STABILITY OF FLUID FLOWS

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WELCOME!

LECTURE CONTENTS:

* Introduction to hydrodynamic stability
* Linear stability: method of normal modes
* Transient energy growth in stable systems

Some examples/results given (only) for hydrodynamic instabilities in incompressible **shear flows**

INTRODUCTION

An example of transition to turbulence

Transition to turbulence

turbulent flow: multi-scale, non-periodic, unpredictable

instabilities develop: the flow loses symmetry & is unsteady

laminar flow: maximum symmetry predictable, usually steady

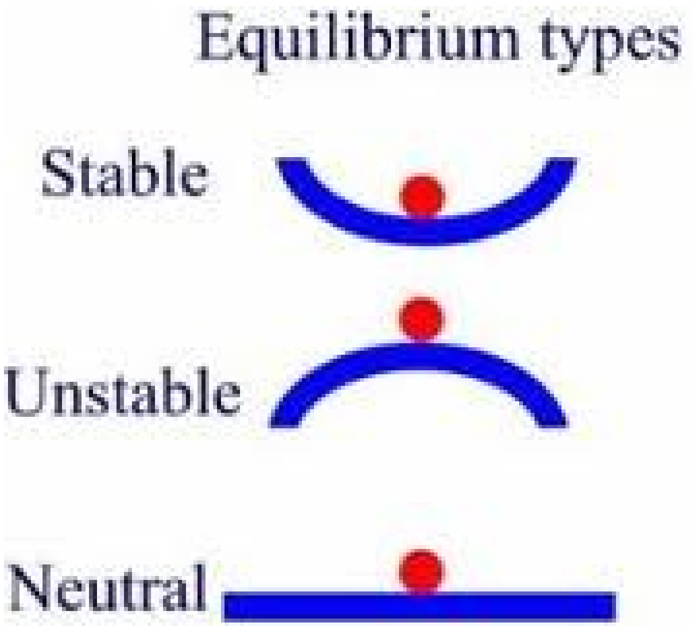
Some questions

Laminar flow solution: same symmetries of problem
Why not often observed?

Why observed solutions are less symmetric than the problem data? (e.g. unsteady if the problem is steady or non axi-sym etc.)

Can we predict when steadiness and symmetry are lost and why?

The hydrodynamic stability main idea



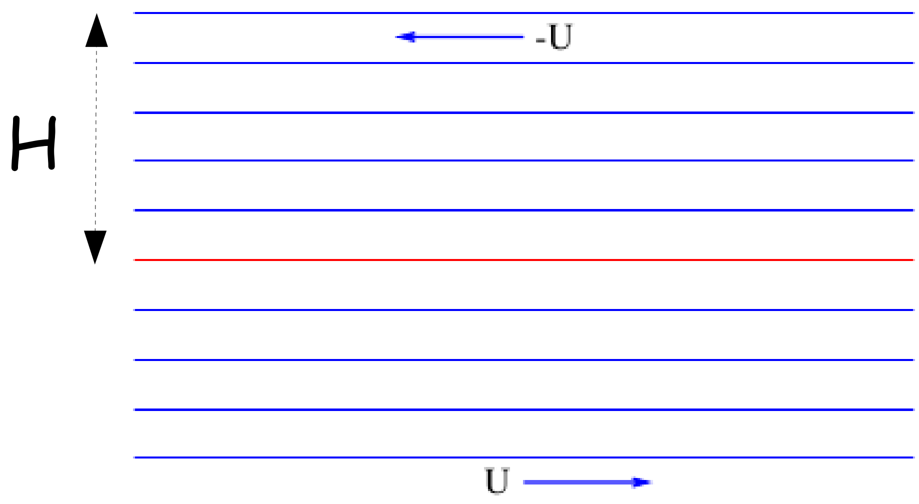
Solutions can be observed only if they are stable!

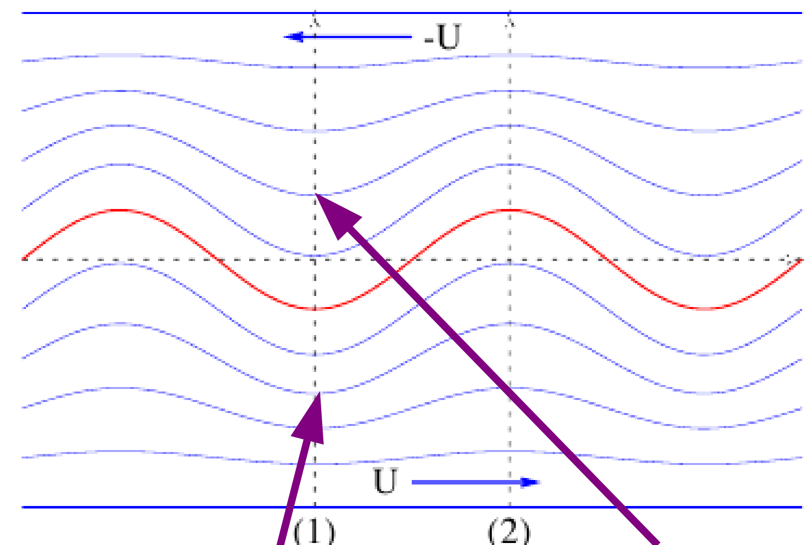
AN EXAMPLE: KELVIN-HELMHOLTZ INSTABILITY

Mechanism: induced forces amplify perturbations

basic flow:
counter streaming flows
with opposite velocities U

here the interface has been
perturbed sinusoidally



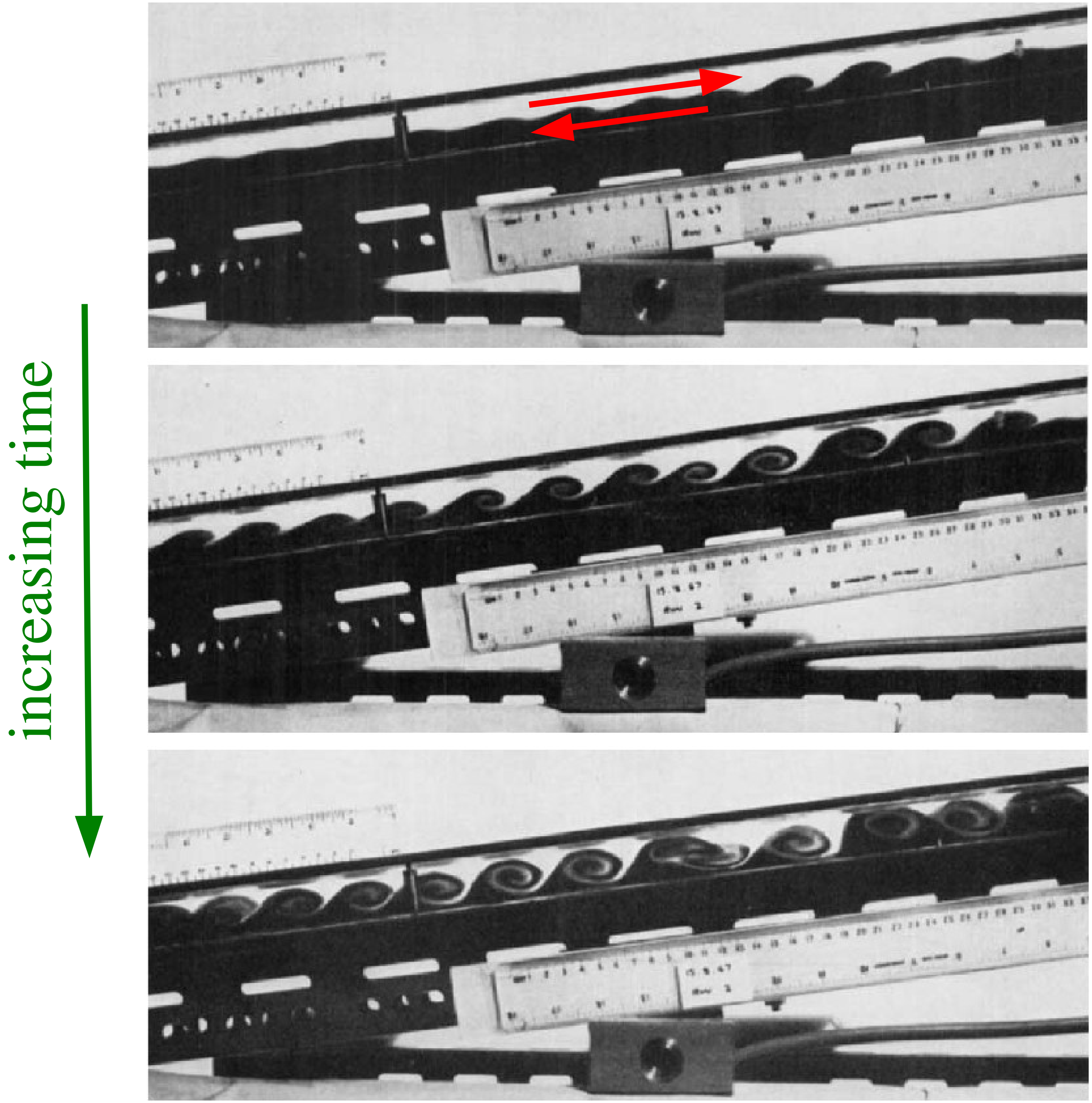


In first approximation:
$h\ u$ = const. (mass conservation)
$p+\rho u^2/2$=const (Bernoulli)

