Fundamentals of Turbulence

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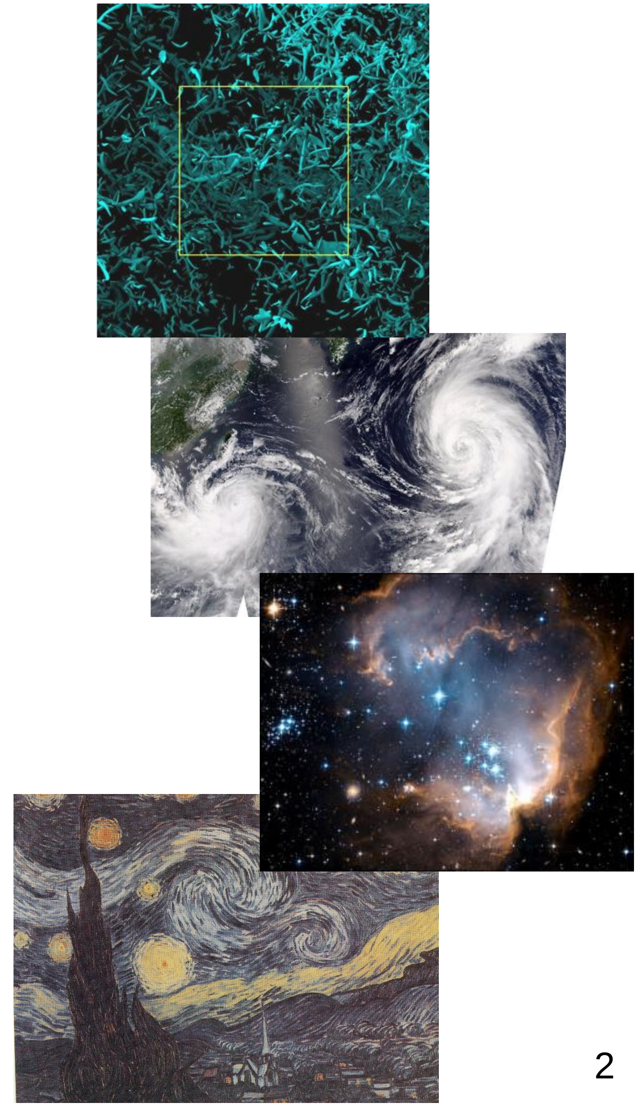
What is turbulence?

No exact definition. Loosely, random motion of fluid where many nonlinearly interacting modes are involved.

Plasma in astrophysical systems is typically turbulent and magnetized:

- magnetic dynamo action
- density structures in the Interstellar Medium
- star formation, scintillation of radio sources
- cosmic ray acceleration and scattering
- heat conduction in galaxy clusters
- transport in fusion devices
- solar corona/solar wind heating
- etc.



Hydrodynamics equations

$\mathbf{v}(\mathbf{x}, t)$ – velocity field $\qquad \nabla \cdot \mathbf{v} = 0$

Navier – Stokes Equation:

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \underbrace{-\nabla p}_{\text{pressure}} + \underbrace{\nu \nabla^2 \mathbf{v}}_{\text{viscosity}} + \underbrace{\mathbf{f}(\mathbf{x}, t)}_{\text{driving force}}$$

Incompressible hydrodynamics does not have linear waves.
Velocity fluctuations are referred to as "eddies"

Steve Morris/AirTeamImages

Hydrodynamics equations

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If L_0 is the typical velocity scale, V_0 is the typical velocity, then

$$\frac{(\mathbf{v}\cdot\nabla)\mathbf{v} \sim V_0^2/L_0}{\nu\nabla^2\mathbf{v} \sim \nu V_0/L_0^2} = V_0 L_0/\nu = Re$$

Reynolds number

A flow is turbulent when Re $\gg$ 1

Hydrodynamics equations

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Navier – Stokes Equation:

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \underbrace{-\nabla p}_{\text{pressure}} + \underbrace{\nu\nabla^2\mathbf{v}}_{\text{viscosity}} + \underbrace{\mathbf{f}(\mathbf{x},t)}_{\text{driving force}}$$

When the viscous and force terms are absent,
the equation is scale invariant:

$$\left.\begin{array}{ccc} \mathbf{x} & \longrightarrow & a\mathbf{x} \\ \mathbf{v} & \longrightarrow & a^h\mathbf{v} \\ t & \longrightarrow & a^{1-h}t \end{array}\right\}$$

this scaling transform does not change the equation, for any h

Scale invariance and correlation functions

$$\begin{aligned}\mathbf{x} &\rightarrow a\mathbf{x}\\ \mathbf{v} &\rightarrow a^h\mathbf{v}\end{aligned}$$

Velocity is not universal since it depends on the reference frame.
Consider velocity difference

$$\delta\mathbf{v}_r = \mathbf{v}(\mathbf{x}+\mathbf{r},t) - \mathbf{v}(\mathbf{x},t)$$

Scale invariance means that

$$\delta\mathbf{v}_r \propto r^h$$

For example, the second-order structure function has the form

$$\boxed{S_2(r) = \langle(\delta v_r)^2\rangle \propto r^{2h}}$$

Structure function and spectrum of turbulence

Fourier transform of velocity field: $\mathbf{v}(\mathbf{k}) = \int_V \mathbf{v}(\mathbf{x}) \exp(-i\mathbf{k}\cdot\mathbf{x})\, d^3x$

Energy spectrum: $E(k) \propto \langle |v(k)|^2 \rangle 4\pi k^2$ ← 3D volume element

Energy spectrum is a Fourier transform of the second-order structure function.

The rule is:

If the structure function scales as: $S_2(r) = \langle (\delta v_r)^2 \rangle \propto r^{2h}, \quad 2h < 2$

then the energy spectrum scales as: $E(k) \propto \langle |\mathbf{v}(\mathbf{k})|^2 \rangle\, k^2 \propto k^{-1-2h}$

Need to find h !

Energy conservation

$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{f}$$

Energy: $E = \frac{1}{2}\int \mathbf{v}^2(\mathbf{x})\, d^3x$

Assume that velocity field vanishes at infinity or has periodic boundary conditions

$$\frac{\partial}{\partial t}\int \frac{1}{2}\mathbf{v}^2\, d^3x + \int \frac{1}{2}(\mathbf{v}\cdot\nabla)\mathbf{v}^2\, d^3x = -\nu\int \frac{1}{2}(\nabla\mathbf{v})^2\, d^3x + \int \mathbf{v}\cdot\mathbf{f}\, d^3x$$

0 (by parts)

Nonlinear interaction does not change energy. Redistributes energy over Fourier harmonics.

