Numerical modeling of MHD turbulence

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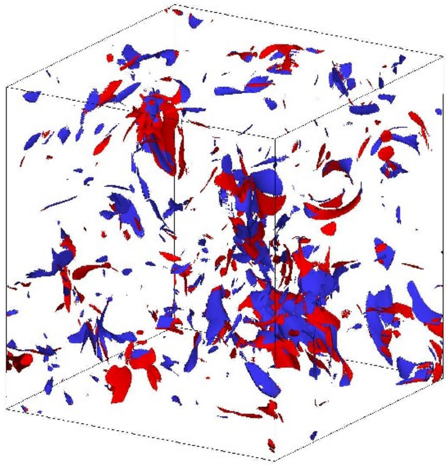


Les Houches, 26.02.2013

Outline

- Why MHD turbulence?
- Why numerical modeling?
- How to model numerically?
- Applications

Why MHD turbulence?

Simplest macroscopic description of a conducting flow

- fluid description (not kinetic)
- one fluid – quasi neutral
- non relativistic

Magnetohydrodynamic field equations

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla P + \boldsymbol{j} \times \boldsymbol{B} + \nu \Delta \boldsymbol{v} + \boldsymbol{F_v}$$ Momentum equation

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{v} + \eta \Delta \boldsymbol{B} + \boldsymbol{F_B}$$ Induction equation

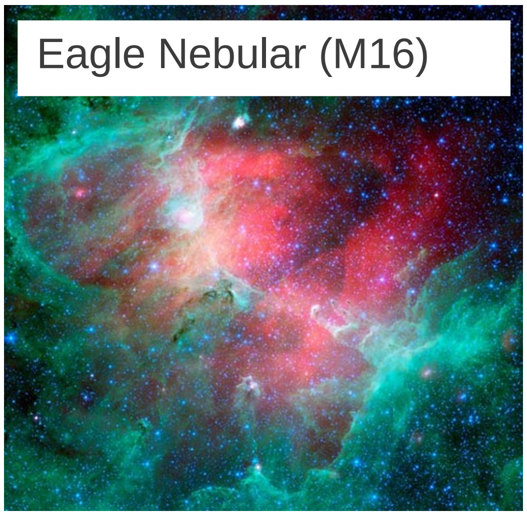
$$\nabla \cdot \boldsymbol{B} = \nabla \cdot \boldsymbol{v} = 0$$ Incompressible flow

$\boldsymbol{F}_v$ and $\boldsymbol{F}_B$: forcing terms

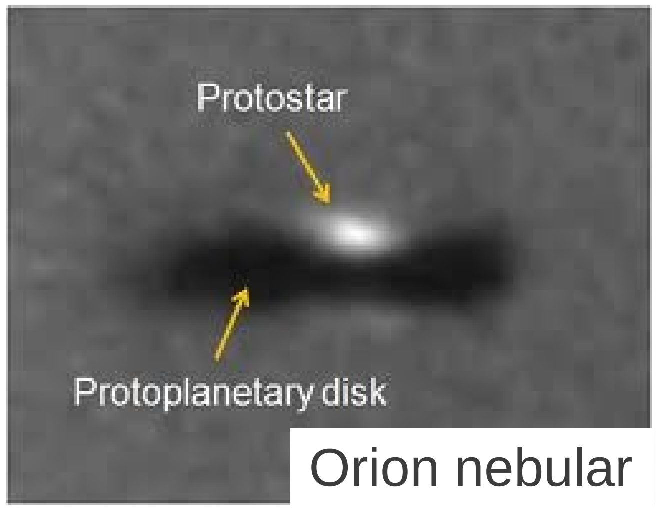
+ boundary conditions!!!

Why MHD turbulence?

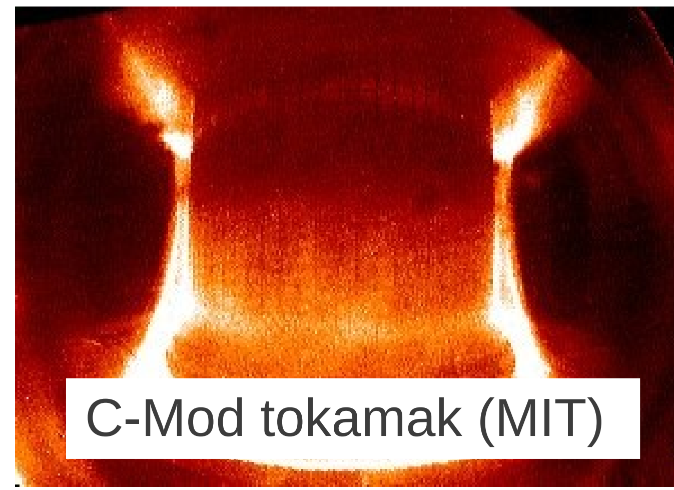
- **Fundamental questions**: universality, scaling laws, isotropy, ...
- **Applications :** Heliosphere, Solar wind, Dynamos, Interstellar medium, Fusion experiments, ...