$h < H$
$\rightarrow |u| > |U|$
$\rightarrow p < P$

$h > H$
$\rightarrow |u| < |U|$
$\rightarrow p > P$

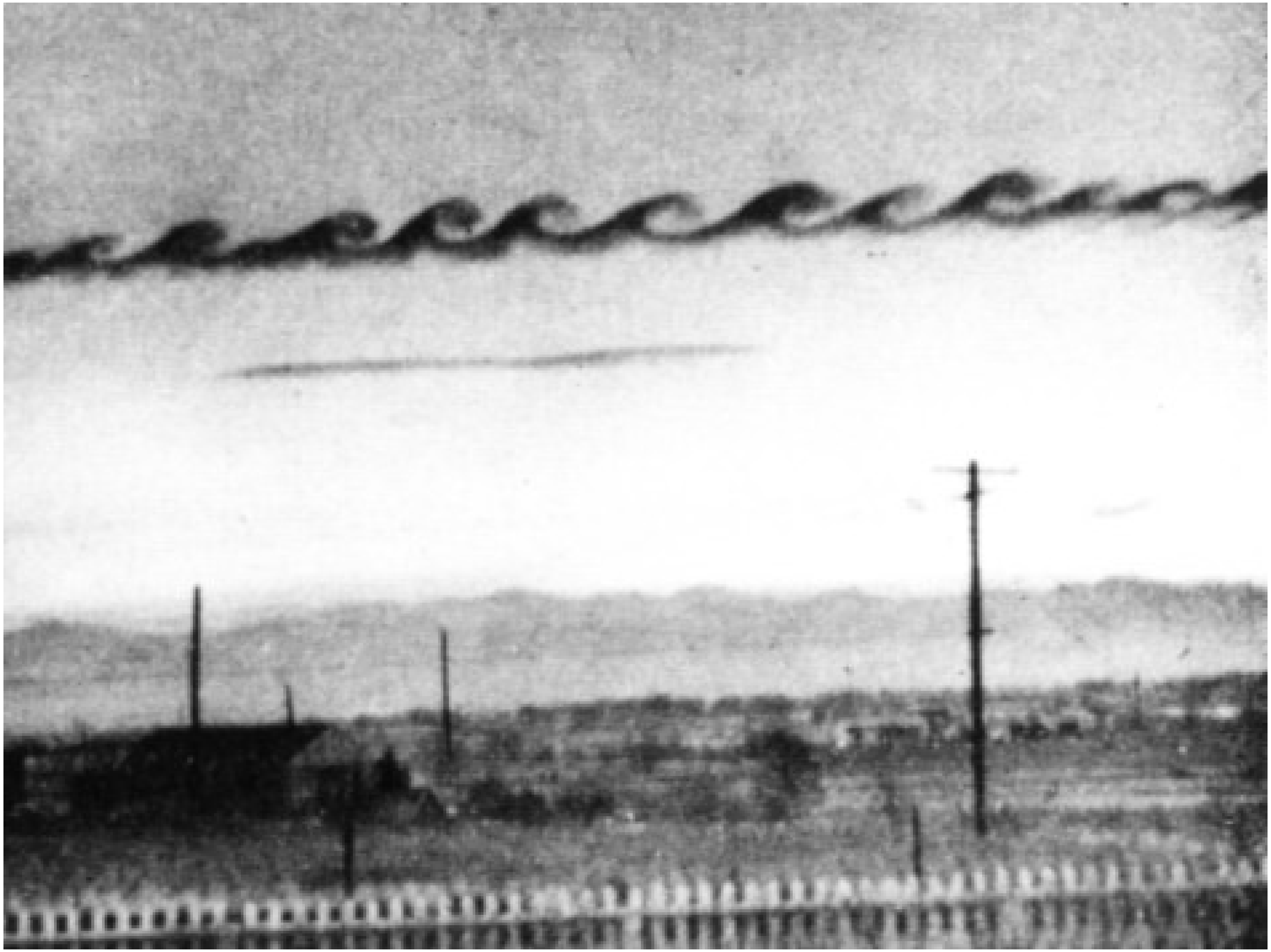
The tilting tank experiment



Thorpe *J. Fluid Mech.* 1968

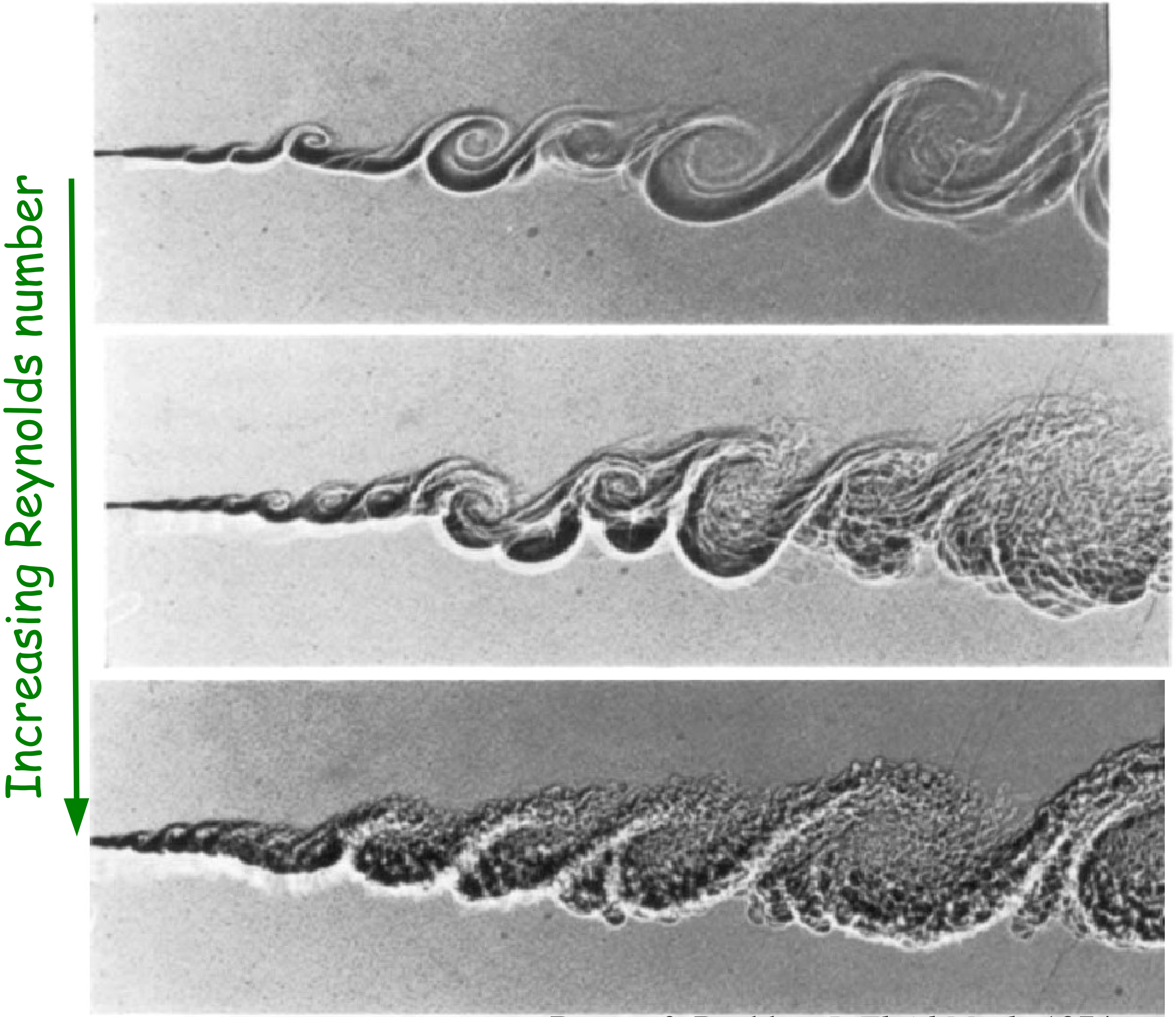
two immiscible fluids in a tilting tank. lighter: transparent, heavier: coloured

tilted tank: lighter fluid pushed up, heavier down → counter-streaming flows for finite time



Billow clouds near Denver, Colorado, (picture by Paul E. Branstine. meteorological details, in Colson 1954)
source: Drazin 2001

Instability active also in turbulent flows



Instability of laminar flow

Transitional flow

Large scale coherent structures in turbulent flow

Brown & Roshko, *J. Fluid Mech.* 1974

DEFINITIONS OF STABILITY

Evolution equations, basic flow, perturbations

evolution equation

state vector

parameter(s)

$$d\phi/dt = \mathbf{f}(\phi, r)$$

basic state / perturbation decomposition

perturbation

$$\phi = \Phi + \phi'$$

basic state

$$d\Phi/dt = \mathbf{f}(\Phi, r)$$

perturbations evolution equation

$$d\phi'/dt = \mathbf{f}(\Phi + \phi') - \mathbf{f}(\Phi)$$

obtained replacing the decomposition in the evolution eqn.
& then removing the basic state evolution eqn.

Norm of the perturbations

need to define a scalar "perturbation amplitude" e.g.

$$||\mathbf{u}'||(t) = \left[\iiint_{\mathcal{V}} (\mathbf{u}' \cdot \mathbf{u}')\, d\mathcal{V}\right]^{1/2}$$

energy

$$||\mathbf{u}'||(t) = \sup_{\mathbf{x}\in\mathcal{V}} |\mathbf{u}'(\mathbf{x};t)|$$

max absolute value

other definitions OK as long as they are norms:

$$||u|| \geq 0;\ ||u|| = 0 \Leftrightarrow u = 0$$

$$||\alpha u|| = |\alpha|\ ||u||,\ \forall \alpha \in C$$

$$||u + v|| \leq ||u|| + ||v||$$

Stability definitions

Lyapunov

Stability (in the sense of Lyapunov): A base solution $\mathbf{\Phi}$ is said to be *stable* if $\forall \varepsilon > 0$, it exists a $\delta(\varepsilon) > 0$ such that if $\|\phi'\|(t = 0) < \delta$ then $\|\phi'\|(t) < \varepsilon, \forall t \geq 0$.

Asymptotic (standard definition used in the following)

Asymptotic stability: A base solution $\mathbf{\Phi}$ is said to be *asymptotically stable* if it is stable (in the sense of Lyapunov) and if furthermore $\lim_{t\to\infty} \|\phi'\| = 0$.

Unconditional

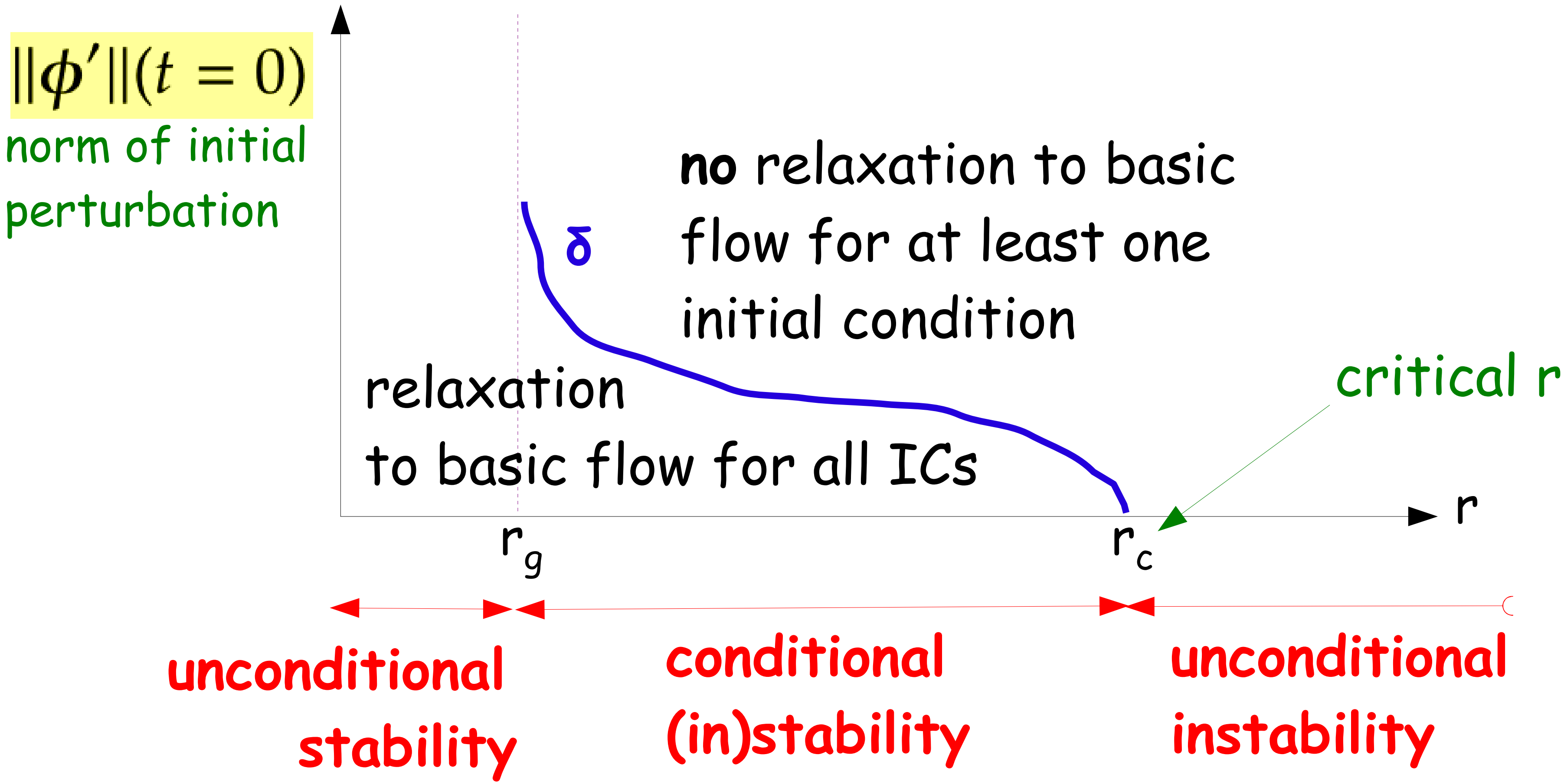
Unconditional (global) stability: A base solution $\mathbf{\Phi}$ is said to be *unconditionally stable* if it is stable and furthermore $\forall \|\phi'\|(t = 0) \Rightarrow \lim_{t\to\infty} \|\phi'\| = 0$.

Remarks: * single initial condition is sufficient to prove instability of the base flow

* to prove stability one has to prove it for ALL possible initial conditions

Dependence on the control parameter

stability properties depends on the **control parameter** r



shown typical case with $r_g < r_c$ but can e.g. happen that r_c is infinite or that $r_g = r_c$

Linearized equations

$$d\phi'/dt = \mathbf{f}(\Phi + \phi') - \mathbf{f}(\Phi)$$

start from perturbation eqns

$$\mathbf{f}(\Phi + \phi') = \mathbf{f}(\Phi) + (d\mathbf{f}/d\phi)(\Phi)\phi' + O(\|\phi'\|^2)$$

Taylor expansion of r.h.s. near basic flow →
replace & neglect higher order terms

$$d\phi'/dt = \mathbf{L}\phi'$$

linearized evolution equations

$$\mathbf{L}(\Phi, r) = (d\mathbf{f}/d\phi)(\Phi, r)$$

linearized (tangent) operator (Jacobian)

Linear stability: A base solution Φ is said to be *linearly stable* if the solutions of the evolution equations linearized about Φ are stable.