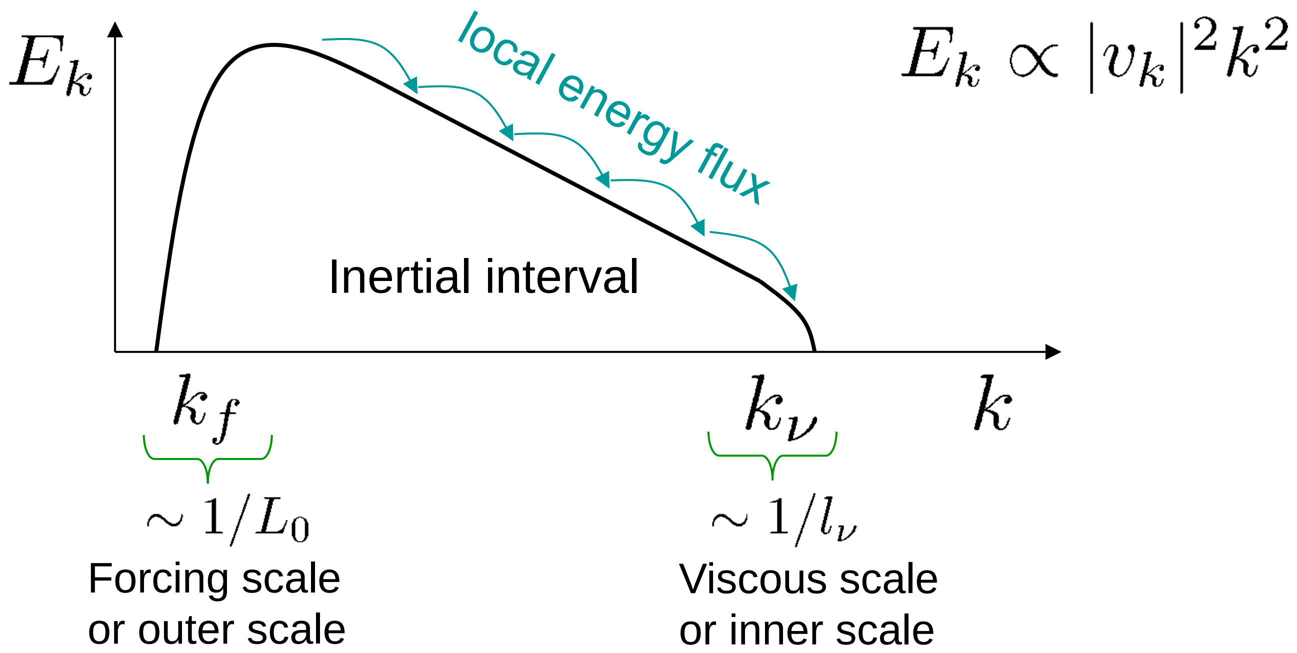
viscous energy dissipation at small scale

energy supply at large scales

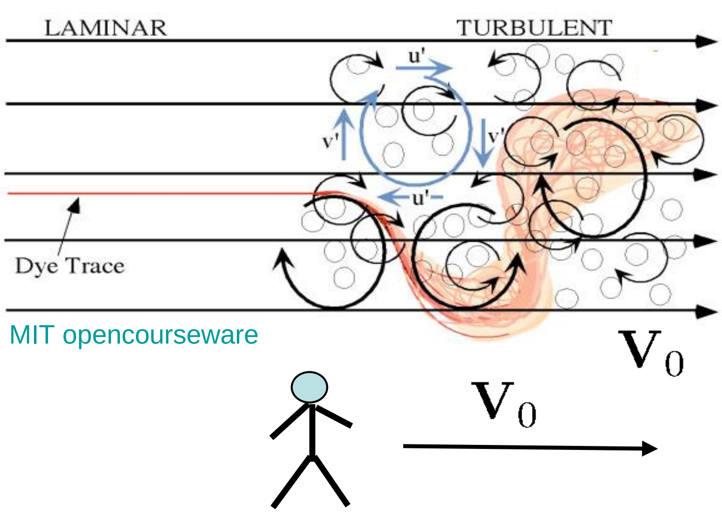
Kolmogorov phenomenology

When energy is supplied at large scales, it gets redistributed over fluctuations of different scales, and removed at small scales due to viscosity.

In a steady state, rate of energy supply = rate of energy transfer = rate of energy dissipation: $\epsilon = \nu \int \frac{1}{2}(\nabla \times \mathbf{v})^2\, d^3x = \mathrm{const}$



"Locality" of turbulence



$$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v}$$

Navier-Stokes equation is invariant under the Galilean transform:

$$\mathbf{v} = \mathbf{V}_0 + \mathbf{v}'$$
$$\mathbf{x} = \mathbf{V}_0 t + \mathbf{x}'$$
$$t = t'$$

Consider eddies at scale r. Much larger eddies, whose velocity is more uniform, do not affect the dynamics at scale r. Neither do much smaller eddies, which are weaker and provide incoherent disturbances.

Kolmogorov spectrum

A.N. Kolmogorov
(1903-1987)

1. Rate of energy transfer $\epsilon = [cm^2/sec^3]$.

   Velocity difference δv_r should depend on r and ϵ. By dimensional analysis:

$$\delta v_r \sim (\epsilon r)^{1/3}$$

2. Energy density $\mathcal{E} \sim (\delta v_r)^2$.

   Time of energy transfer $t_r \sim r/\delta v_r$.

   Rate of energy transfer $\epsilon \sim \mathcal{E}/t_r = \text{const}$

$$\Longrightarrow \delta v_r \sim (\epsilon r)^{1/3}$$

$$\boxed{E_k \propto k^{-5/3}}$$

Kolmogorov spectrum of turbulence

Kolmogorov theory

An exact relation for scales $r \ll L_0$, where L_0 is the scale where energy is supplied to the system.

and, additionally, $r \gg l_\nu$, where l_ν is the scale where energy is removed from the system by viscosity:

$$\boxed{S_{rrr} \equiv \langle (\delta \mathbf{v}_r \cdot \mathbf{n})^3 \rangle = -\frac{4}{5} \epsilon r}$$

Kolmogorov 4/5 law

The range of scales, $l_\nu \ll r \ll L_0$, is called the inertial interval. Homogeneous and isotropic turbulence in the inertial interval obeys the Kolmogorov 4/5 law.

This is the exact law. It explains in what sense the phenomenological relation $\delta v_r \sim (\epsilon r)^{1/3}$ must be understood.

Kolmogorov spectrum: numerical simulations

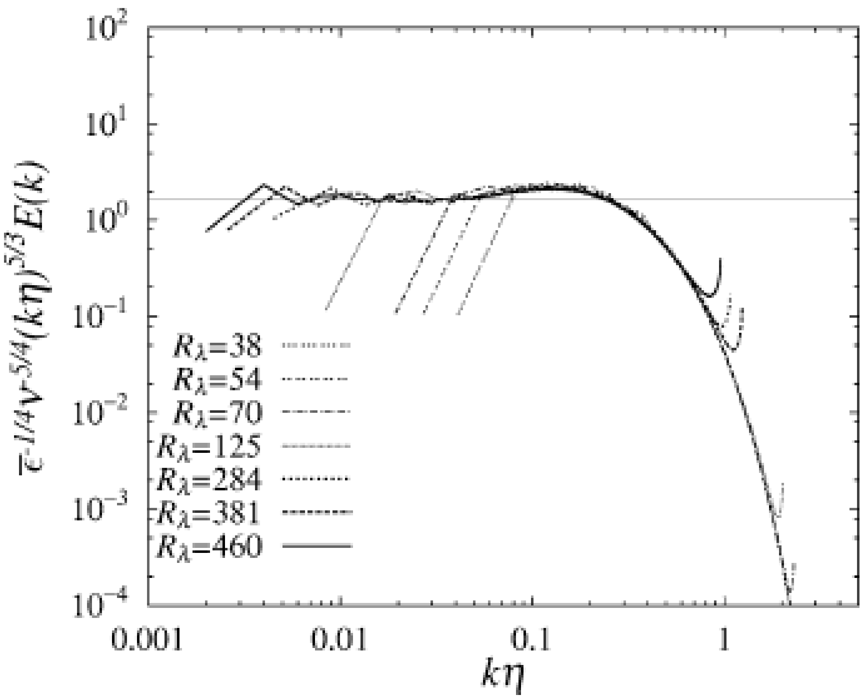


Fig. 1. Scaled energy spectra, $\bar{\epsilon}^{-1/4}\nu^{-5/4}(k\eta)^{5/3}E(k)$. The inertial range is between $0.007 \leqslant k\eta \leqslant 0.04$. $K = 1.64 \pm 0.04$. The horizontal line indicates $K = 1.64$.

