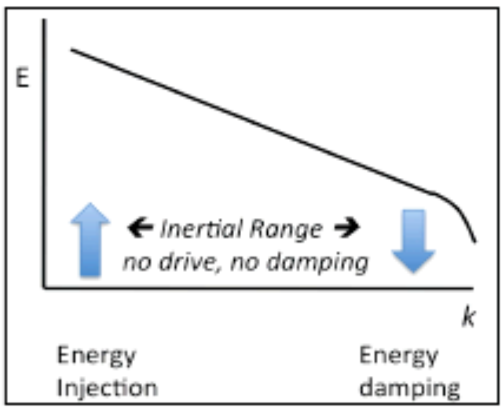


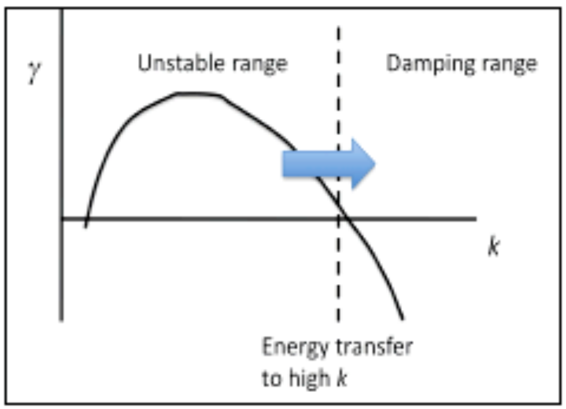


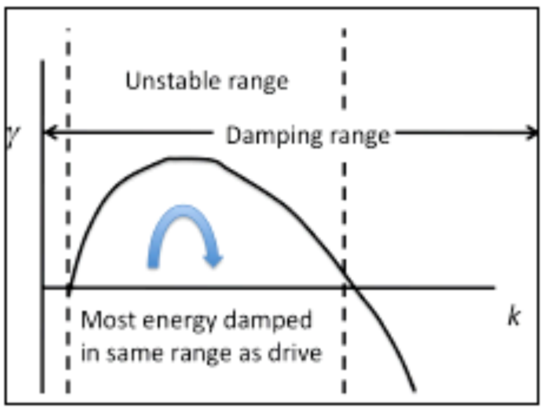
Kinetic turbulence in laboratory and astrophysical systems

Turbulence in fluids and plasmas – Three basic scenarios

1.Hydrodynamic cascade	2. Conventional μ-turbulence	3. Saturation by damped eigenmode







Inertial range
→ no dissipation
→scale invariant dynamics
→power law spectrum

Energy transfer to high k like hydro – no inertial range adjacent unstable, damping ranges

Energy can go to high k but most of it is lost at low k in driving range

Saturation via damped eigenmodes

Plasma dispersion relation has multiple roots

- One root unstable → drives turbulence (TEM, ITG, ETG…)
- Other roots can be damped for all *k*
- Fluid models: one root per equation
- Gyrokinetics: infinite in principle;

  discretization yields large but finite number

3-wave interactions drive damped eigenmodes

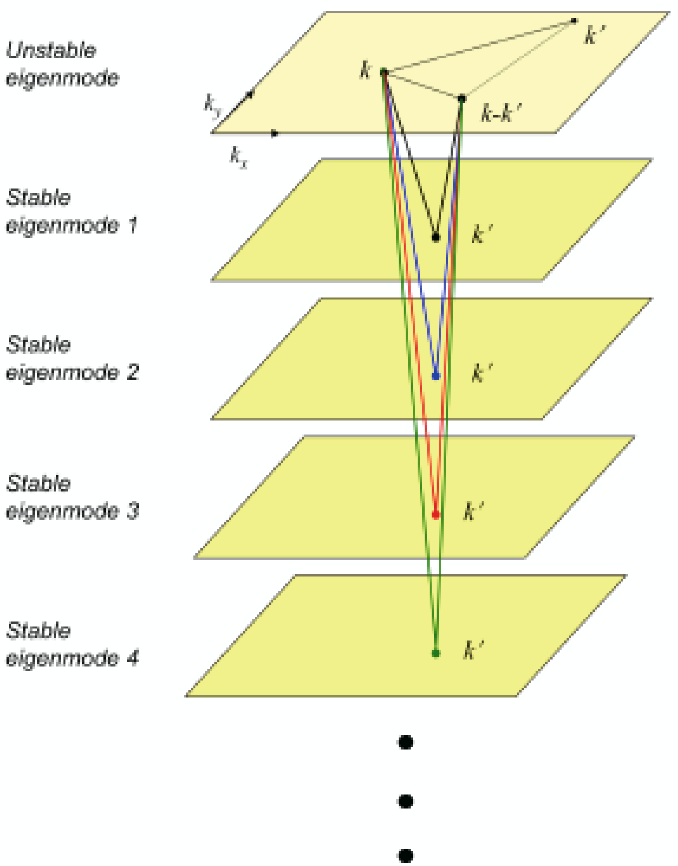
- Pumped by unstable mode through parametric instability