Linear stability analysis

linear problem → decomposition of generic solution on basis of `fundamental' solutions

$$\phi'(t) = \sum c_j \phi'^{(j)}(t)$$

complete set of linearly independent solutions

constants depend on the initial condition

linear instability if at least one fundamental solution is unstable

linear stability if ALL fundamental solutions are stable

new problem: compute the basis of linearly independent solutions and analyze their stability

general method available for steady basic flows (steady L):
method of normal modes

THE METHOD OF NORMAL MODES: LINEAR SYSTEMS OF ODEs

System of autonomous ODEs

Consider a system of N 1st-order ordinary differential equations with constant coefficients:

$$d\phi'/dt = \mathbf{L}\phi'$$

state:
N-dimensional
vector

Linear operator:
NxN matrix
L does not depend on t!

Eigenvalues & eigenvectors

$$\mathbf{L}\,\psi = s\,\psi$$

eigenvector
(direction left
unchanged by L)

eigenvalue (scalar)

Homogeneous system:
non-trivial solutions if

$$det(\mathbf{L} - s\,\mathbf{I}) = 0$$

characteristic eqn → algebraic
N-th order → N roots for s →
N eigenvalue-eigenvector pairs

$$\mathbf{L}\,\psi^{(j)} = s^{(j)}\,\psi^{(j)}$$

Modal decomposition for ODEs

assume N *distinct* eigenvalues → N *linearly independent* eigenvectors = eigenvector basis → express ϕ' in the modal basis:

$$\phi'(t) = \sum_{j=1}^{N} \psi^{(j)} q_j(t)$$

modal amplitudes q

$$d\phi'/dt = \mathbf{L}\phi'$$

$$\sum_{j=1}^{N} \psi^{(j)} dq_j/dt = \sum_{j=1}^{N} L\psi^{(j)} q_j$$

$$\sum_{j=1}^{N} \psi^{(j)} dq_j/dt = \sum_{j=1}^{N} s^{(j)}\psi^{(j)} q_j$$

$$\sum_{j=1}^{N} (dq_j/dt - s^{(j)} q_j)\psi^{(j)}$$

N **independent** equations for each modal amplitude

$$dq_j/dt = s^{(j)} q_j$$

Modal solution to the IVP: stability

$$dq_j/dt = s^{(j)} q_j$$

$$q_j(t) = e^{t\,s^{(j)}} q_j(0)$$

$$\phi'(t) = \sum_{j=1}^{N} e^{t\,s^{(j)}} \psi^{(j)} q_j(0)$$

envelope

$$e^{t\,s_r^{(j)}} (\cos s_i^{(j)} t + i \sin s_i^{(j)} t)$$

oscillations

coefficients from initial condition

Linear instability if at least one eigenvalue with **s_r>0**
(unbounded growth of a fundamental solution)
Linear stability if *ALL* eigenvalues have **s_r<0**

Linear stability analysis: given L compute its eigenvalues and check their real parts

THE METHOD OF NORMAL MODES FOR LINEAR PDEs: AN EXAMPLE

The case of a `parallel' unconfined system

$$\frac{\partial \phi'}{\partial t} = r\phi' + \frac{\partial^2 \phi'}{\partial x^2}$$

1D reaction-diffusion eqn

BC: solution bounded as |x| → ∞

$$L = r\,I + d^2/d\,x^2 \rightarrow e^{px} \rightarrow$$

functions for which $L\psi=s\psi$ BUT they remain bounded only if $p_r=0$

$$\psi(k,x) = e^{ikx}$$

eigenfunctions = Fourier modes of (real) wavenumber k → uncountable infinity of modes (derives from translational invariance of the system)

$$s = r - k^2$$

eigenvalues found replacing the eigenfunction in in $L\psi=s\psi$

dispersion relation relating the complex temporal eigenvalue to the wavenumber k and the control parameter r

Modal decomposition for unconfined parallel flows

$$\phi'(x,t) = \int_{-\infty}^{\infty} q(k,t)\psi(k,x)dk = \int_{-\infty}^{\infty} q(k,t)e^{ikx}dk$$

modal decomposition:
sum → integral on index k

modal decomposition =
inverse Fourier transform

modal decomposition → usual procedure → usual solution:

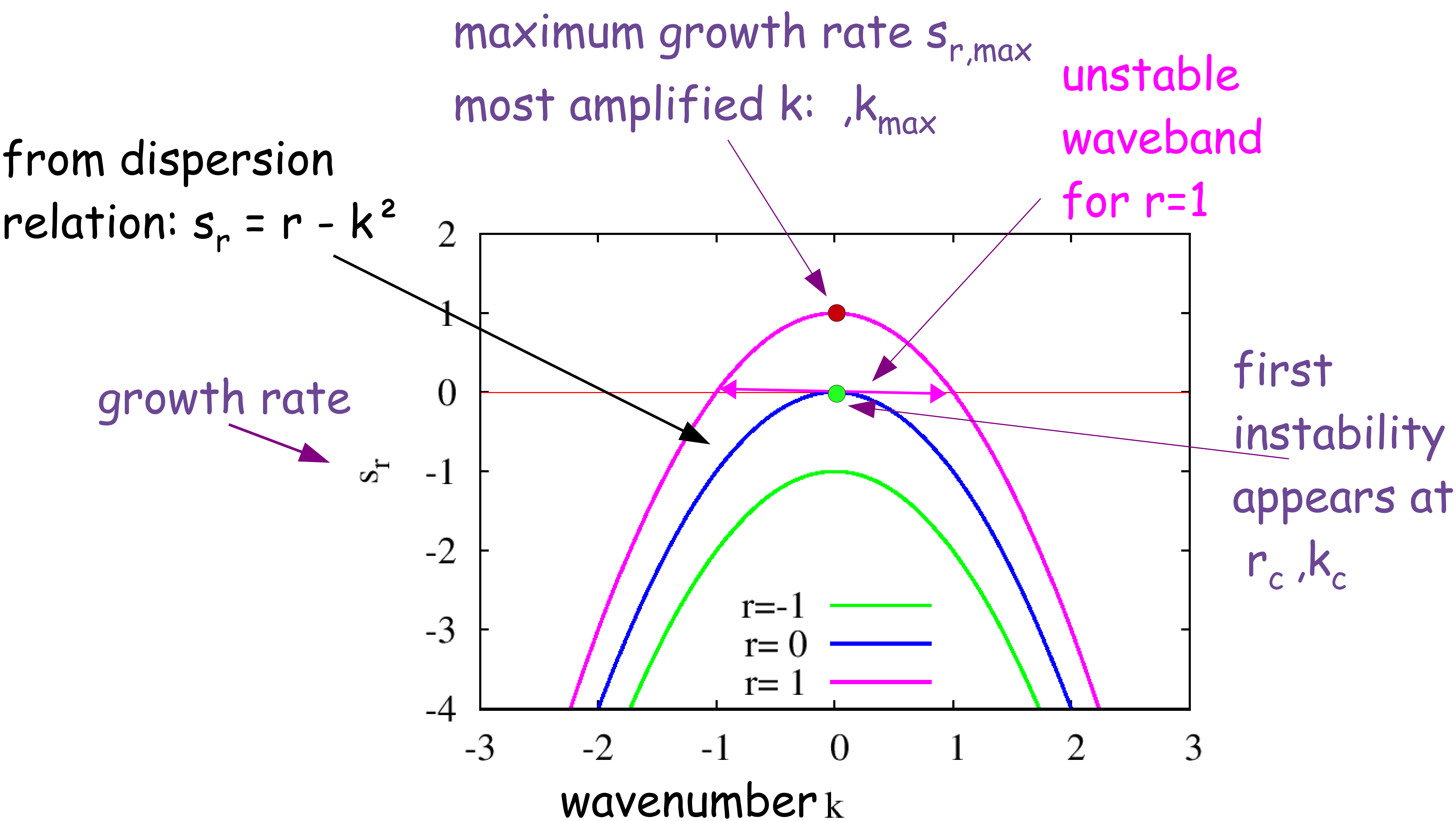
$$q_j(t) = e^{t\, s^{(j)}} q_j(0)$$

Linear instability: if at least one k exists for which **s_r(k)>0**

Linear stability: if **s_r(k)<0** for all k

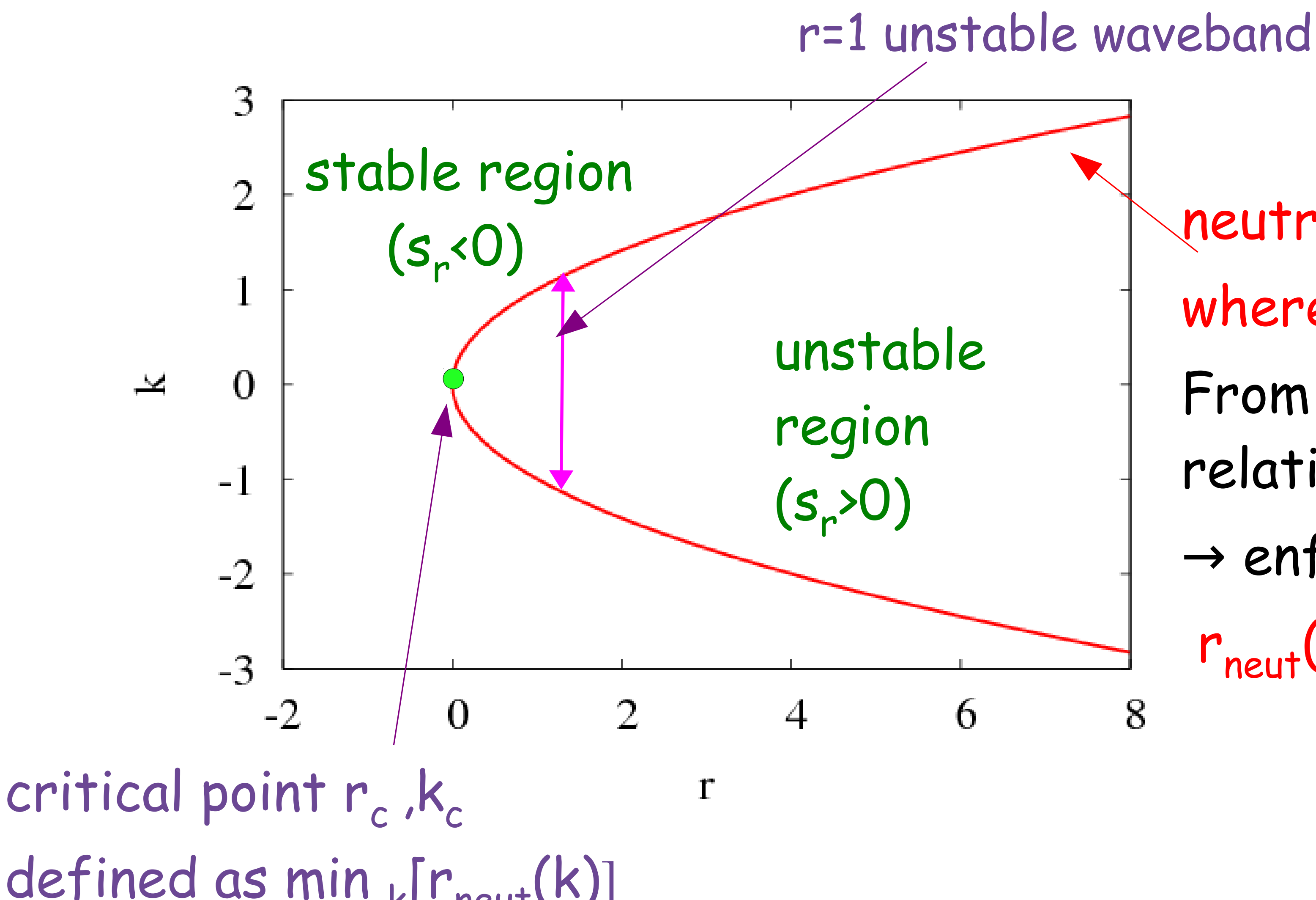
Typical stability plots: growth rate

growth rate vs. wavenumber for selected values of r



Typical stability plots: neutral curve

neutral curve s_r=0:   separates regions with positive growth rate from regions of negative growth rate in the r-k plane



neutral curve $r_{neut}(k)$

where s_r=0.