Gotoh 2002

Hydrodynamic turbulence is universal. Has same spectrum (close to -5/3) independent of large-scale driving and small-scale dissipation

Turbulence of a conducting fluid

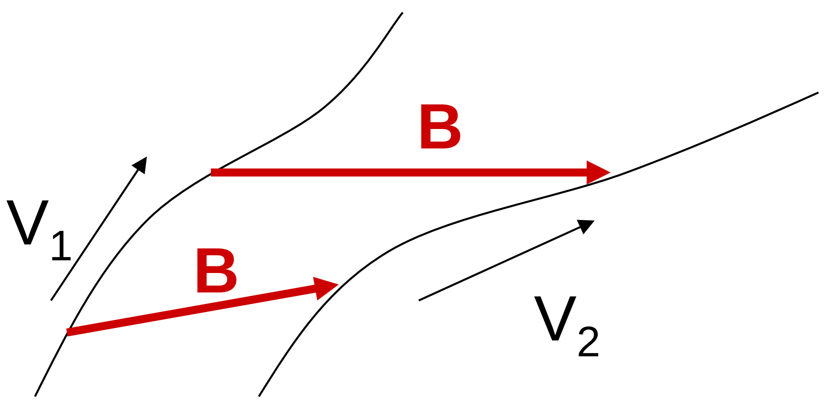
always gets magnetized and turns into MHD turbulence

$$\begin{cases} \partial_\tau v + (v \cdot \nabla) v = -\nabla p + \cancel{(\nabla \times B) \times B} + \nu \Delta v + f \\ \partial_\tau B = \nabla \times (v \times B) + \eta \Delta B \end{cases}$$

Kinematic dynamo

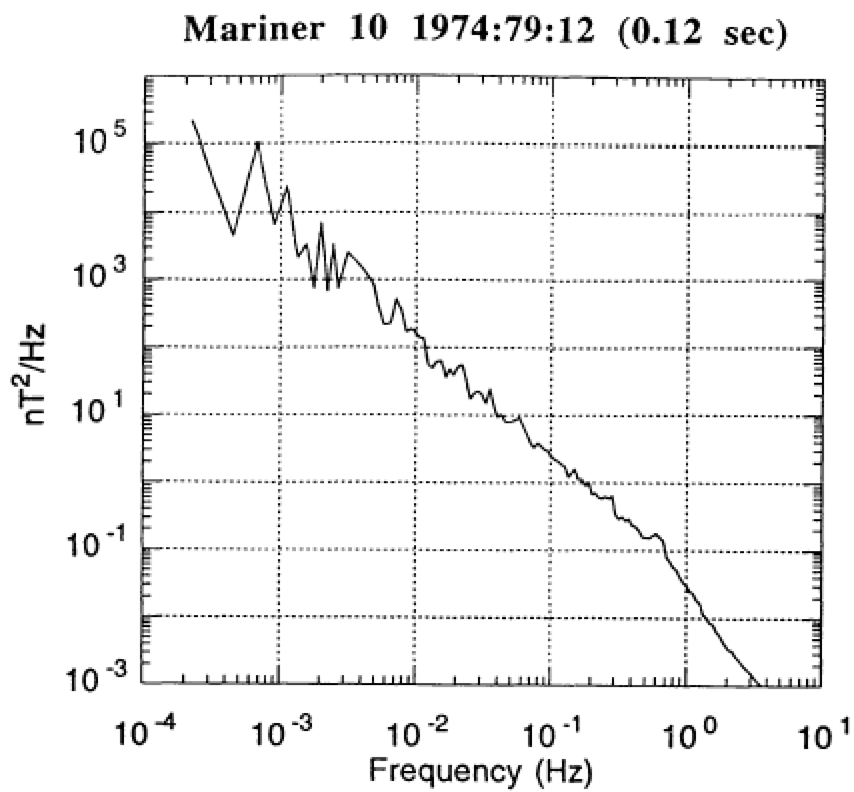
$Re = VL/\nu$ - Reynolds number

$R_m = VL/\eta$ - magnetic Reynolds number



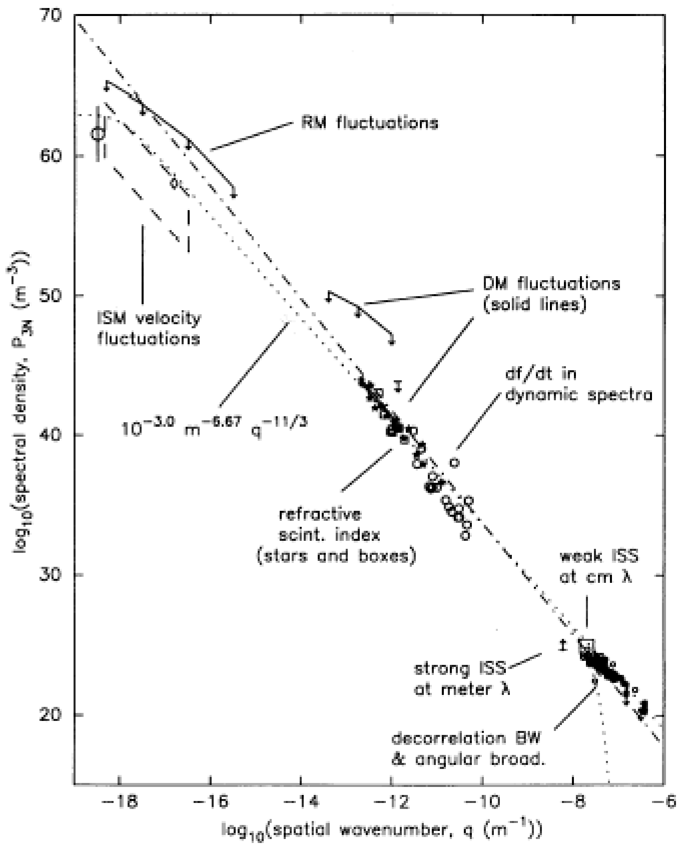
Magnetic turbulence in nature

energy spectra



Solar wind

[Goldstein, Roberts, Matthaeus (1995)]



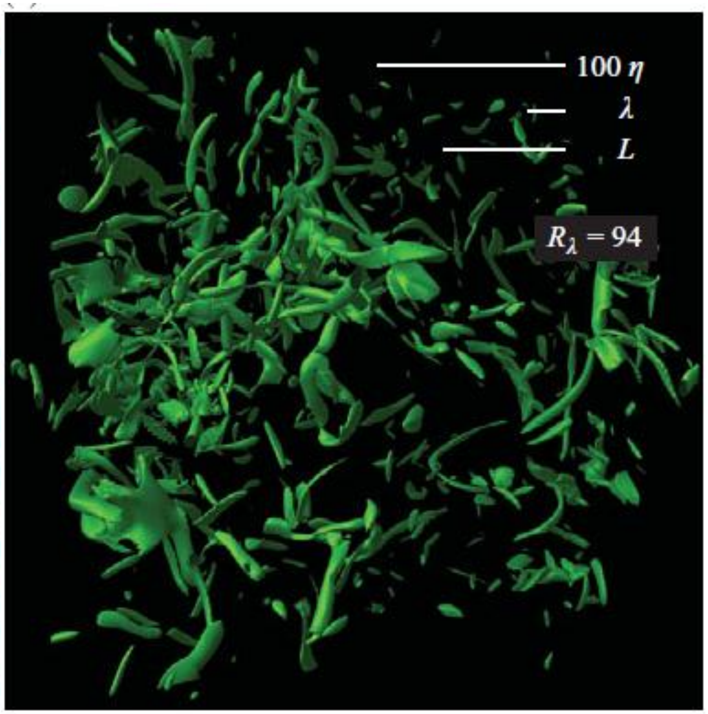
ISM

[Armstrong, Rickett, Spangler (1995)]