  Only condition: $\mathrm{Amp}_{damp} << \mathrm{Amp}_{ustable}$ initially
- Each eigenmode driven by combo of all nonlinearities

  => Large *multiplicity* of coupling channels

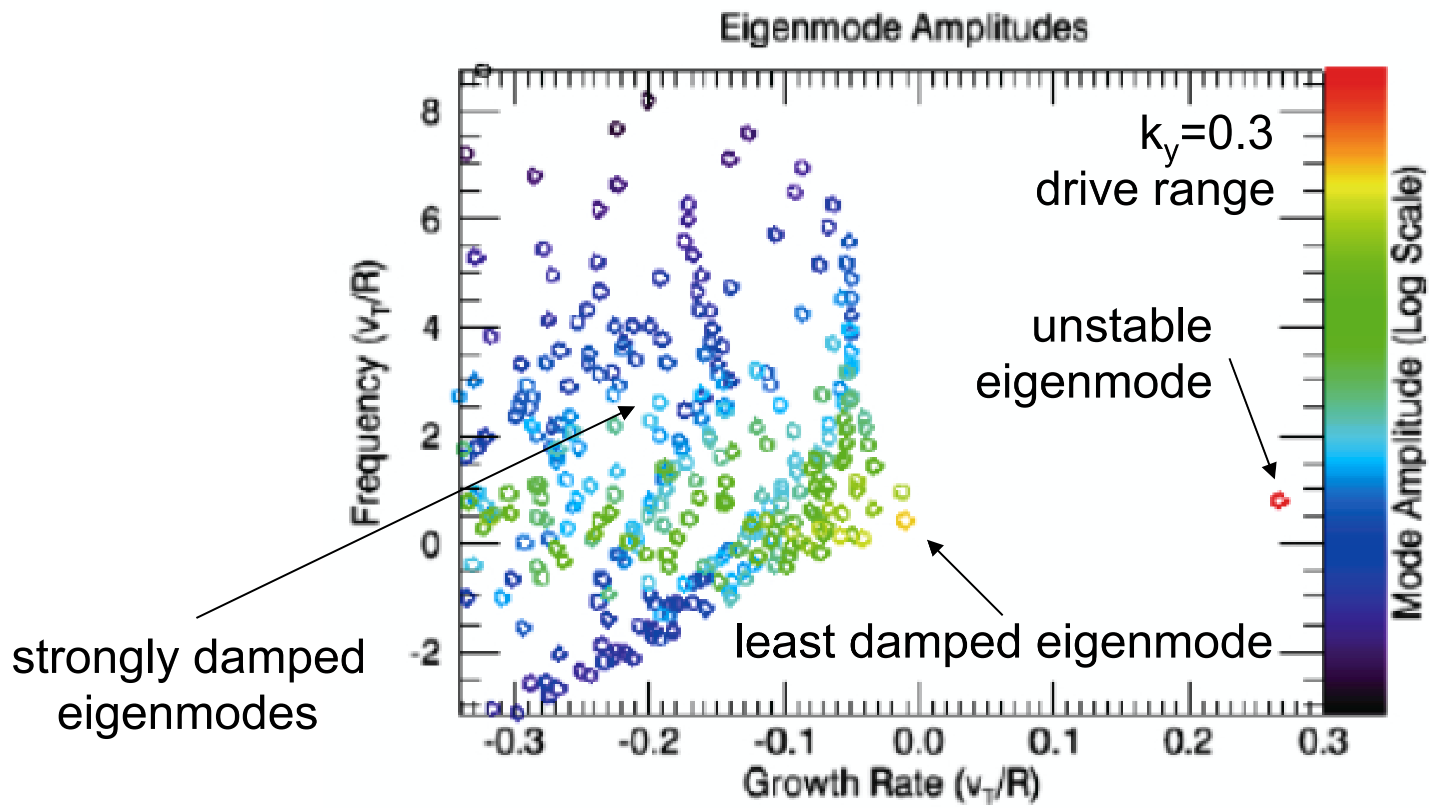
  => Many eigenmodes are excited

Consistent phenomenology across many models

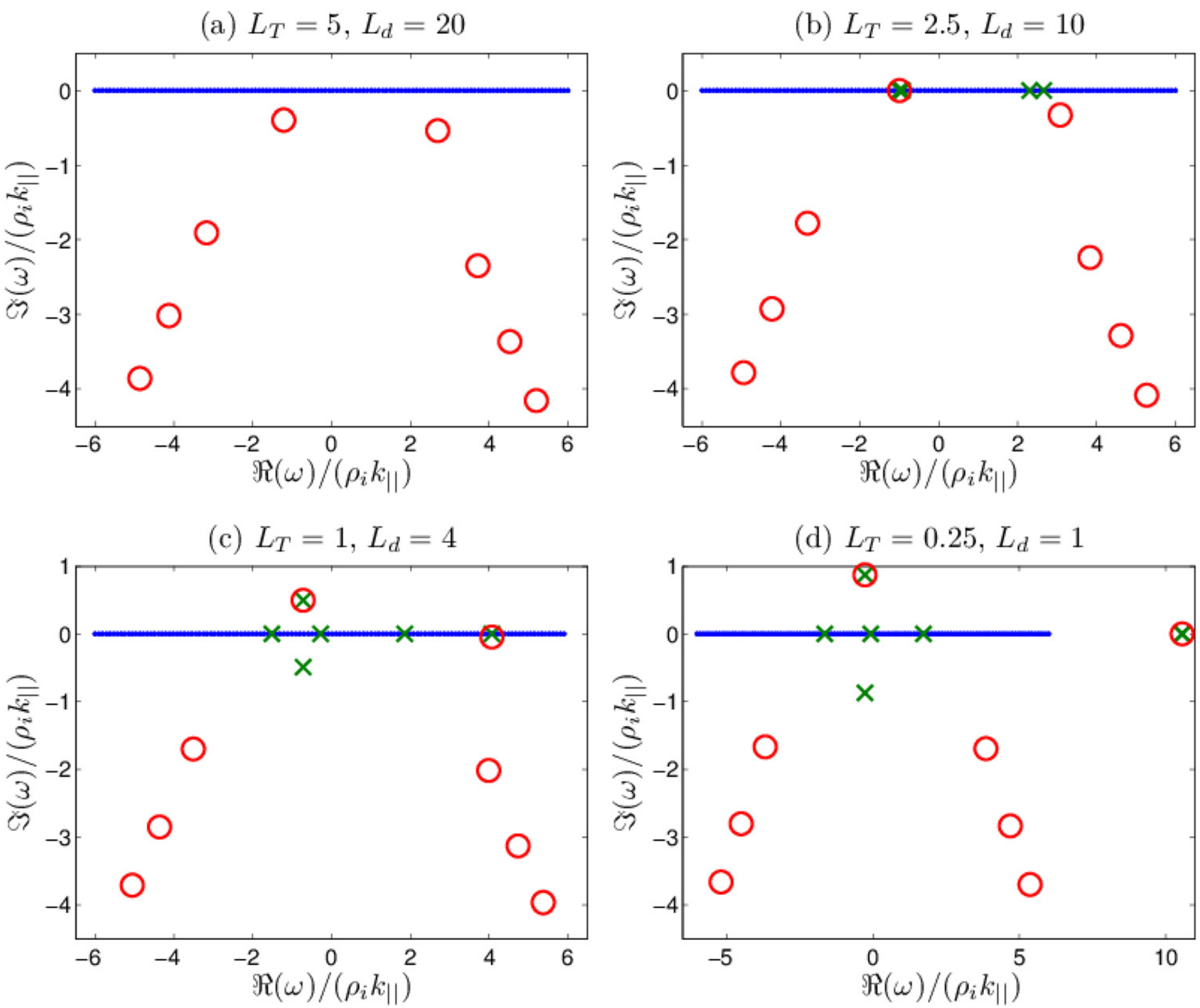


Excitation of damped eigenmodes

Using GENE as a linear eigenvalue solver to analyze nonlinear ITG runs via projection methods, one finds…



Eigenvalue spectra for (slab) ITG modes



Case-Van Kampen
vs
Landau

FIG. 1. Evolution of the discrete collisionless Case-Van Kampen eigenmodes (green crosses) and the collisionless Landau solutions (red circles) with respect to the temperature and density gradient lengths. For large gradients the ion acoustic mode emerges which is almost marginally stable and practically coincides with one of the discrete Case-Van Kampen eigenmodes. The unstable discrete Case-Van Kampen modes match the unstable Landau solutions and the blue asterisks represent the continuum part of the Van Kampen spectrum. $k_y = 0.3$, $k_{||} = 0.03$.

Bratanov, Jenko, Hatch, and Brunner, Phys. Plasmas 20, 022108 (2013)

Energetics

Turbulent free energy consists of two parts:

$$\mathcal{E}_f = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} \frac{f_j^2}{2}, \qquad \mathcal{E}_\phi = \sum_j \int d\Lambda q_j \frac{\bar{\phi}_1 f_j}{2}.$$

Drive and damping terms:

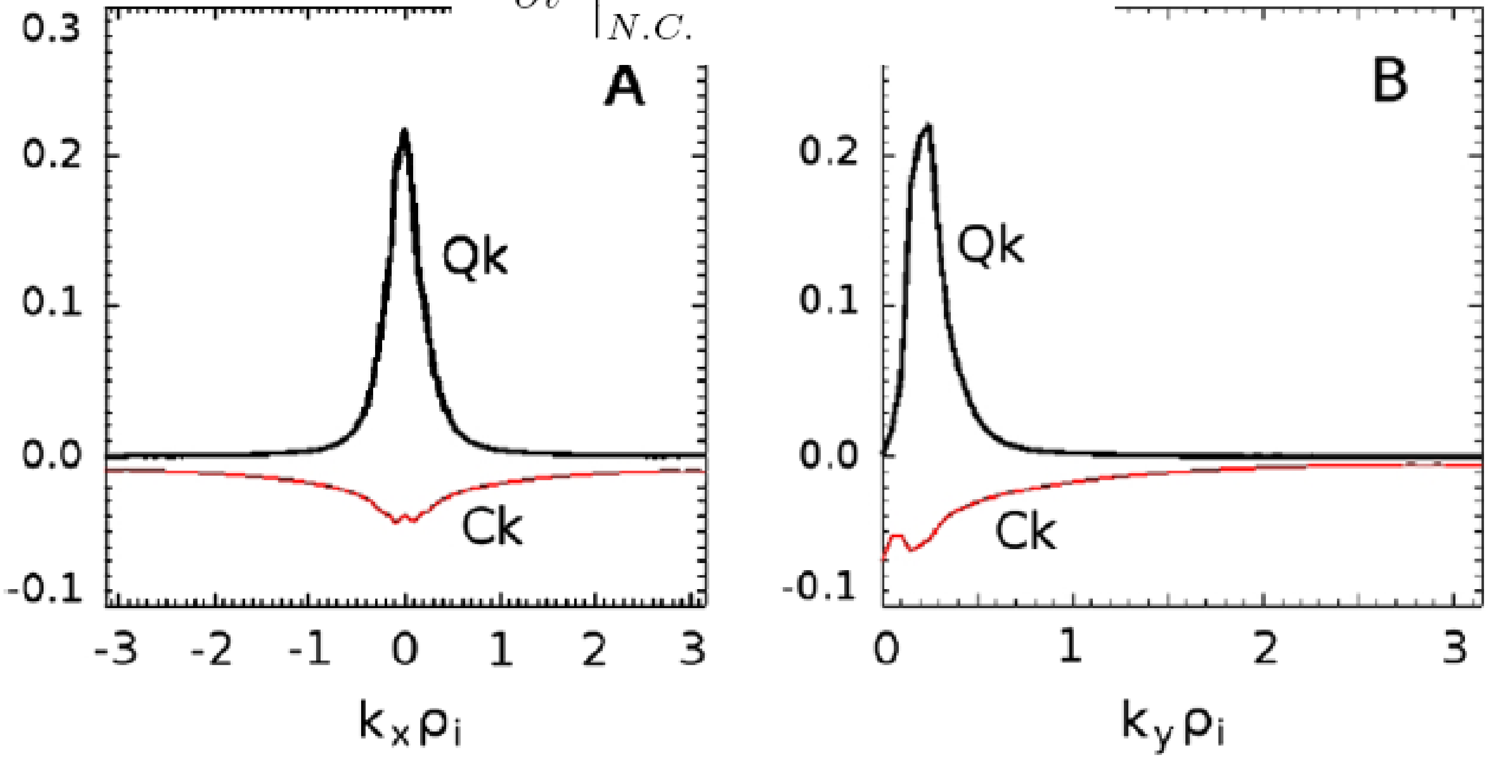
$$\frac{\partial \mathcal{E}}{\partial t} = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} h_j \frac{\partial f_j}{\partial t} = \mathcal{G} - \mathcal{D} \qquad h_j = f_j + (q_j \bar{\phi}_1 / T_{0j}) F_{0j}$$

$$\mathcal{G} = -\sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} h_j \cdot \left[\omega_n + \left(v_\parallel^2 + \mu B_0 - \frac{3}{2}\right)\omega_{Tj}\right] \times F_{0j} \frac{\partial \bar{\phi}_1}{\partial y}$$

$$\mathcal{D} = -\sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} h_j (\mathcal{D}_z f_j + \mathcal{D}_{v_\parallel} f_j).$$

Energetics in wavenumber space

$$\left.\frac{\partial E_k}{\partial t}\right|_{N.C.} = Q_k + C_k + N_k$$



Damped eigenmodes are responsible for significant dissipation in the drive range (!)

Hatch, Terry, Jenko, Merz & Nevins, PRL 2011

Resulting spectrum decays exponentially @lo k, asymptotes to power law @hi k

Spectrum from k space attenuation of $T(k)$ by dissipation $\alpha E(k)$:

$$\frac{dT(k)}{dk} = \frac{d(v_k^3 k)}{dk} = aE(k)$$

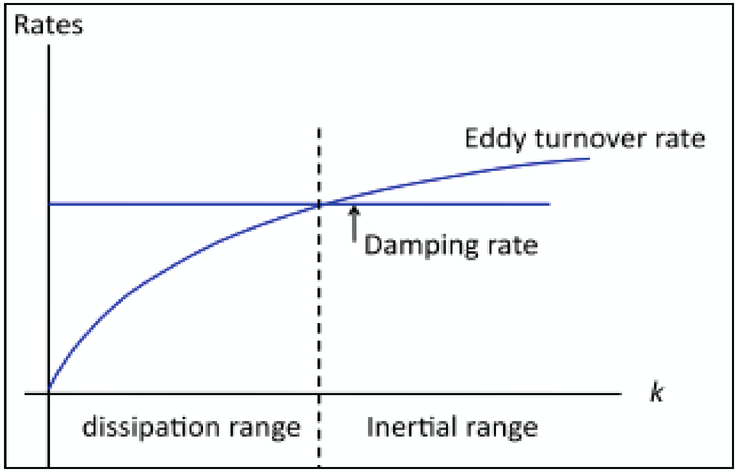
nonlinear energy transfer rate

Corrsin closure procedure: $v_k^3 k = v_k^2 \cdot v_k k = E(k)k \cdot \varepsilon^{1/3} k^{-1/3} k$

Solving attenuation ODE:

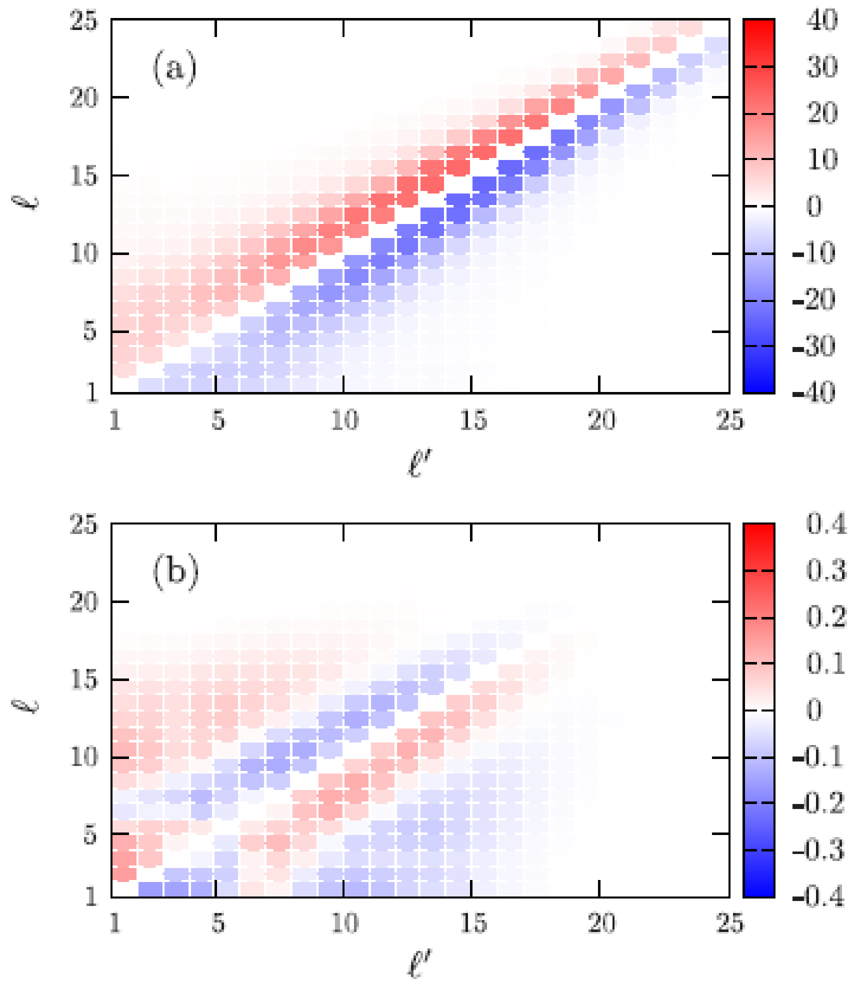
$$E(k) = \beta \varepsilon^{2/3} k^{-5/3} \exp\left[\frac{3}{2}\alpha \varepsilon^{-1/3} k^{-2/3}\right]$$

Spectrum becomes power law in range where eddy turnover rate exceeds constant dissipation rate



Hatch *et al.*, PRL 2011
Terry *et al.*, PoP 2012

Shell-to-shell transfer of free energy



$$\mathcal{E}_f = \sum_j \int d\Lambda \frac{T_{0j}}{F_{0j}} \frac{f_j^2}{2},$$

ITG turbulence (adiabatic electrons); logarithmically spaced shells

Entropy contribution dominates; exhibits very local, forward cascade

$$\mathcal{E}_\phi = \sum_j \int d\Lambda q_j \frac{\bar{\phi}_1 f_j}{2}.$$

Bañón Navarro *et al.*, PRL 2011

Large Eddy Simulation techniques in kinetic turbulence

Gyrokinetic LES models

Apply LES filter: $\partial_t f_{ki} = L[f_{ki}] + N[\phi_k, f_{ki}] - D[f_{ki}]$

$$\partial_t \overline{f_k} = L[\overline{f_k}] + N[\overline{\phi_k}, \overline{f_k}] + T_{\overline{\Delta},\Delta^{\mathrm{DNS}}} - D[\overline{f_k}]$$