From the dispersion relation: $s_r = r - k^2$

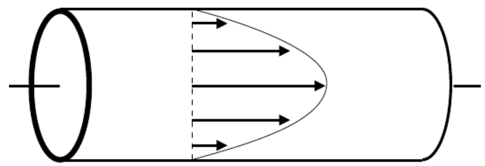
→ enforcing s_r=0 →

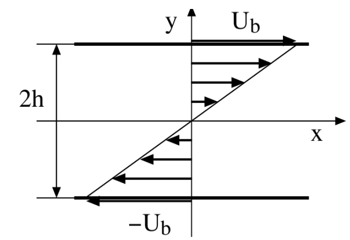
$r_{neut}(k) = k^2$

MODAL STABILITY OF PARALLEL SHEAR FLOWS

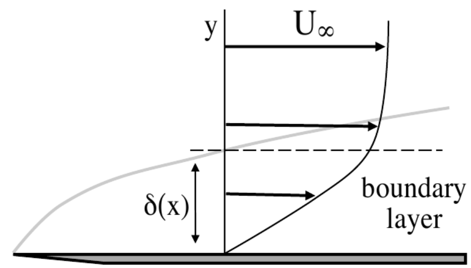
Parallel incompressible shear flows

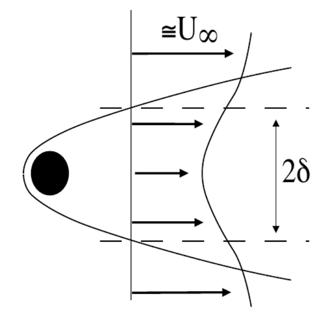
Some flows of interest are exactly parallel:





Other are weakly non parallel
→ 'local' analysis at some x





Will consider parallel basic flows $\mathbf{U} = \{U(y),0,0\}$

Control parameter:
Reynolds number
= ratio btw viscous
time scale δ^2/ν and
time scale of shear $\delta/\Delta U$

$$Re = \Delta U\, \delta \,/\, \nu$$

max velocity variation

length scale of shear region

kinematic viscosity

Linearized Navier-Stokes eqns.

linearized Navier-Stokes equations

$$\frac{\partial \mathbf{u}'}{\partial t} + (\nabla \mathbf{U})\,\mathbf{u}' + (\nabla \mathbf{u}')\,(\mathbf{U} + \mathbf{u}') \quad = \quad -\nabla p' + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u}'$$

usually transformed into v-η form (exploiting div **u**'=0):

wall-normal velocity v

$$\nabla^2 \frac{\partial v}{\partial t} = -\left(U\nabla^2 - \frac{d^2 U}{dy^2}\right)\frac{\partial v}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^4 v$$

wall-normal vorticity η

$\eta = \partial u/\partial z - \partial w/\partial x$

$$\frac{\partial \eta}{\partial t} = -U\frac{\partial \eta}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 \eta - \frac{dU}{dy}\frac{\partial v}{\partial z}$$

Orr-Sommerfeld-Squire system

consider Fourier modes in x-z (unconfined homogeneous direction)s;

$$\widehat{v}(y,t;\alpha,\beta)e^{i(\alpha x+\beta z)}, \widehat{\eta}(y,t;\alpha,\beta)e^{i(\alpha x+\beta z)}$$

D := d/dy

replace and find the Orr-Sommerfeld-Squire system

$$\begin{bmatrix} D^2-k^2 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial t} \begin{Bmatrix} \widehat{v} \\ \widehat{\omega_y} \end{Bmatrix} = \begin{bmatrix} \mathcal{L}_{\mathcal{OS}} & 0 \\ -i\beta U' & \mathcal{L}_{\mathcal{SQ}} \end{bmatrix} \begin{Bmatrix} \widehat{v} \\ \widehat{\omega_y} \end{Bmatrix}$$

Orr-Sommerfeld operator

$$\mathcal{L}_{OS} = -i\alpha\left[U(\mathcal{D}^2-k^2)-\frac{d^2U}{dy^2}\right]+\frac{1}{\mathrm{Re}}(\mathcal{D}^2-k^2)^2$$

Squire operator

$$\mathcal{L}_{SQ} = -i\alpha U+\frac{1}{\mathrm{Re}}(\mathcal{D}^2-k^2),$$

eigenvalues – eigenfunctions found numerically solving the problem in the y direction

Fundamental results for parallel shear flows U(y)

Squire (viscous) theorem: The critical mode, becoming unstable at the lowest Reynolds number, is two-dimensional

Squire (inviscid) theorem: In the inviscid case, given an unstable 3D mode, a 2D more unstable mode can always be found

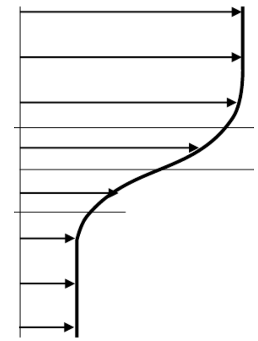
Confirmed by experimental observations → consider 2D perturbations in modal stability analyses

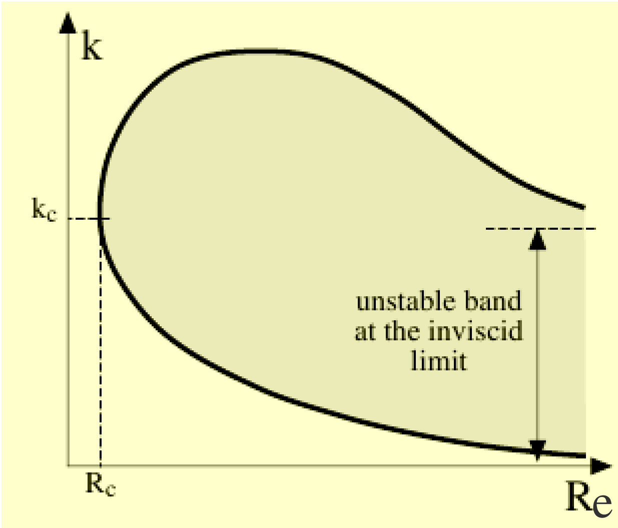
Rayleigh inflection point theorem: A necessary condition for U(y) of class at least C^2 to be unstable to 2D inviscid perturbations satisfying $v(y_b) = 0$, $v(y_a) = 0$ is that U(y) admits at least one inflection point in $]y_b, y_a[$

Inviscid instabilities only for inflectional profiles!

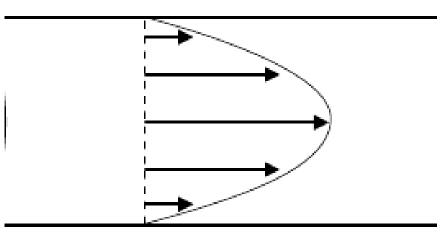
Typical neutral curves

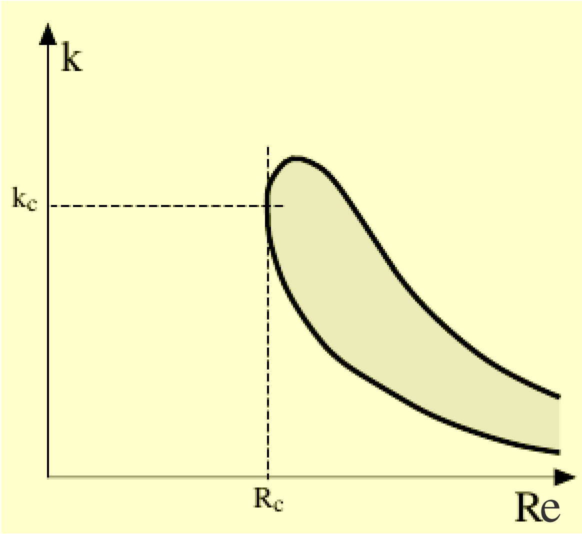
typical **free shear flows** (no walls) → **inviscid instabilities** allowed by inflectional profiles
→ instability develops on **short** (shear) **time scales**





typical **wall-bounded flows** → **no inflection points** → may have viscous instabilities (Tollmien Schlichting) vanishing as Re → ∞
develop on long (viscous time scales)

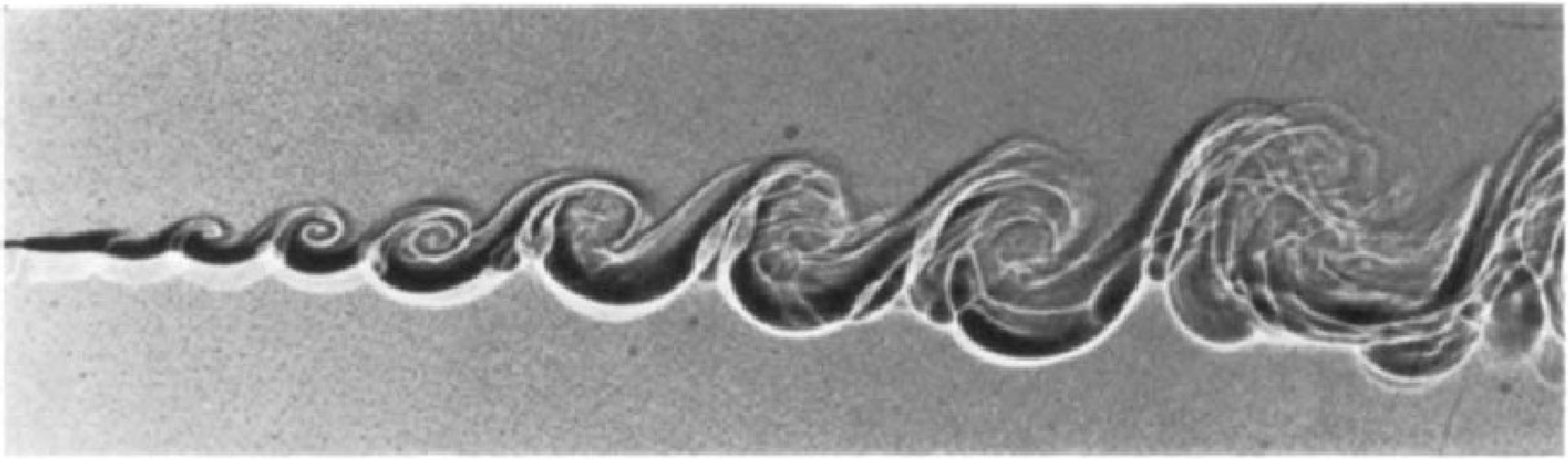




figures: from Manneville, 2010

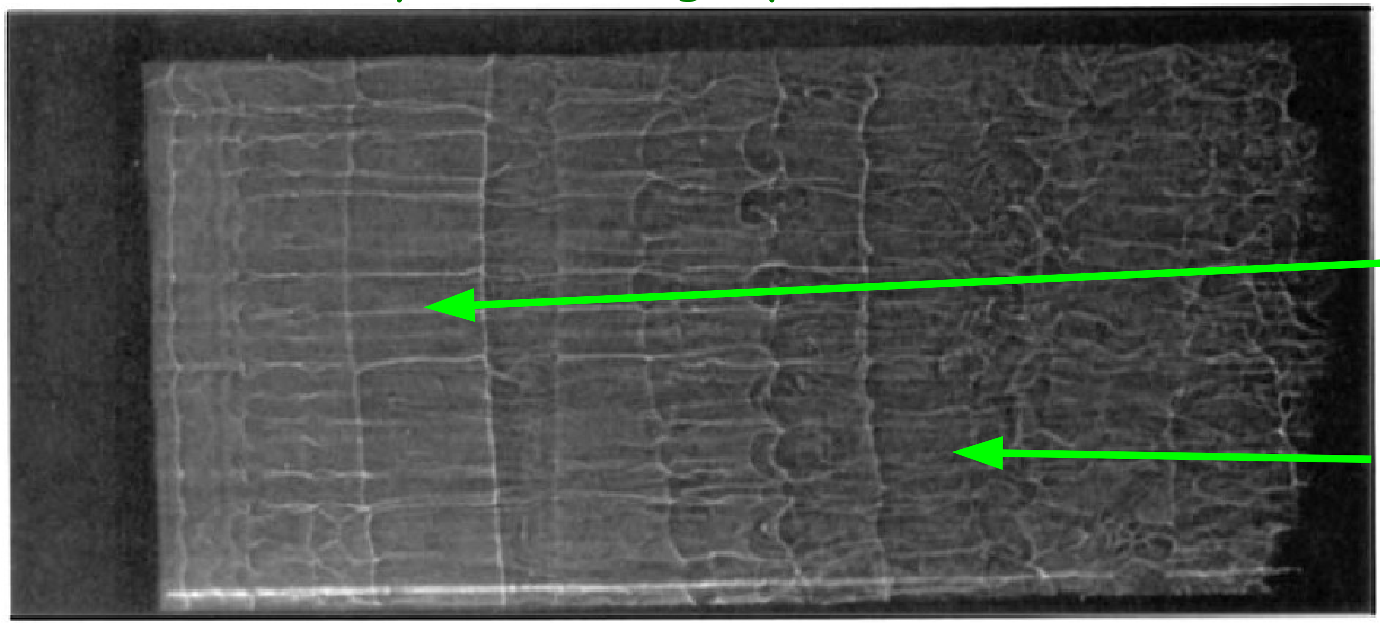
TRANSITION TO TURBULENCE IN SHEAR FLOWS

Transition to turbulence in mixing layers



Side view (xy)