Magnetic turbulence in numerical simulations

structures

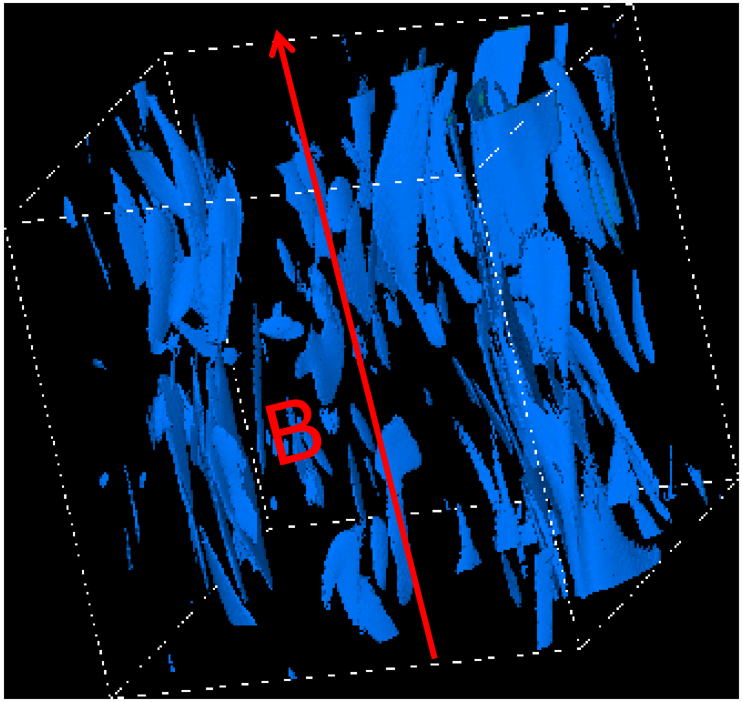
Neutral fluid, B=0



Filaments

Ishihara et al (2007)

MHD, B $\neq$ 0



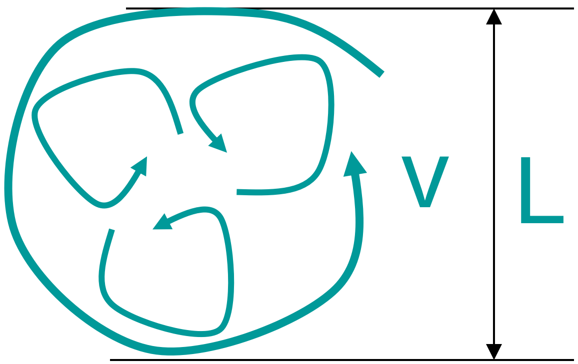
"Ribbons" stretched along B

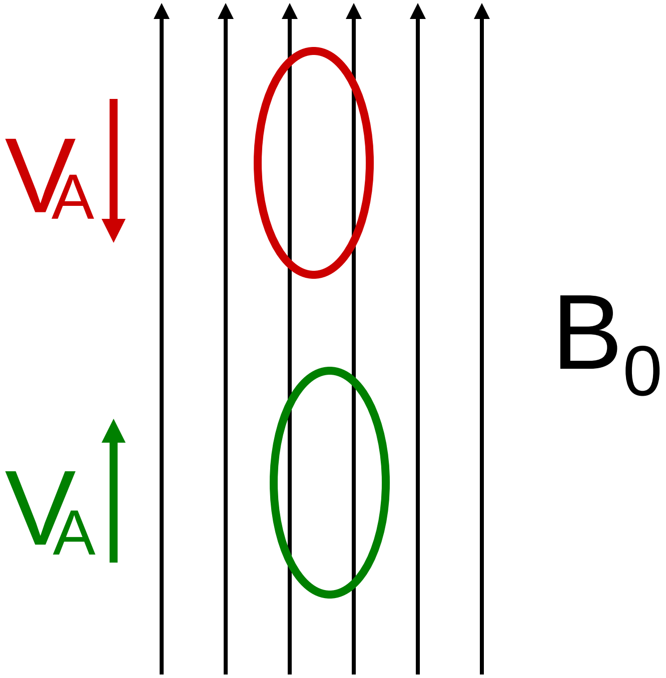
Biskamp & Muller (2000)

Nature of Magnetohydrodynamic (MHD) turbulence

HD turbulence:
interaction of eddies

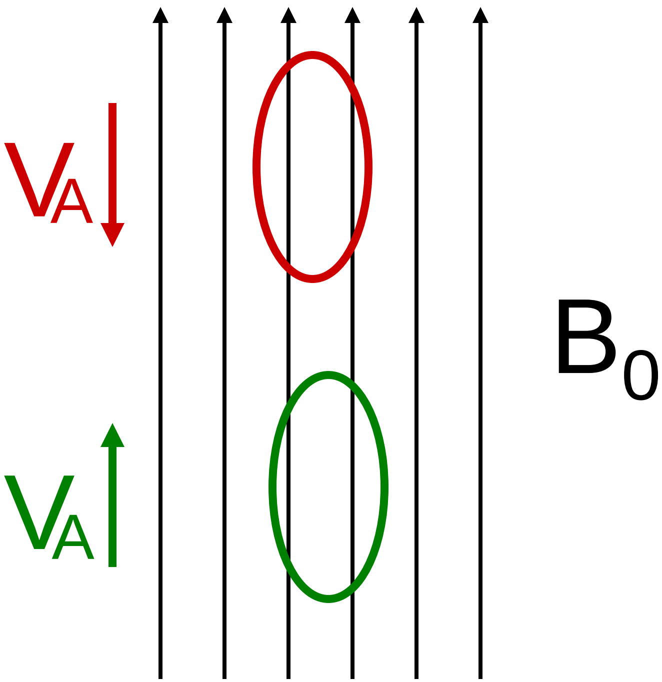
MHD turbulence:
interaction of wave packets
moving with Alfven velocities



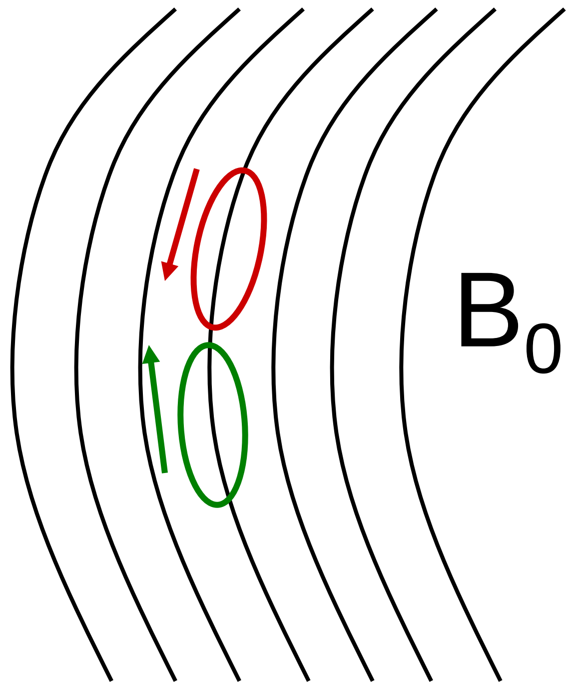


$$\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$$

Guide field in MHD turbulence



B_0 imposed by
external sources



B_0 created by
large-scale eddies

Magnetohydrodynamic (MHD) turbulence

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{4\pi\rho_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{v}$$
$$\partial_t \mathbf{B} = \nabla\times[\mathbf{v}\times\mathbf{B}] + \eta\nabla^2\mathbf{B}$$