Sub-grid term: $T_{\overline{\Delta},\Delta^{\mathrm{DNS}}} = \overline{N}[\phi_k, f_k] - N[\overline{\phi_k}, \overline{f_k}] \approx c_\perp k_\perp^4 h_{ki}$

Free energy spectra vs $c_\perp$:

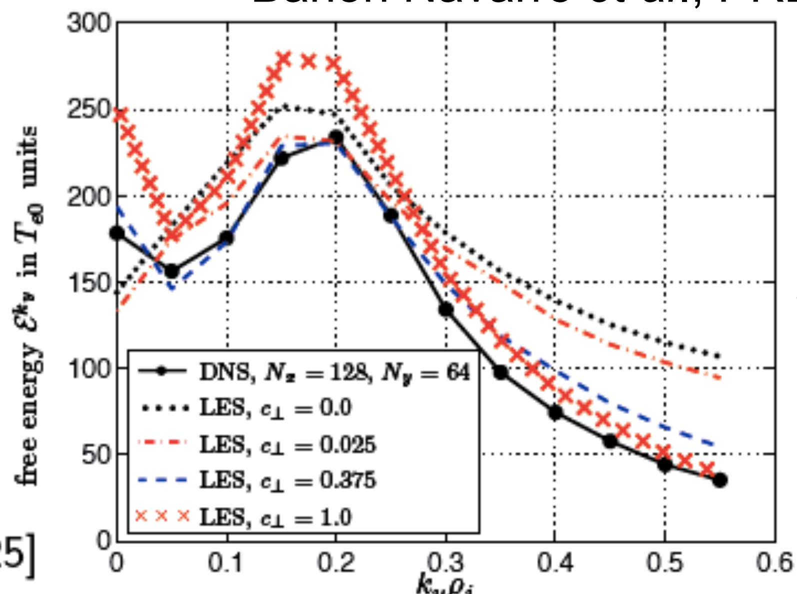
Cyclone Base Case (ITG)

- ⋆ $c_\perp$ too small
  - ⇒ not enough dissipation
- ⋆ $c_\perp$ too strong
  - ⇒ overestimates injection
- ⋆ $c_\perp = 0.375$ good agreement

→ "plateau" for $c_\perp \in [0.25, 0.625]$

→ holds for k_x

Bañón Navarro *et al.*, PRL 2011



Morel *et al.*, PoP 2011

Substantial savings in computational cost: Here, a factor of 20

Self-adjustment of model parameters

Test filter in DNS domain: $\partial_t \widehat{f}_k = L[\widehat{f}_k] + N[\widehat{\phi_k}, \widehat{f}_k] - D[\widehat{f}_k] + T_{\widehat{\Delta},\Delta^{\mathrm{DNS}}}$

Test filter in LES domain: $\partial_t \widehat{f}_k = L[\widehat{f}_k] + \widehat{N}[\overline{\phi_k}, \overline{f_k}] - D[\widehat{f}_k] + \widehat{T}_{\overline{\Delta},\Delta^{\mathrm{DNS}}}$

$\widehat{\overline{\cdots}} = \widehat{\cdots}$ ...for the Fourier cut-off filters used here

One thus obtains the (Germano) identity:

$$T_{\widehat{\Delta},\Delta^{\mathrm{DNS}}} = \widehat{T}_{\overline{\Delta},\Delta^{\mathrm{DNS}}} + \widehat{N}[\overline{\phi_k}, \overline{f_k}] - N[\widehat{\phi_k}, \widehat{f}_k] = \widehat{T}_{\overline{\Delta},\Delta^{\mathrm{DNS}}} + T_{\widehat{\Delta},\overline{\Delta}}$$

Approximate sub-grid terms and minimize error:

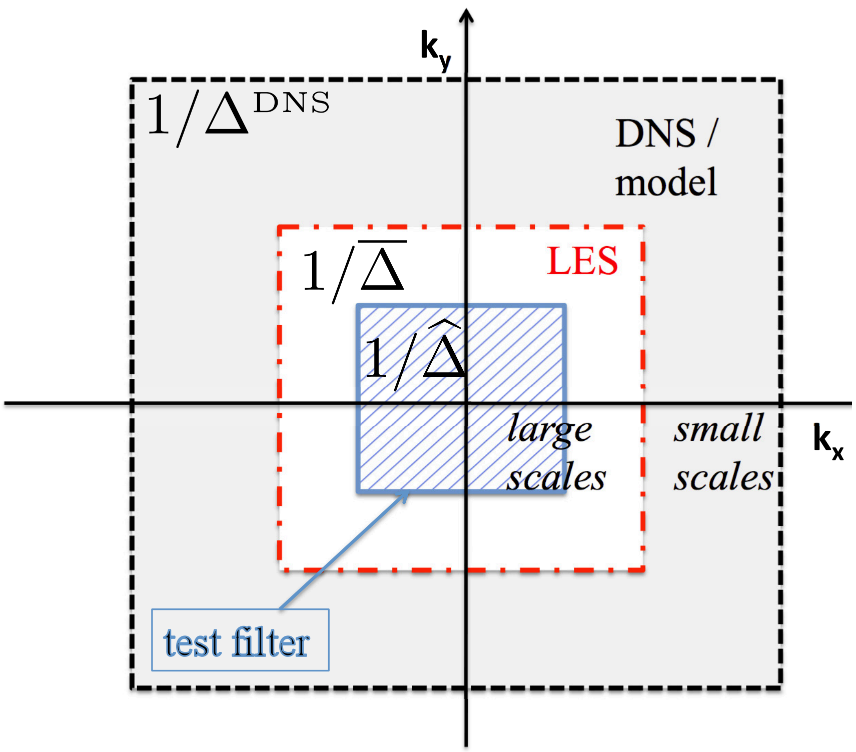
$$T_{\widehat{\Delta},\Delta^{\mathrm{DNS}}} \approx M_{\widehat{\Delta}} \quad ; \quad T_{\overline{\Delta},\Delta^{\mathrm{DNS}}} \approx M_{\overline{\Delta}} \qquad M_{\widehat{\Delta}} \approx \widehat{M}_{\overline{\Delta}} + T_{\widehat{\Delta},\overline{\Delta}}$$

$$d^2 = \left\langle \left(T_{\widehat{\Delta},\overline{\Delta}} + \widehat{M}_{\overline{\Delta}} - M_{\widehat{\Delta}} \right)^2 \right\rangle_\Lambda$$

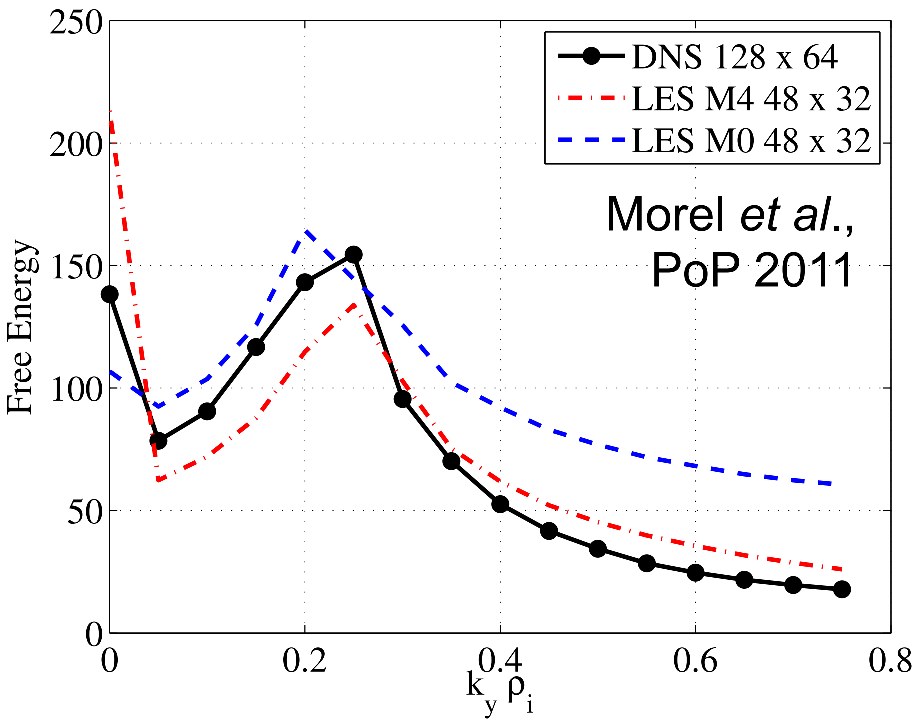
...this procedure yields explicit expressions for the model parameter(s)

The “dynamic procedure” in practice

Schematic of „dynamic procedure“



Free energy spectra (w/ and w/o model)



LES techniques are likely to reduce the simulation effort substantially without introducing many free parameters. This offers interesting perspectives…

Final remarks: The bigger picture

Multiscale drive/damping: Universality?

Kuramoto-Sivashinsky equation (linked to complex Ginzburg-Landau equation)

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t)\frac{\partial u(x,t)}{\partial x} - \mu\frac{\partial^2 u(x,t)}{\partial x^2} - \nu\frac{\partial^4 u(x,t)}{\partial x^4}$$

$$\frac{\partial \widehat{u}(k_n,t)}{\partial t} = -\frac{1}{2} i k_n \sum_{m\in\mathbb{Z}} \widehat{u}(k_n - k_m, t)\widehat{u}(k_m,t) + (\mu k_n^2 - \nu k_n^4)\widehat{u}(k_n,t)$$

Modification: Constant damping rate at high wavenumbers

$$\mu k_n^2 - \nu k_n^4 \to (\mu k_n^2 - \nu k_n^4)/(a + b k_n^4)$$

Energy balance (in quasi-stationary turbulent state)

$$k_n \sum_{m\in\mathbb{Z}} \Im\left(\langle \overline{\widehat{u}(k_n,t)}\widehat{u}(k_n - k_m,t)\widehat{u}(k_m,t)\rangle_\tau\right) + 2\frac{\mu k_n^2 - \nu k_n^4}{1 + b k_n^4} E(k_n) = 0$$

Multiscale drive/damping: Universality?