Transition in a spatial mixing layer (Brown & Roshko *J. Flud Mech.* 1974)



Top view (xz)

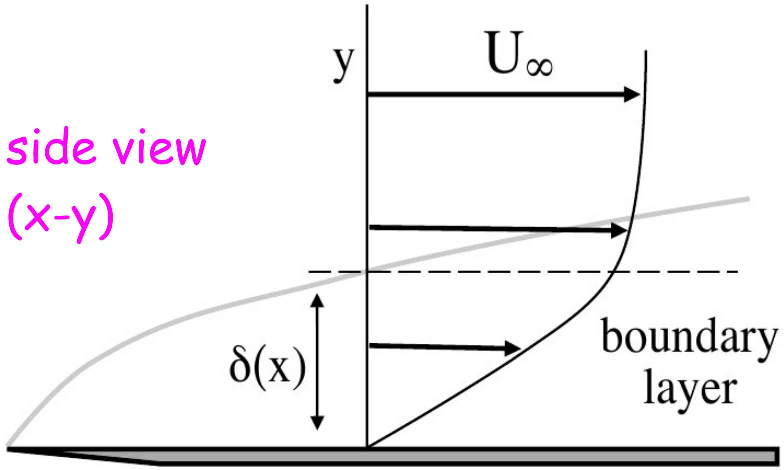
2D fronts in transition

3D structures appear later & important in turbulent flow

Similar trends observed for modal instabilities of jets, wakes,

`Classical' transition scenario in boundary layers

side view (x-y)

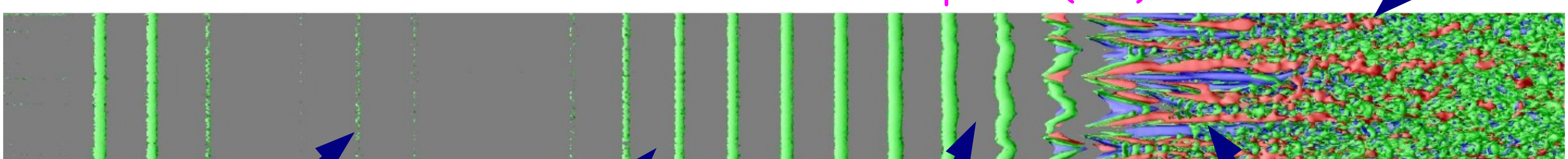


scenario observed in the presence of very low external noise levels

Re increases downstream

top view (x-z)

developed turbulence



Primary instability of $U(y)$ profile → 2D Tollmien-Schlichting waves

exponential amplification of unstable 2D TS waves

TS waves reach critical amplitude ~ 1% Ue → **secondary instability** to 3D waves

amplification of 3D waves → new instability

figure: from DNS of Schlatter, KTH

HOWEVER...

some flows do not follow this scenario:

Plane Couette & circular pipe flow: linearly stable for all Re but become turbulent at finite Re

Plane Poiseuille flow: transition almost always observed for Re well below Re_c

Bypass transition in boundary layers

Transition for
$Re < Re_c$
No evidence
of 2D TS waves
→ subcritical

Purely nonlinear mechanism?

Matsubara & Alfredsson,2001

Scenario observed in noisy environments
Structures observed before transition: streaks
Streaks: uniform in z / ~ periodic in x
TS waves: ~periodic in x / uniform in z

in the 1990"-2000':

a lot about bypass transition understood reconsidering classical linear stability analysis

ENERGY AMPLIFICATION IN LINEARLY STABLE SYSTEMS

Reminder: 'classical' linear stability analysis

perturbations state vector

depends on Re & basic flow

Linear IVP

$$d\phi'/dt = \mathbf{L}\phi'$$

$$\phi'(t = 0) = \phi'_0$$

Linear asymptotic stability requirement

$$\lim_{t\to\infty} \|\phi'\| = 0$$

Standard modal stability analysis:

* Compute the spectrum of L
* Linear asymptotic stability if spectrum is in the stable complex half-plane

Results apply when $t \to \infty$

What happens at finite times?

Can $||\phi'||^2$ become large during transients?

Not forbidden by asymptotic stability... but

does energy actually grow? How much?

Find worst case disturbances → optimal growth

Optimal energy growth in the IVP

Formal solution of linear IVP

propagator from t=0 to actual t

initial condition

$$\phi'(t) = \mathbf{P}\phi'_0$$

$$\mathbf{P} = e^{t\mathbf{L}}$$

input- output definition of operator norm

Optimal energy growth:

$$G(t) = \sup_{\phi'_0} \frac{\|\phi'\|^2}{\|\phi'_0\|^2} = \sup_{\phi'_0} \frac{\|\mathbf{P}\phi'_0\|^2}{\|\phi'_0\|^2} := \|\mathbf{P}\|^2$$

* to each t corresponds a different G & optimal IC

* G(t): envelope of all energy gain curves

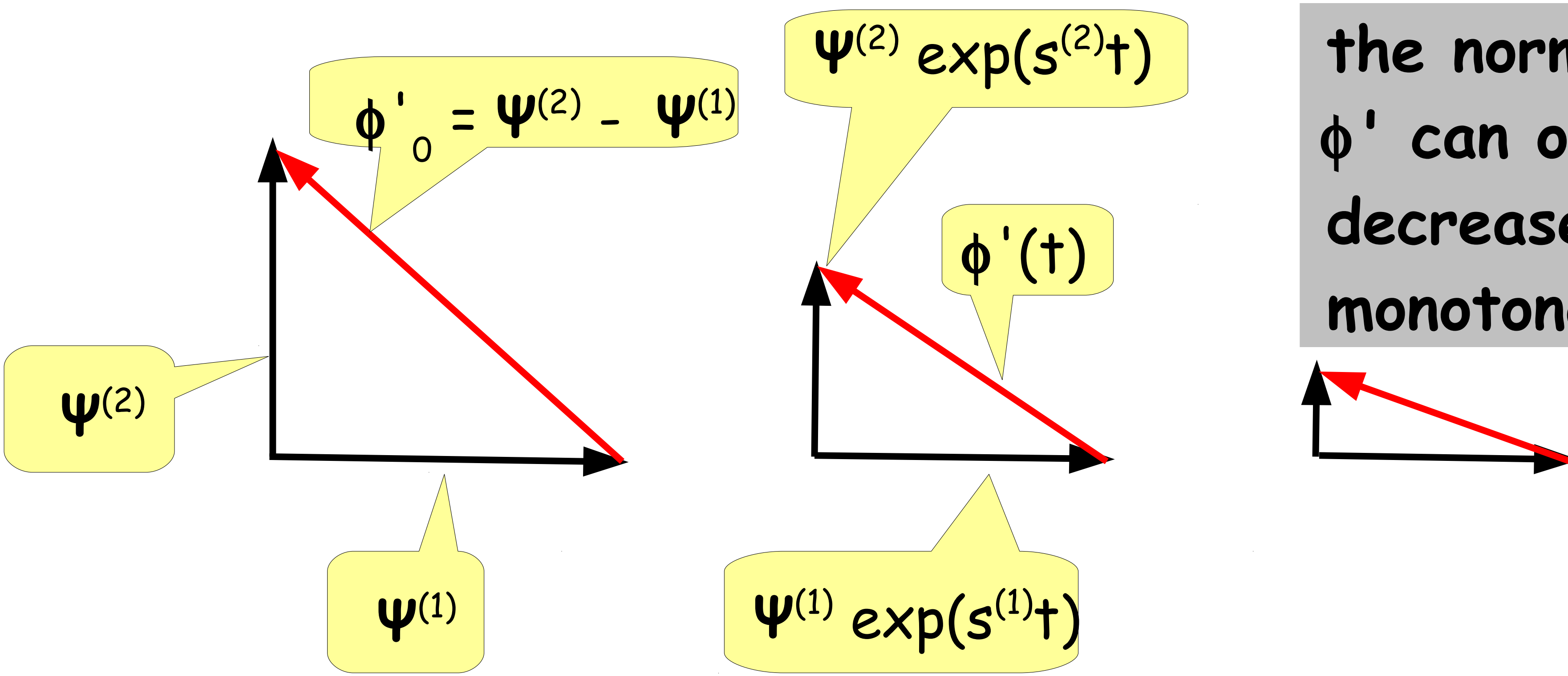
No energy growth if L is stable & normal

Simple 2D case with real eigenvalues-eigenvectors:

$\phi'(t) = q_1(0)\,\Psi^{(1)} \exp(s^{(1)}t) + q_2(0)\,\Psi^{(2)} \exp(s^{(2)}t)$

Assume **stable eigenvalues** with $s^{(2)}$ the most stable and **orthogonal eigenvectors** $\Psi^{(j)}$.

Choose $q_1(0)=-1$, $q_2(0)=1 \rightarrow \phi'(0) = \Psi^{(2)} - \Psi^{(1)}$



the norm of ϕ' can only decrease monotonously

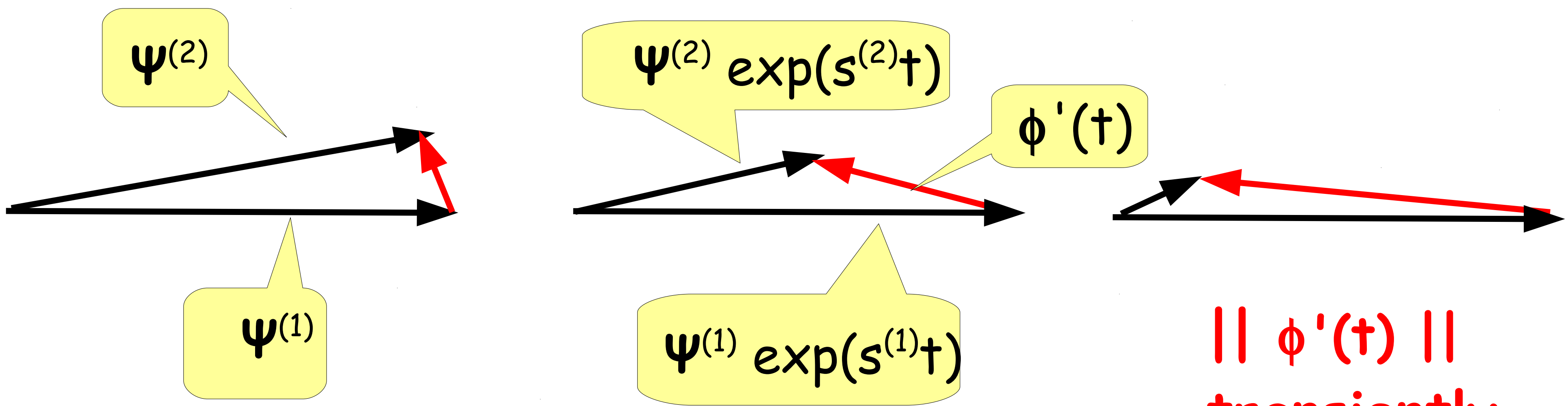
Non-normality & transient growth

If stable eigenvalues & orthogonal eigenvectors

→ energy growth impossible → $G=1$

What happens if the eigenvectors are non-orthogonal?