Separate the uniform magnetic field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

And introduce the Elsasser variables $\mathbf{z}^{\pm} = \mathbf{v} \pm \frac{1}{\sqrt{4\pi\rho_0}}\mathbf{b}$

Then the equations take a symmetric form:

$$\partial_t \mathbf{z}^{+} - (\mathbf{v}_A\cdot\nabla)\mathbf{z}^{+} + (\mathbf{z}^{-}\cdot\nabla)\mathbf{z}^{+} = -\nabla P$$
$$\partial_t \mathbf{z}^{-} + (\mathbf{v}_A\cdot\nabla)\mathbf{z}^{-} + (\mathbf{z}^{+}\cdot\nabla)\mathbf{z}^{-} = -\nabla P$$

With the Alfven velocity $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$

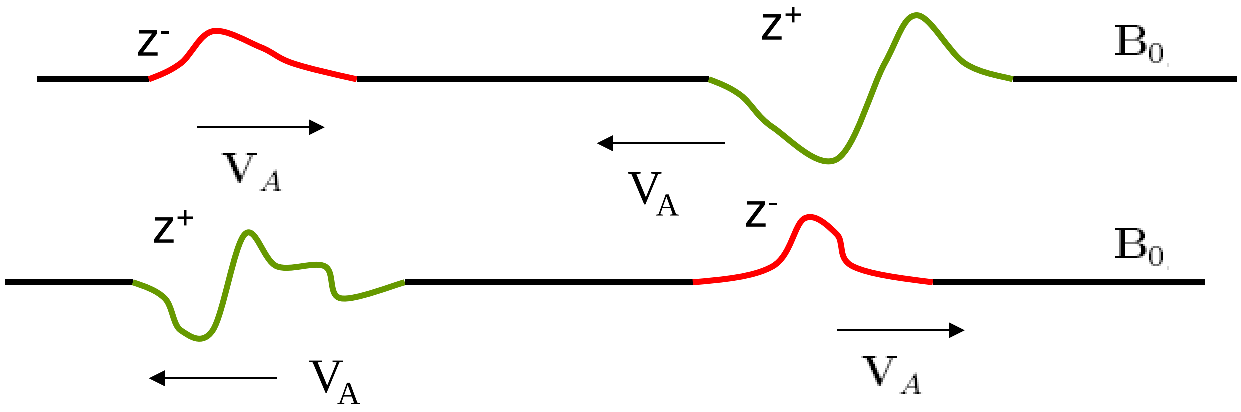
The uniform magnetic field cannot be removed by a Galilean transform!

Alfvenic turbulence

$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

Ideal system conserves the Elsasser energies

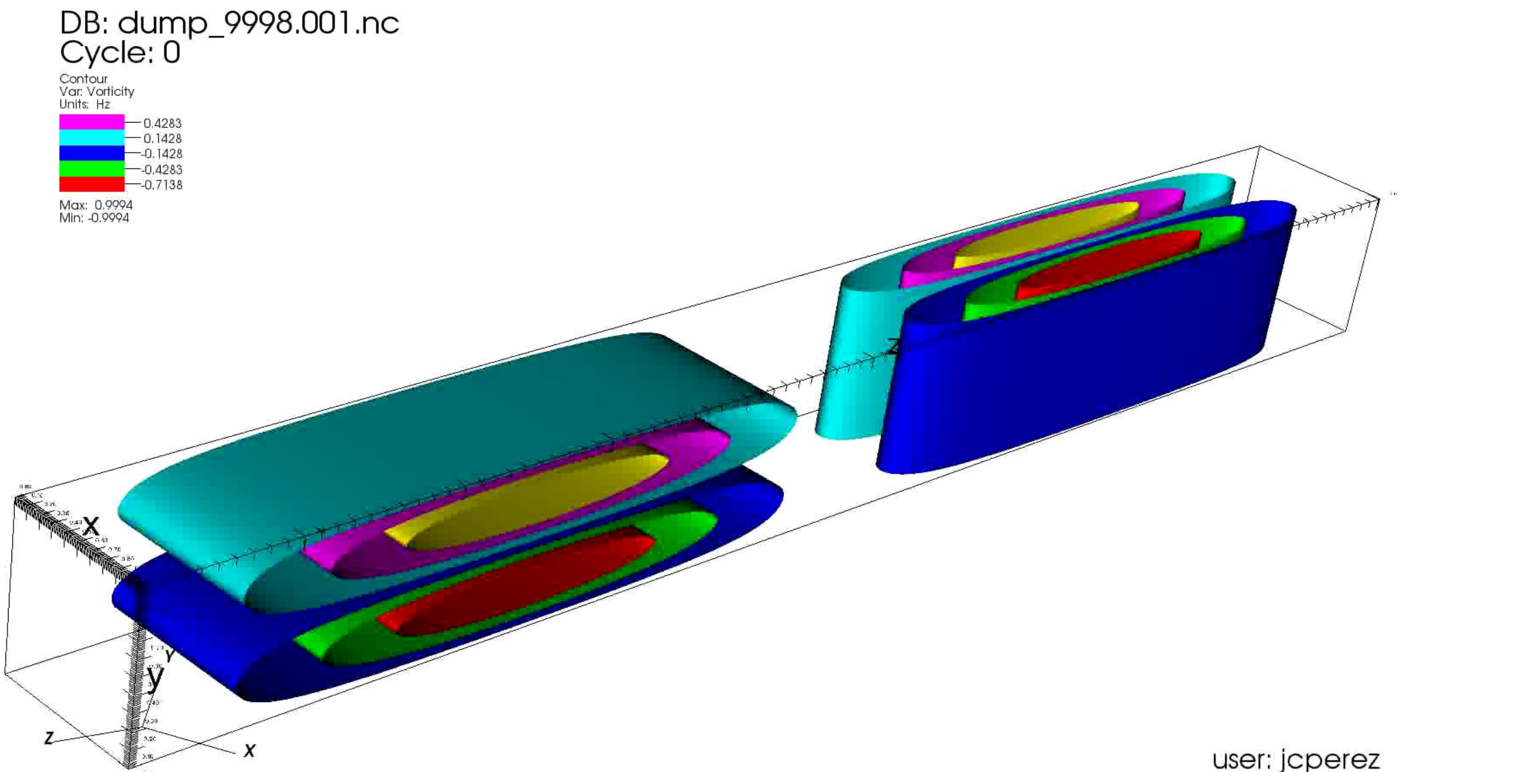
$$E^{+} = \int (\mathbf{z}^{+})^2 \, d^3x \qquad\qquad E = \frac{1}{2}\int (v^2 + b^2) d^3x$$
$$\equiv$$
$$E^{-} = \int (\mathbf{z}^{-})^2 \, d^3x \qquad\qquad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$