An analytical closure yields...

$$-2\frac{a_1 a_2}{\Delta k}\sqrt{2\pi a_3}\,k\frac{dE}{dk} + 2\frac{\mu k^2 - \nu k^4}{1 + bk^4}E(k) = 0$$

The exact solution of this equation reads...

$$E(k) = \widetilde{E}_0 \exp\left(\frac{\lambda\mu}{\sqrt{b}}\arctan(\sqrt{b}k^2) - \frac{\lambda\nu}{2b}\ln(1 + bk^4)\right)$$

$$\lambda = \Delta k/(2a_1 a_2\sqrt{2\pi a_3})$$

High-k limit: $E(k) = E_0 k^{-2\lambda\nu/b}$

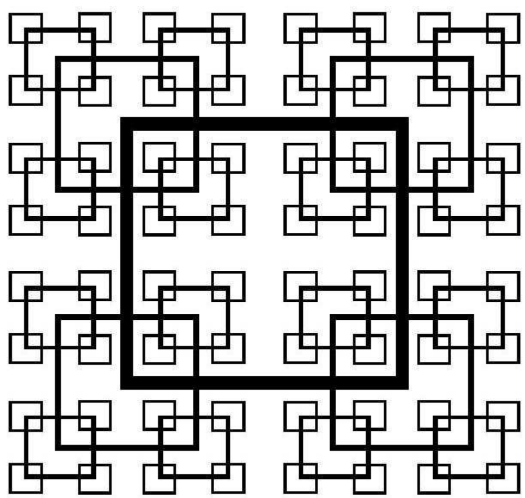
Spectral exponent is proportional to high-k damping rate!

Bratanov et al., to be submitted

Beyond Richardson and Kolmogorov: Multi-scale driven/damped turbulence

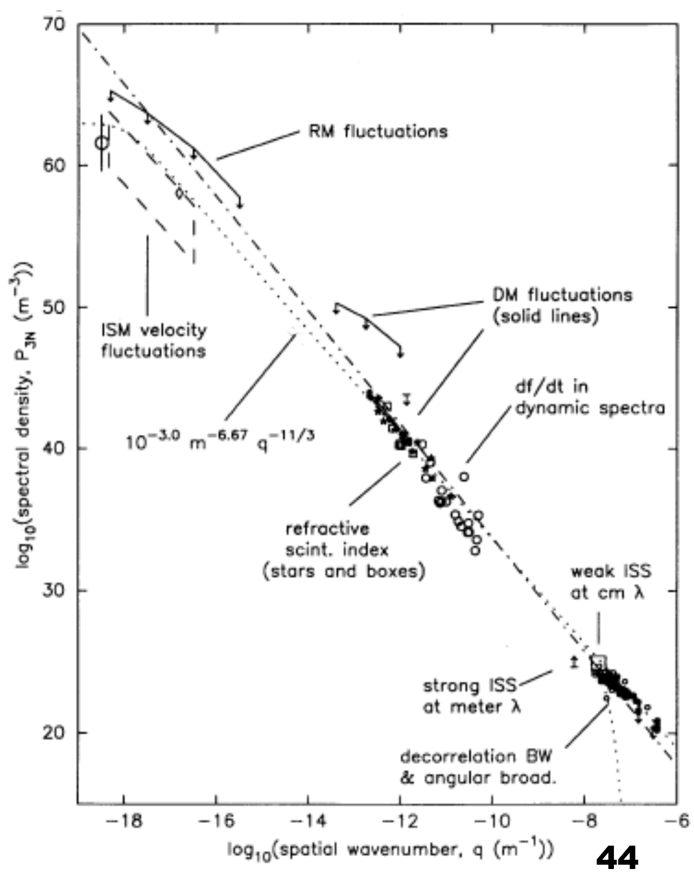
- Kinetic turbulence in lab plasmas
- ISM turbulence in galaxies?!
- Turbulence in biological systems
- Turbulence behind fractal grids

Space-filling fractal grids in wind tunnels



Defining a New Class of Turbulent Flows, Stresing et al., PRL 104, 194501 (2010)

ISM e-density power spectrum: The big power law in the sky



Recommended reading

The gyrokinetic description of microturbulence in magnetized plasmas

Annual Review of Fluid Mechanics 44, 175-201 (2012)

John A. Krommes

Fundamental statistical descriptions of plasma turbulence in magnetic fields

Physics Reports 360 (2002) 1–352

John A. Krommes