(i.e. if L is non-normal: $LL^+ \neq L^+L$)



$|| \phi'(t) ||$ **transiently grows & changes direction**

Necessary condition for transient growth ($G>1$): non-normal L (non-orthogonal eigenvectors)

Do-it-yourself: a simple 2x2 example

non normal matrix
~ Orr-Sommerfeld-
-Squire system

$$\mathbf{L} = \begin{bmatrix} -1/\mathrm{Re} & 0 \\ 1 & -3/\mathrm{Re} \end{bmatrix}$$

eigenvalues: -1/Re and -3/Re → linearly stable
eigenvectors: non-orthogonal with angle decreasing with Re

$G(t) = || e^{tL} ||$ computed in a few matlab (or octave!) lines:

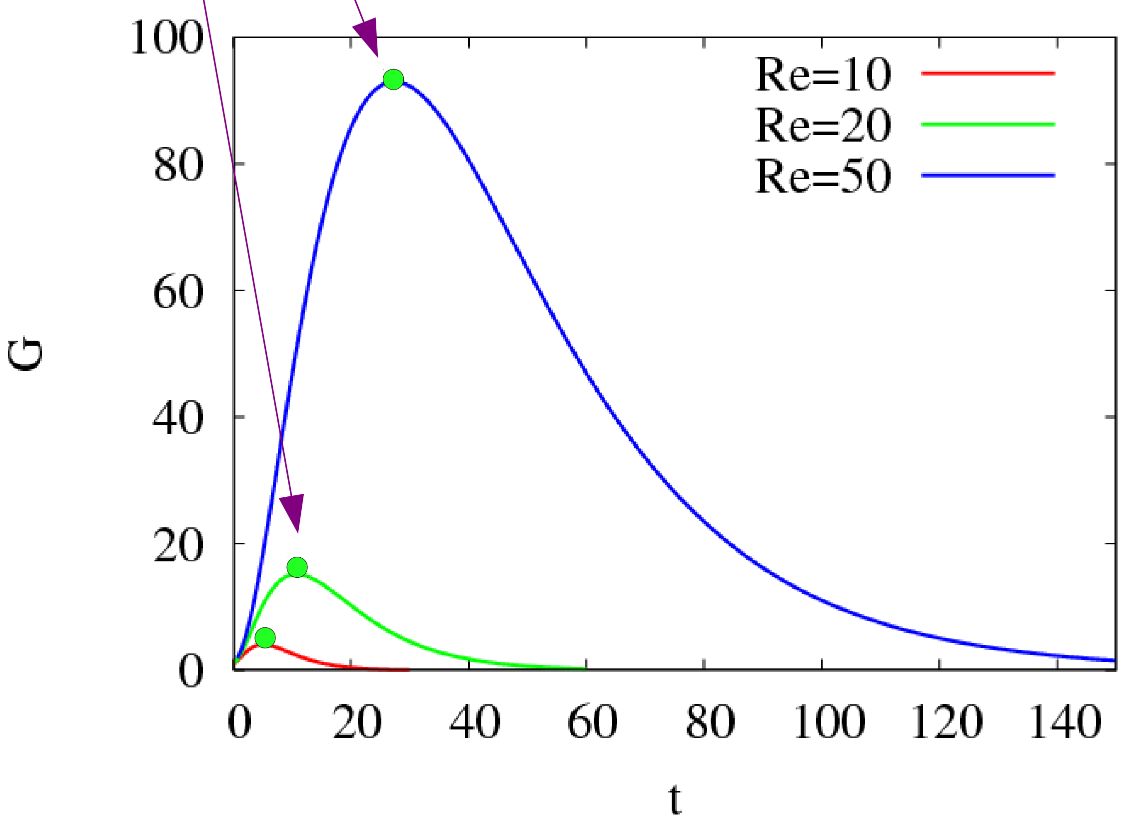
```
Reynolds=20
L=[-1/Reynolds, 0 ; 1 , -3/Reynolds]
t=linspace(0,60,60);
for j=1:60
 P=expm(t(j)*L);
 G(j)=(norm(P))^2;
end
plot(t,G);
```

2x2 example: scaling of growth with Re

$G_{max} \sim Re^2$

$t_{max} \sim Re$

same Re scalings as genuine Navier-Stokes case



optimal perturbations associated to Gmax:

optimal IC →

first component ~ v

response at t_{max}:

second component ~ η

Optimal amplification in forced responses

linear forced system

$$\frac{\partial \phi'}{\partial t} = \mathbf{L}\phi' + \mathbf{f}'$$

complex harmonic

forcing/response (L assumed stable)

$$\mathbf{f}' = \widetilde{\mathbf{f}}\, e^{\zeta t} \qquad \phi' = \widetilde{\phi}\, e^{\zeta t}$$

$$(\zeta\mathbf{I} - \mathbf{L})\,\widetilde{\phi} = \widetilde{\mathbf{f}}$$

resolvent operator

$$\mathbf{R}_\zeta = (\zeta\mathbf{I} - \mathbf{L})^{-1}$$

$$\widetilde{\phi} = \mathbf{R}_\zeta \widetilde{\mathbf{f}}$$

also called pseudospectrum

infinite if ζ=eigenvalue

$$R_\zeta = \sup_{\widetilde{\mathbf{f}}} \frac{||\widetilde{\phi}||^2}{||\widetilde{\mathbf{f}}||^2} = \sup_{\widetilde{\mathbf{f}}} \frac{||\mathbf{R}_\zeta \widetilde{\mathbf{f}}||^2}{||\widetilde{\mathbf{f}}||^2} := ||\mathbf{R}_\zeta||^2$$

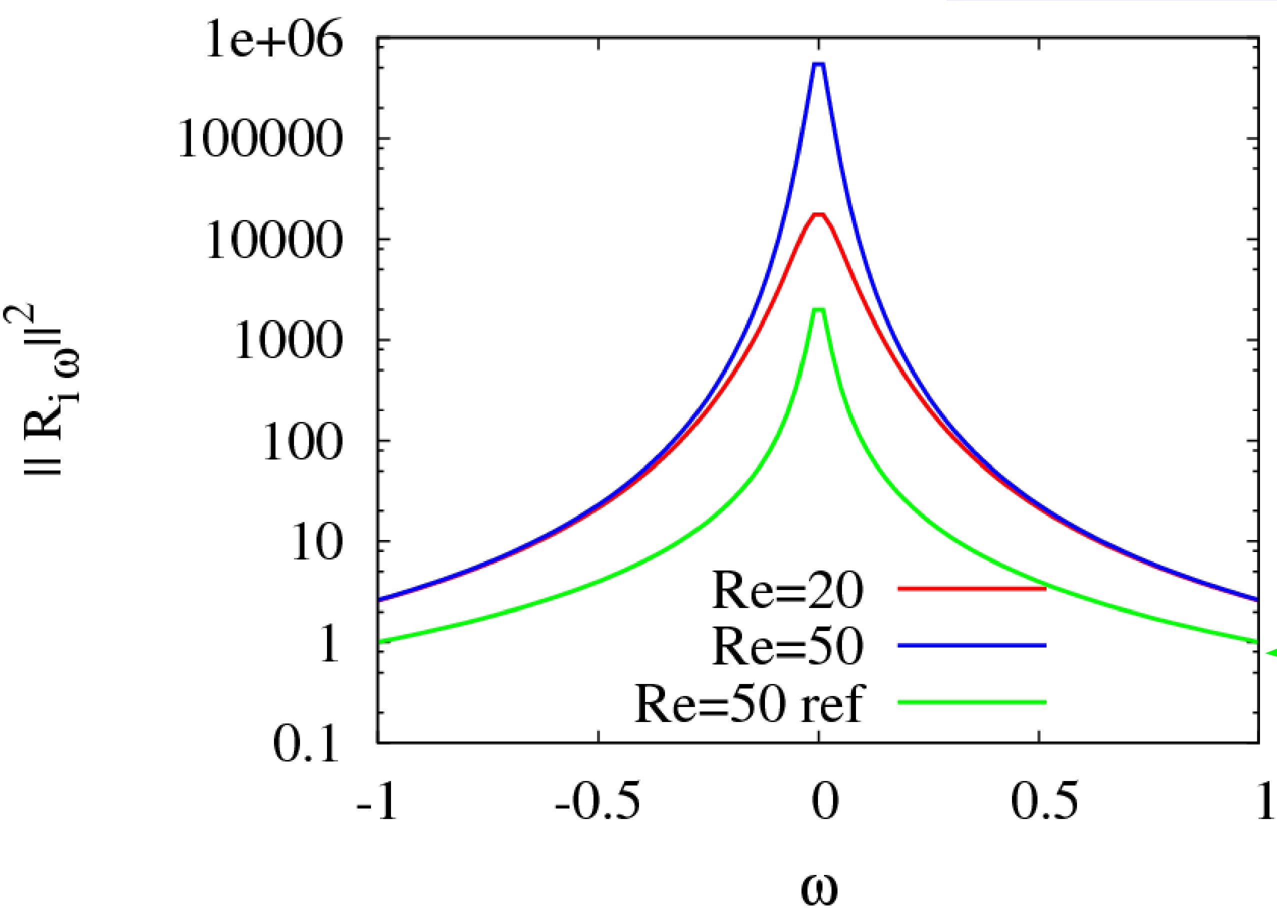
optimal amplification of forcing energy

Response to harmonic forcing

harmonic forcing: $\zeta = i\,\omega$
resolvent norm computed for 2x2 toy model:

$$\mathbf{L} = \begin{bmatrix} -1/\mathrm{Re} & 0 \\ 1 & -3/\mathrm{Re} \end{bmatrix}$$

```
Reynolds=20
L=[-1/Reynolds, 0 ; 1 , -3/Reynolds]
freq=linspace(-2,2,40);
for j=1:40
 Resol=inv(freq(j)*eye(2)-L);
 R(j)=(norm(Resol))^2;
end
plot(t,R);
```

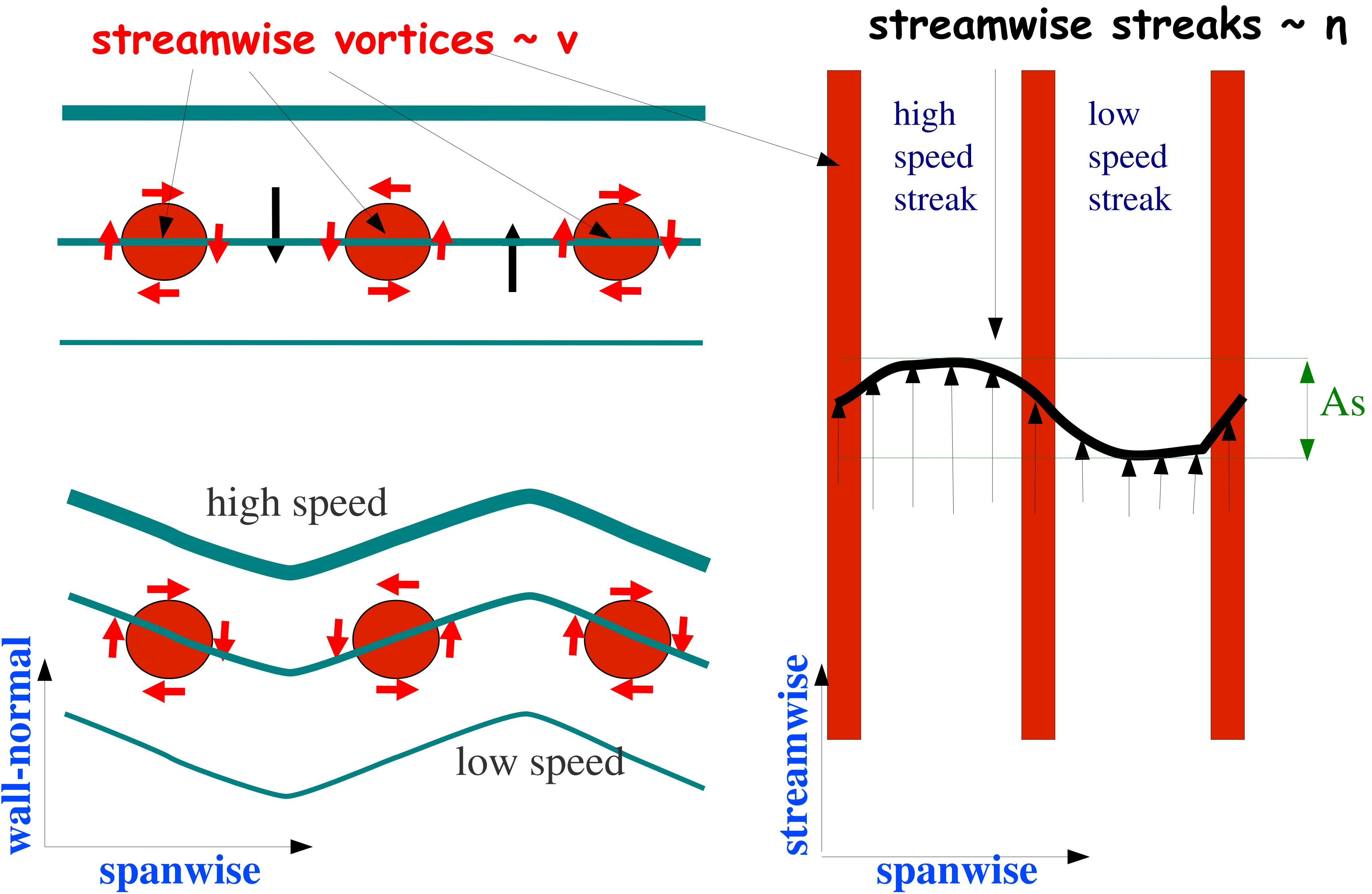


non-normality → large excitability far from resonance!