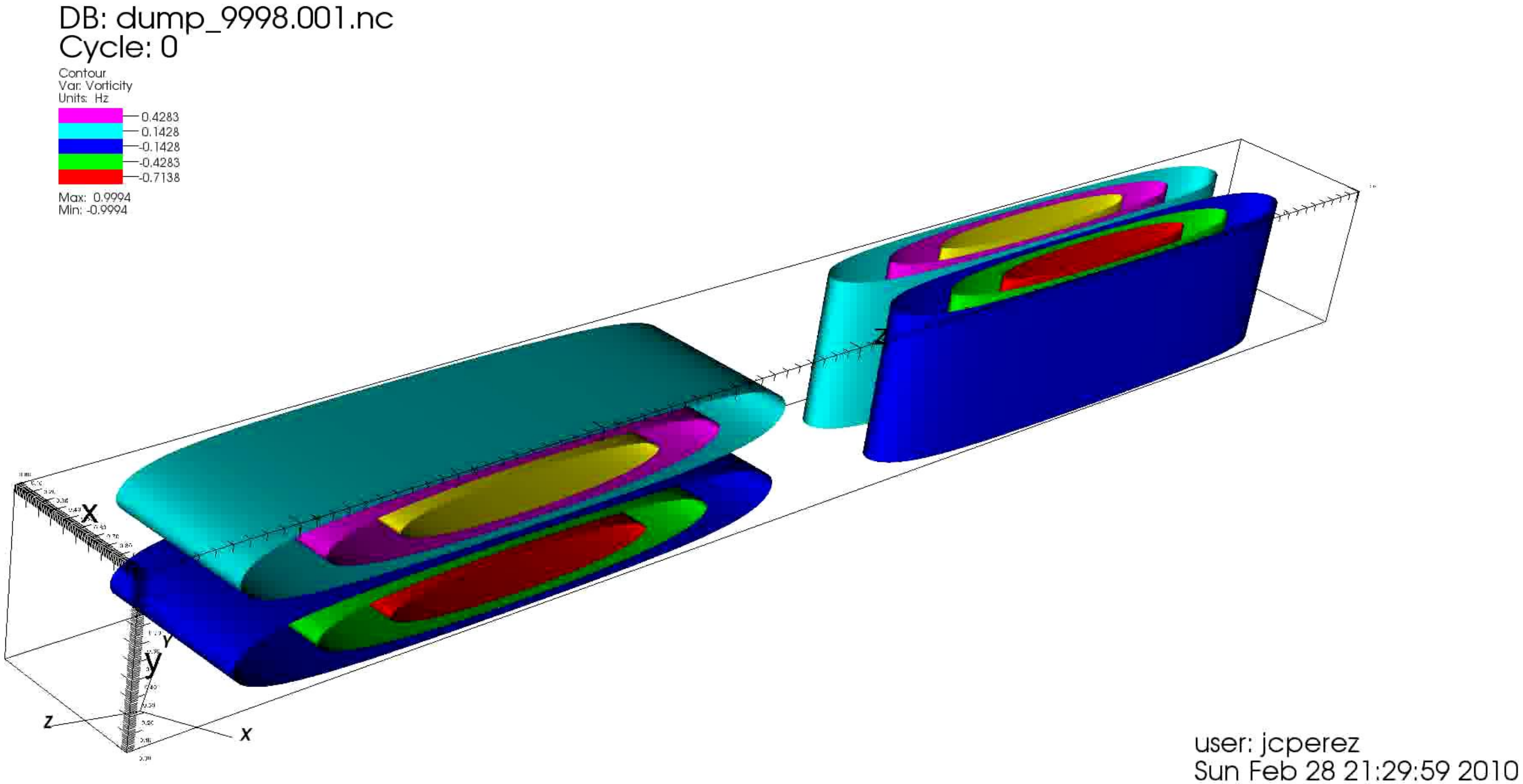
After interaction, shape of each packet changes, but energy does not.

$E^+ \sim E^-$: balanced case. $\quad E^+ \gg E^-$: imbalanced case

Strong MHD turbulence: collision of eddies

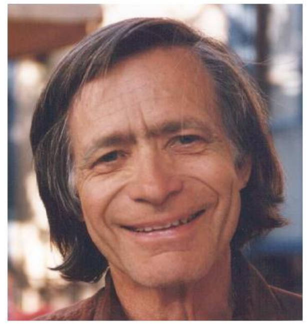


Strong MHD turbulence: collision of eddies

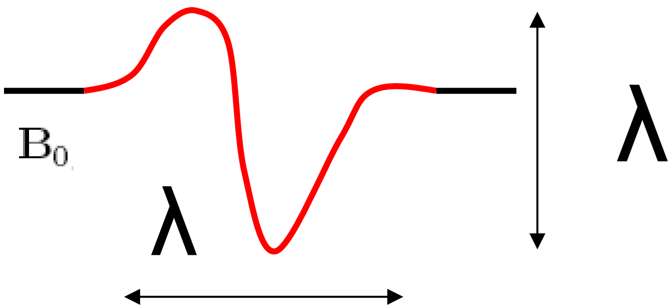


Iroshnikov-Kraichnan spectrum

R..S.Iroshnikov (1937-1991)



R. H. Kraichnan (1928-2008)



$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$
$$\partial_t \mathbf{z}^- + (\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)\mathbf{z}^- = -\nabla P$$

during one collision:

$$\Delta\delta v_\lambda \sim (\delta v_\lambda^2/\lambda)(\lambda/V_A)$$

number of collisions required to deform packet considerably:

$$N \sim (\delta v_\lambda/\Delta\delta v_\lambda)^2 \sim (V_A/\delta v_\lambda)^2$$

$$\boxed{\tau_{IK}(\lambda) \sim N\lambda/V_A \sim \lambda/\delta v_\lambda (V_A/\delta v_\lambda)}$$