reference curve: response that would be obtained with a normal matrix with same eigenvalues

SHEAR FLOWS: STREAKS, VORTICES & SELF-SUSTAINED PROCESSES

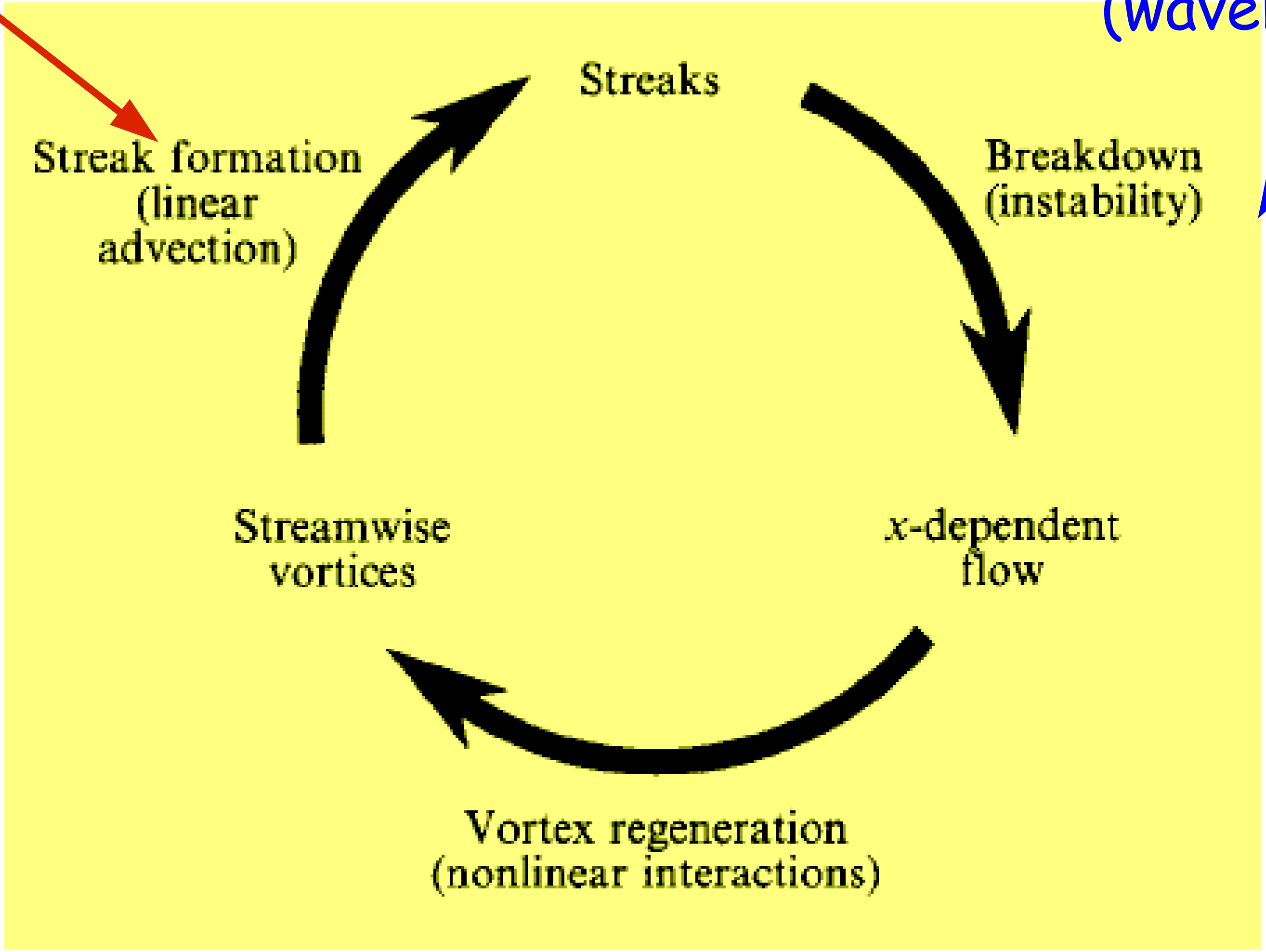
Vortices & streaks: the lift-up effect



Tentative explanation: the self-sustained process

lift-up effect: selection of amplified spanwise scales λ_z (waveband β)

selection of unstable streamwise scales λ_x (waveband α)

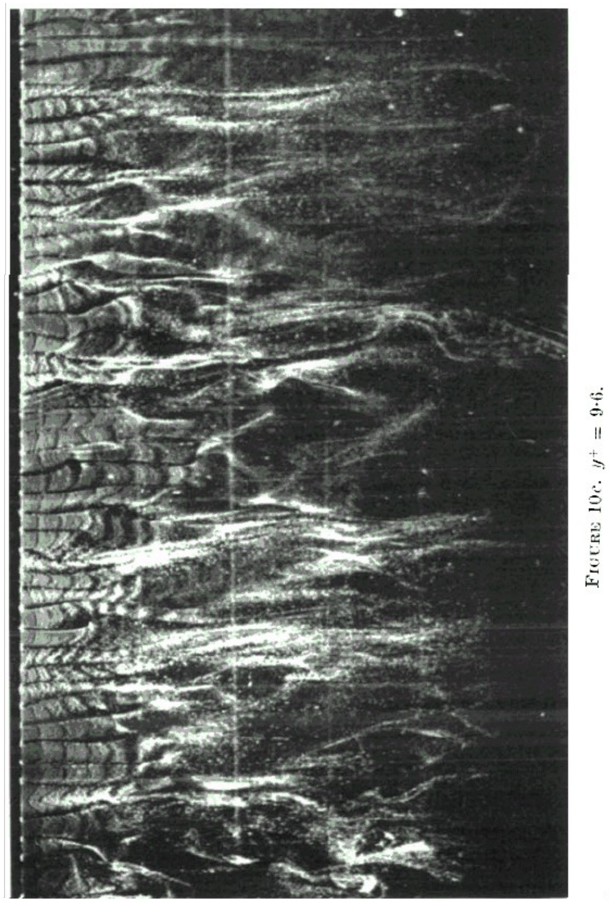


SSP
self-
sustained
process

Figure from Hamilton et al. .JFM 1995

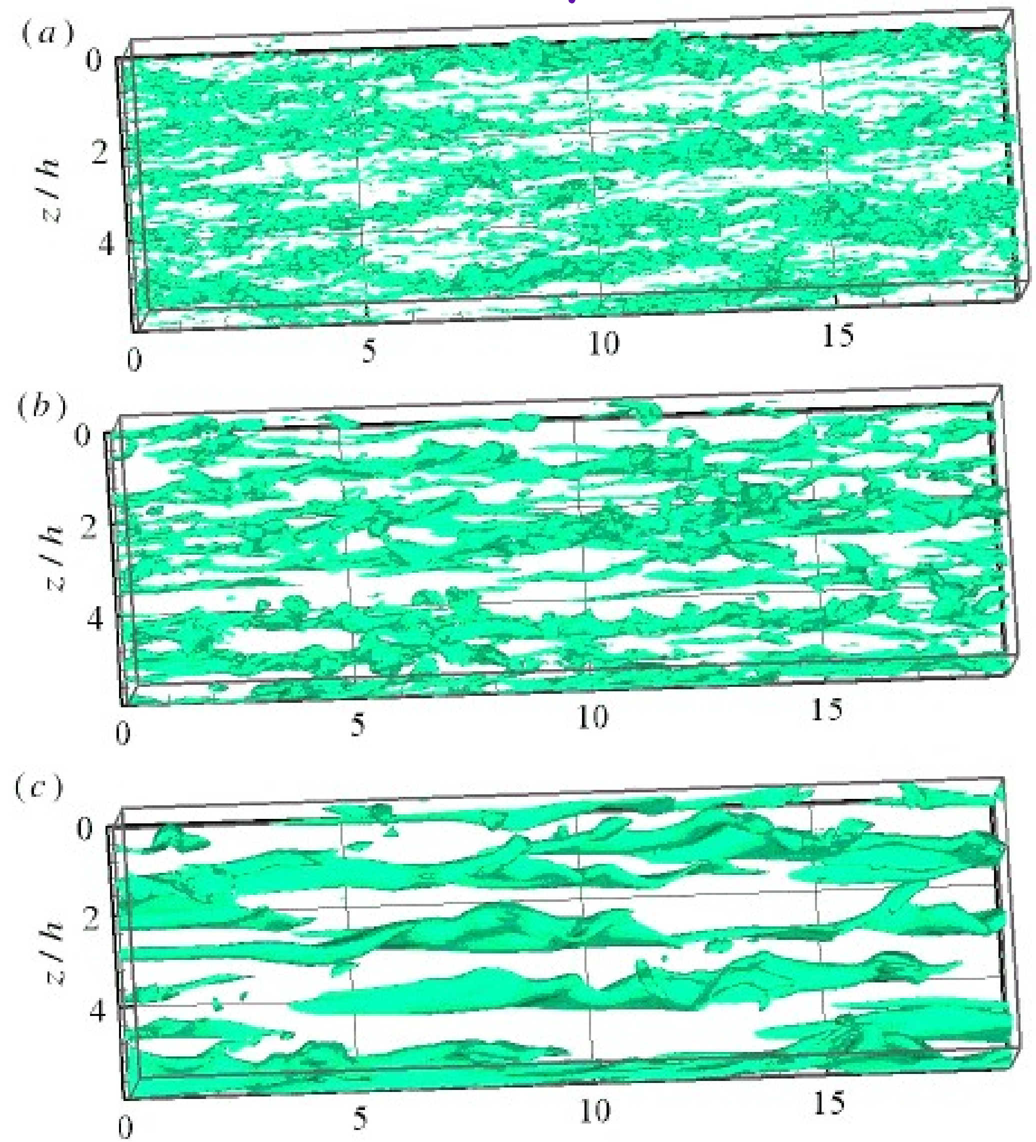
Streaks in wall-bounded turbulent flows

SSP mechanism active at small scale near walls



Kline et al. JFM 1967

SSP active also at large scale (not induced by small scale)



Hwang & Cossu ,*Phys. Rev. Lett.* 2010.

USING EXCITABILITY TO MANIPULATE THE FLOW

streaks & flow control

stable streaks (below critical amplitude):