Constant energy flux: $\delta v_\lambda^2/\tau_{IK}(\lambda) = \text{const}$

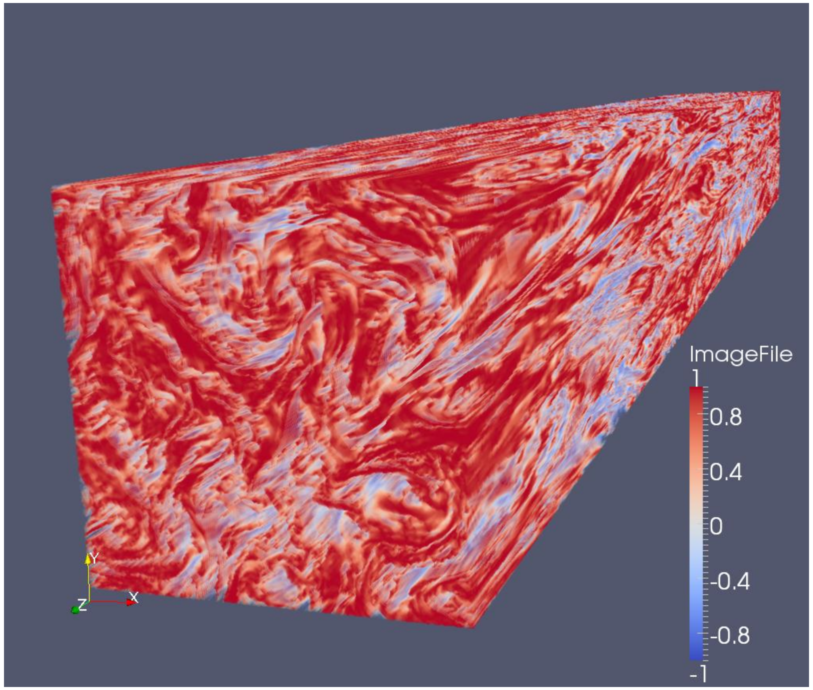
$$\Downarrow$$

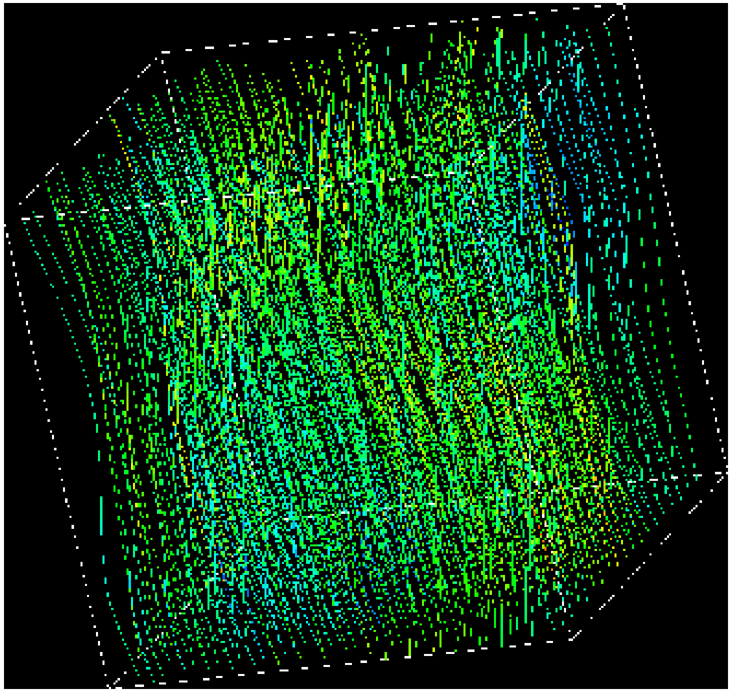
$$\Leftarrow \quad \delta v_\lambda \propto \lambda^{1/4},$$

$$\boxed{E_{IK}(k) = |\delta v_k|^2 k^2 \propto k^{-3/2}.}$$

MHD spectrum is isotropic [Iroshnikov (1963); Kraichnan (1965)]

MHD turbulence is locally anisotropic





W.-C. Muller et al (2005)

Goldreich-Sridhar spectrum

Anisotropy of “eddies”

P. Goldreich

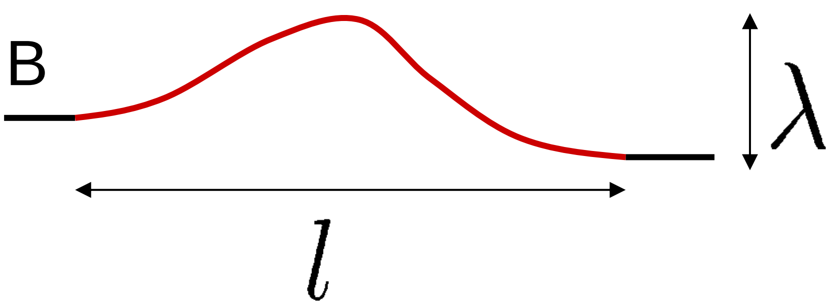
S.Sridhar

$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$

$$\partial_t \mathbf{z}^- + \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)\mathbf{z}^-}_{\boxed{V_A/l \sim \delta b_\lambda/\lambda}} = -\nabla P$$

Critical Balance

$$l \gg \lambda \qquad \delta v_\lambda \sim \delta b_\lambda \propto \lambda^{1/3}$$

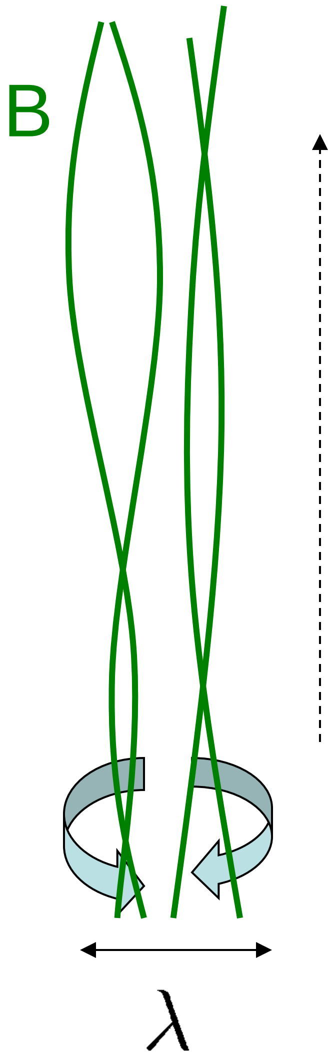


Energy spectrum:

$$\boxed{E_{GS}(k_\perp) \propto k_\perp^{-5/3}}$$

[Goldreich & Sridhar 1995]

Goldreich-Sridhar theory: critical balance



$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$
$$\partial_t \mathbf{z}^- + \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)\mathbf{z}^-}_{V_A/l \sim \delta b_\lambda/\lambda} = -\nabla P$$

$$l \sim V_A \tau_\lambda \quad \Rightarrow \quad l \propto \lambda^{2/3}$$

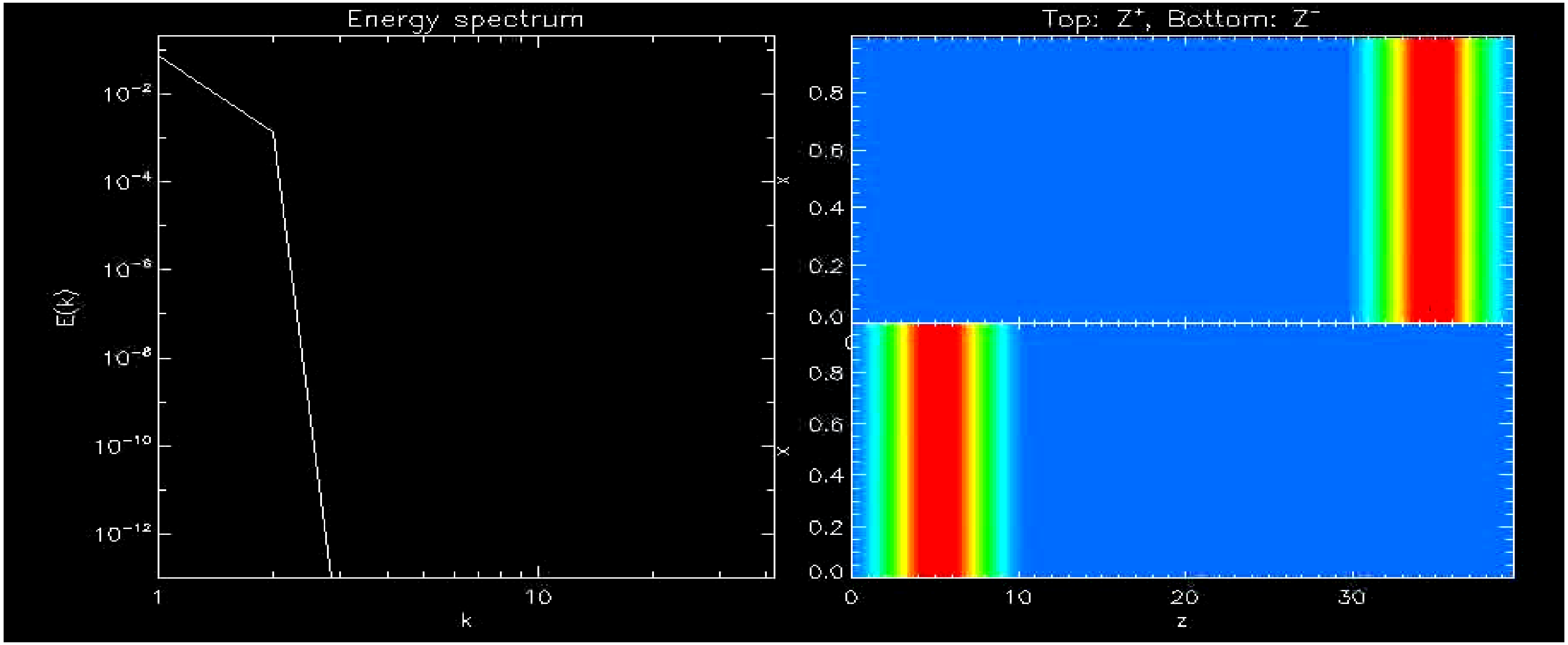
Causality

GS Critical balance

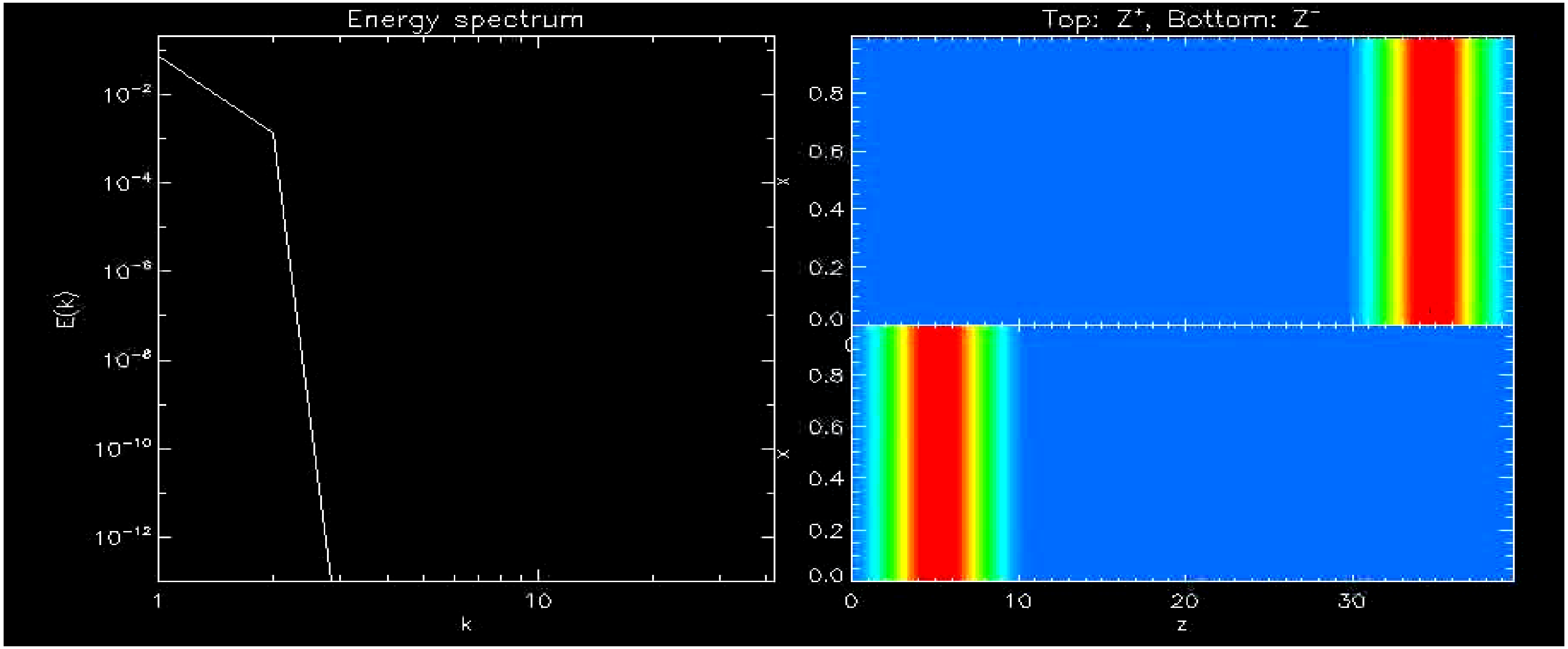
$$\tau_\lambda \sim \lambda/\delta v_\lambda$$

Correlation time of fluctuations, or eddy turnover time

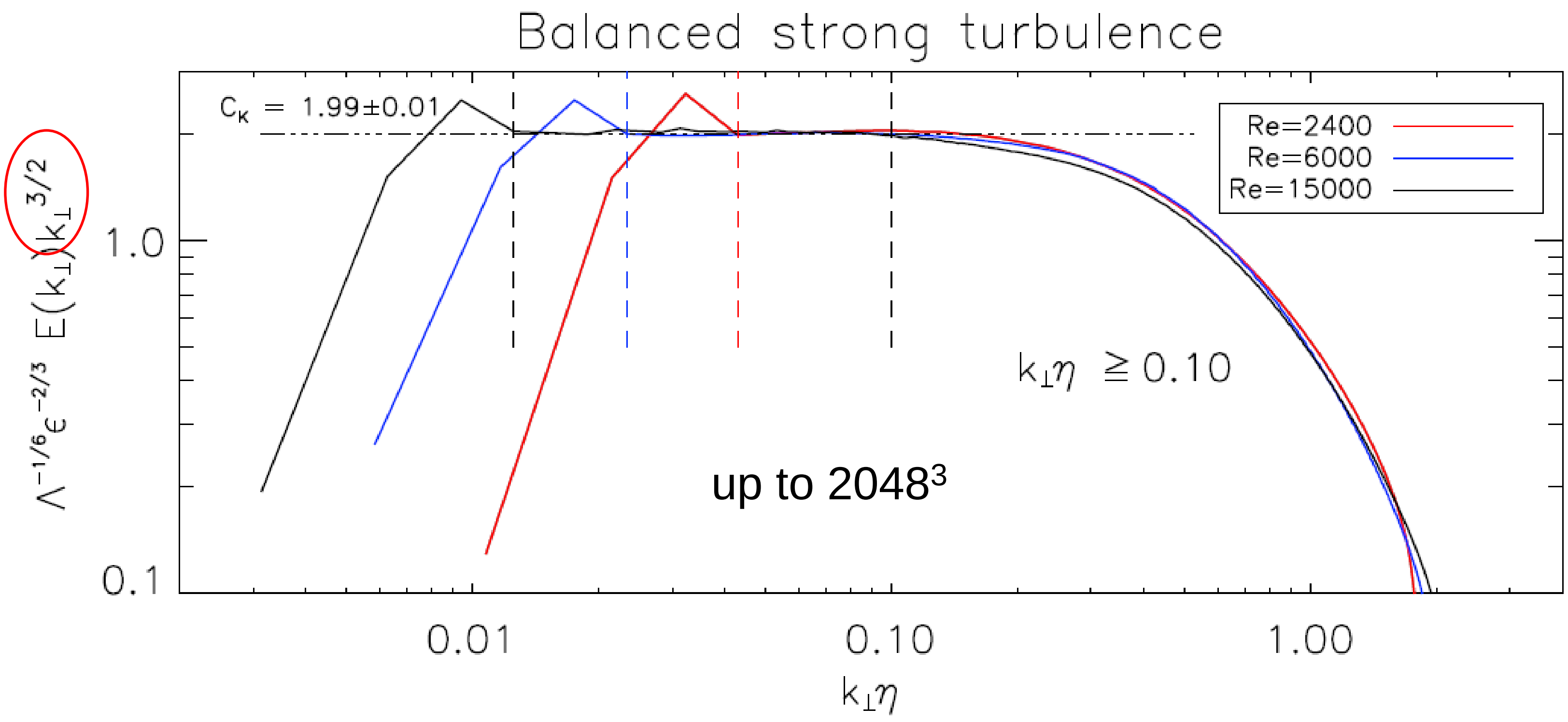
Strong collision of eddies



Strong collision of eddies



Spectrum of strong MHD turbulence: balanced case



Computational resources: DoE 2010 INCITE,
Machine: Intrepid, IBM BG/P at Argonne Leadership Computing Facility

Perez et al, Phys Rev X (2012)

Conclusions of Part I

- We have reviewed phenomenological description of hydrodynamic and MHD turbulence

- Astrophysical turbulence is typically magnetic

- ---See Part II for MDH turbulence