→ strongly modify the basic flow 2D $U(y)$ → 3D $U(y,z)$

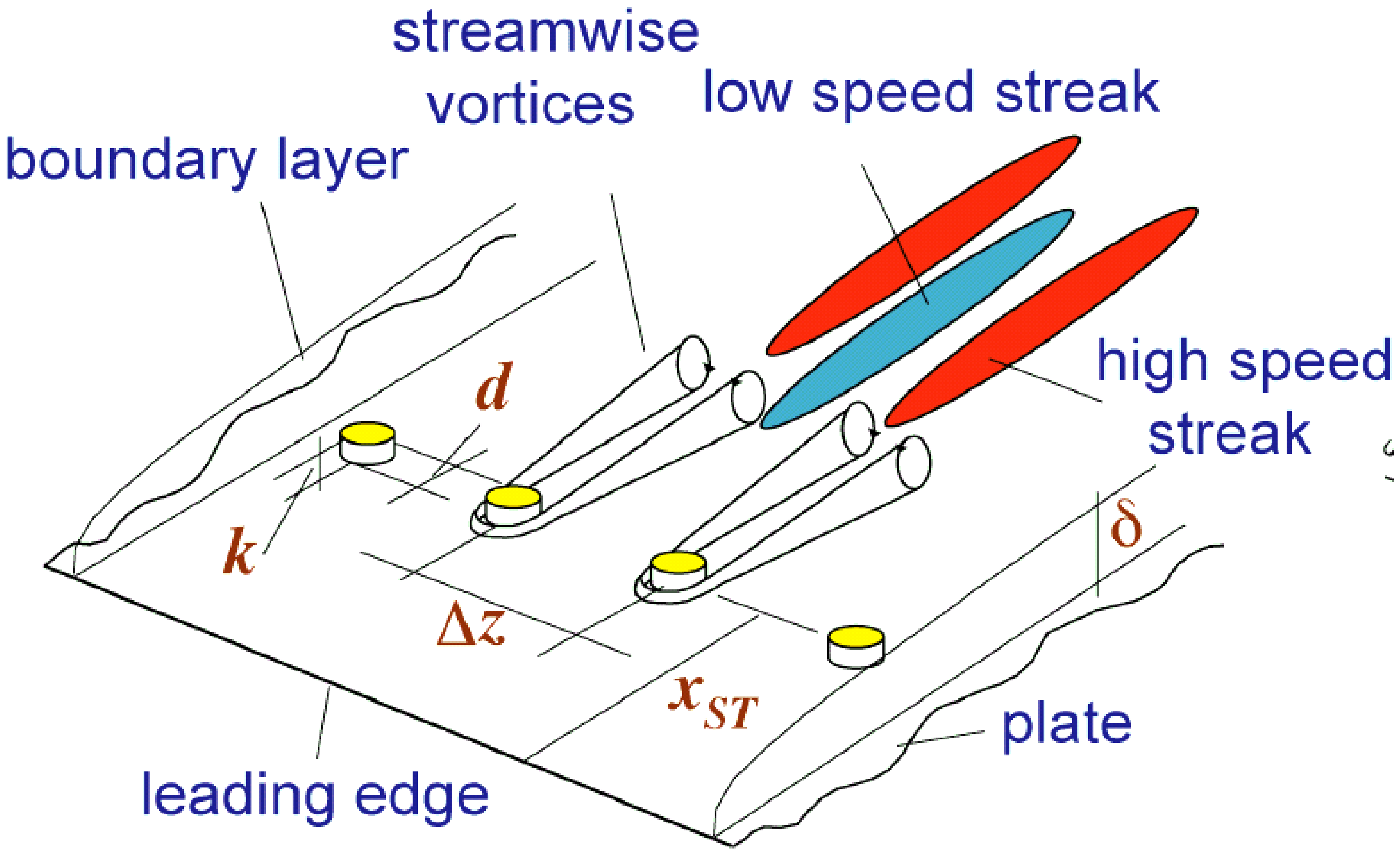
→ can be forced with low $O(1/Re^2)$ input energy

transient growth efficiently used to
modify basic flow

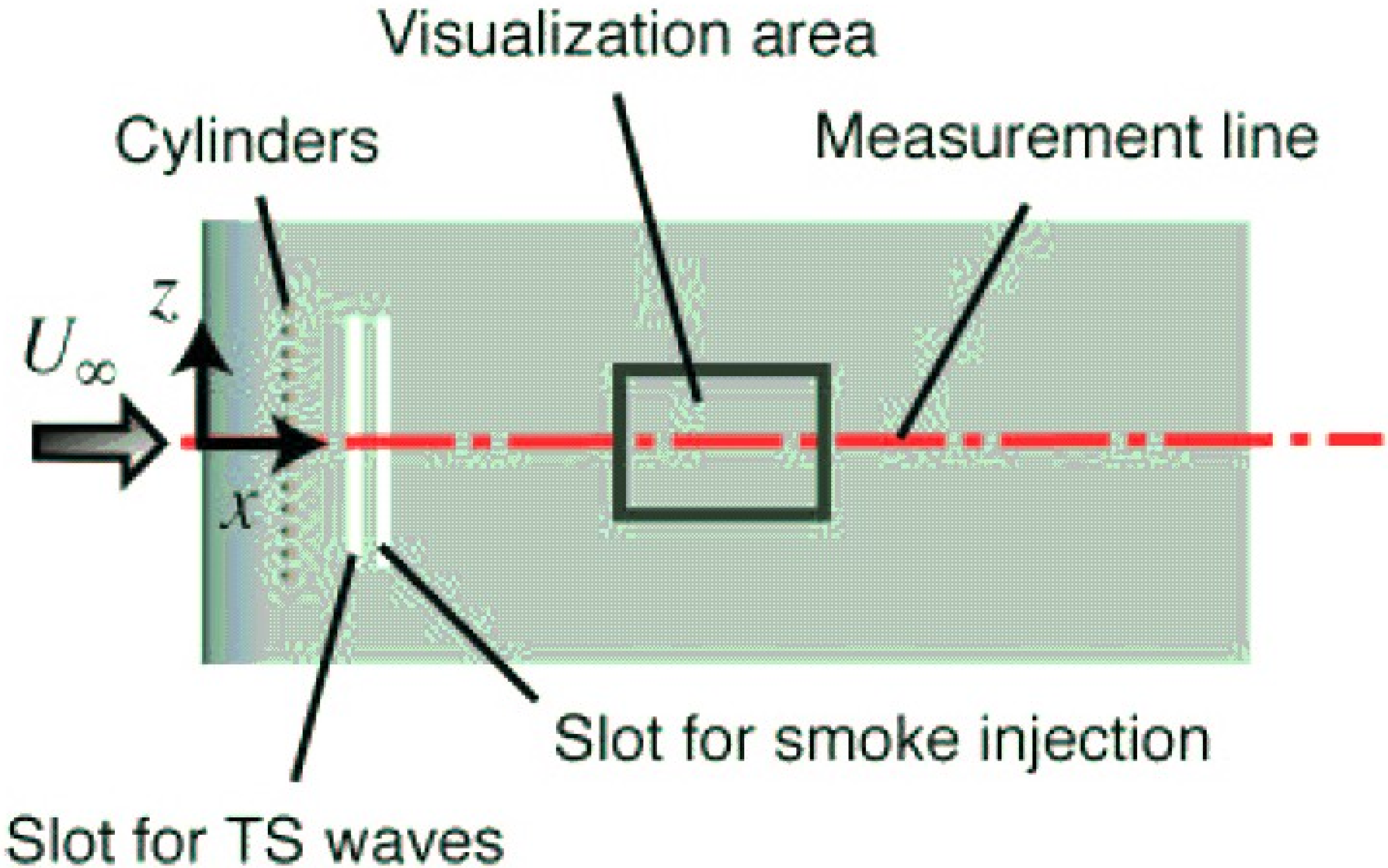
→ used to stabilize the flow

* lift-up effect: $O(Re^2)$ actuator energy amplifier
* kill one instability with another = vaccination

Forcing streaks with roughness elements



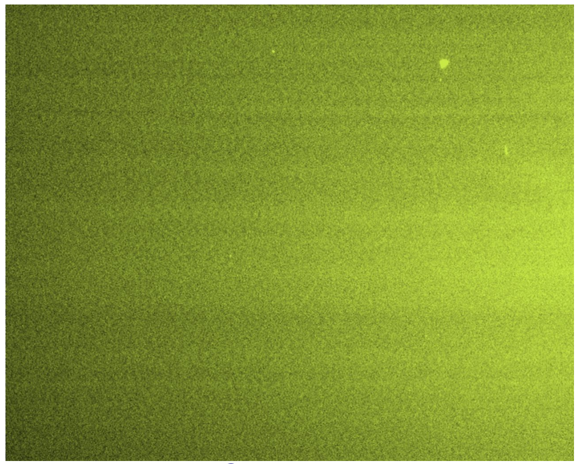
Experimental test of flow vaccination



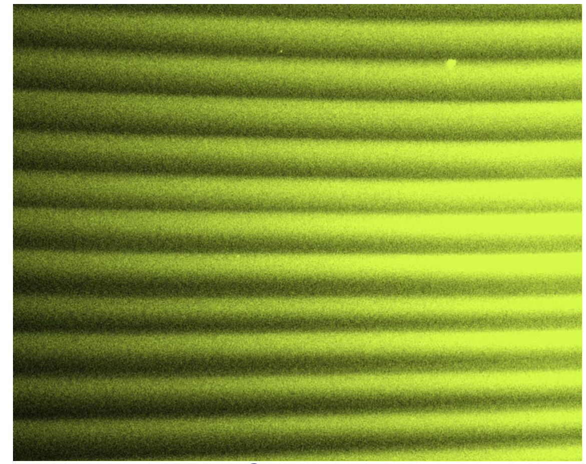
Transition delay with forced streaks

No streaks forced | Streaks forced

Basic flow

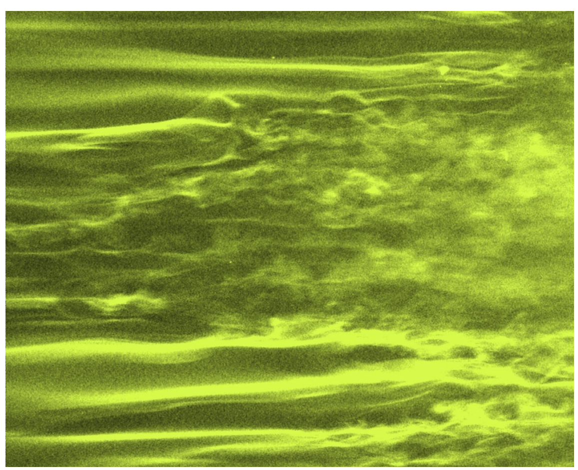


0 mV

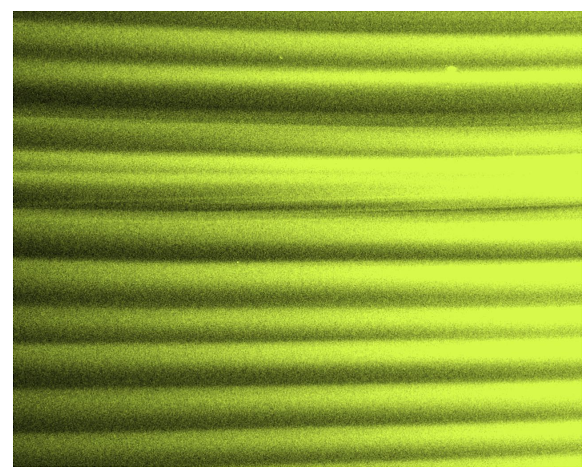


0 mV

Basic flow + tripping (unsteady) forcing



205 mV

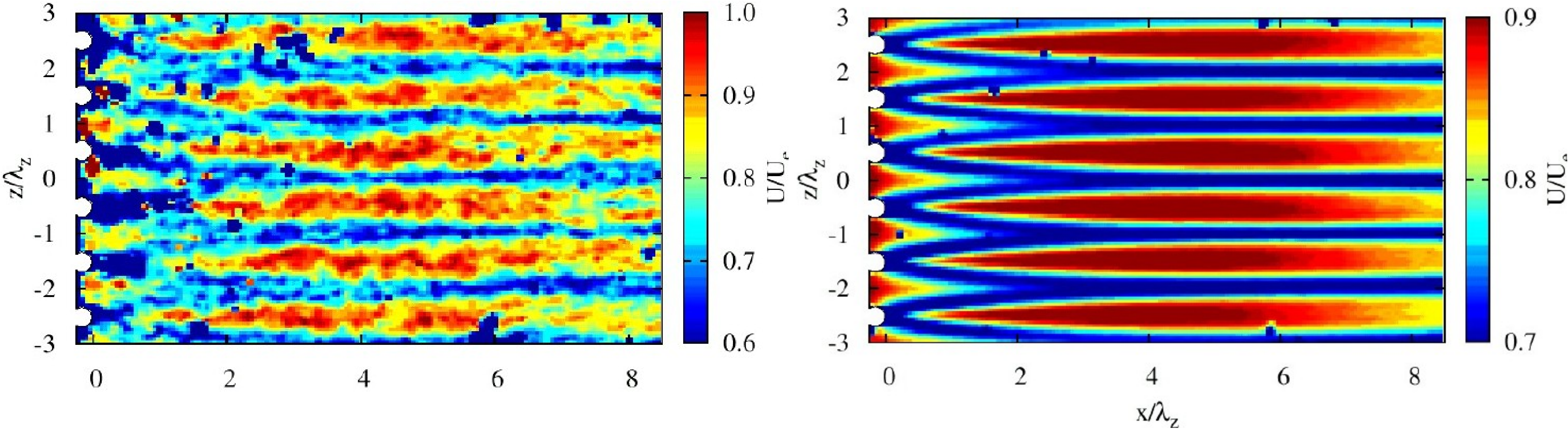


450 mV

Fransson, Talamelli, Brandt & Cossu, *Phys. Rev. Lett.* 2006

general idea: use optimal energy amplification to manipulate flows

idea extended to coherent non-normal amplification in turbulent flows



can probably be extended to MHD applications

THANK YOU FOR YOUR ATTENTION

http://www.imft.fr/Carlo-Cossu/
http://www.enseignement.polytechnique.fr/profs/mecanique/Carlo.Cossu