Eagle Nebular (M16)



Orion nebular



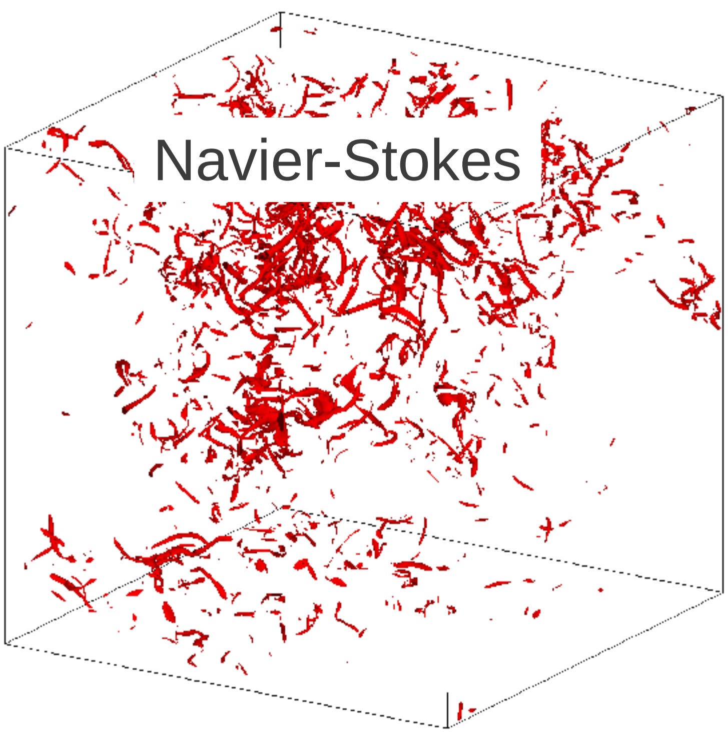
C-Mod tokamak (MIT)

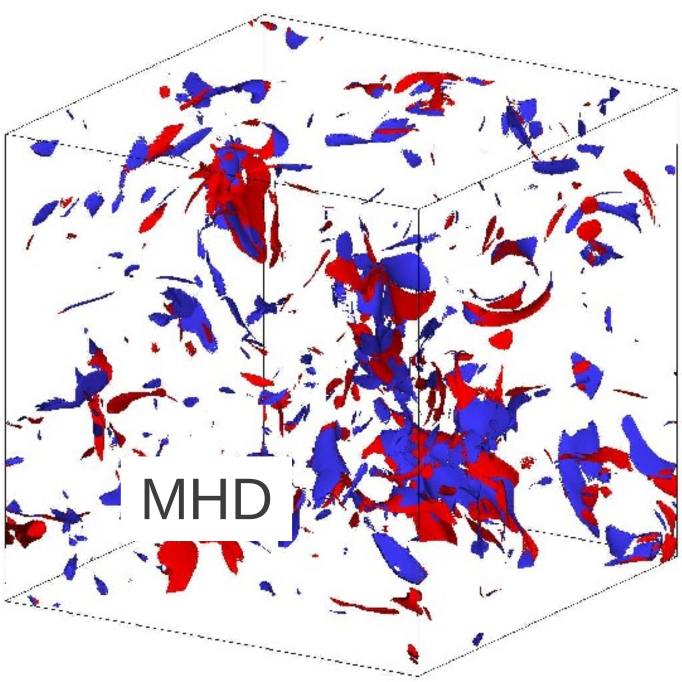
Caution: MHD turbulence is a very complicated state

- non-linear
- high Re → many degrees of freedom $\sim Re^{9/4}$
- Proving uniqueness and existence of solution to Navier-Stokes equations → 1 million Dollar

Why numerical modeling?

- Access to detailed data
- Focus on idealized situation
- Understanding nature by simplified models
- Supercomputer provide remote laboratory
- might be much cheaper than experiments
- free (peer-reviewed) access to European resources





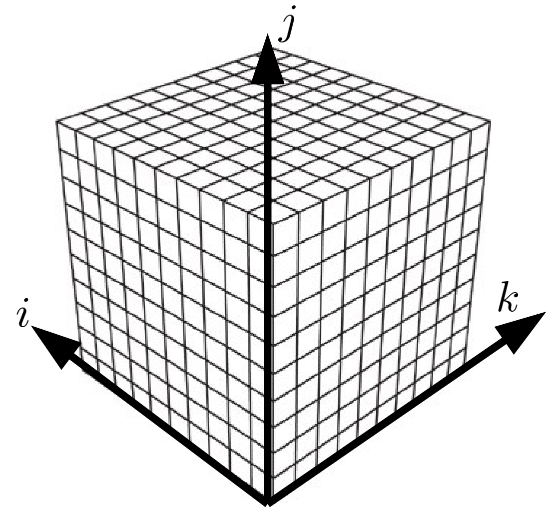
How to model numerically

- Numerical derivatives
- Fast Fourier Transforms
- Resolution issues: inertial and dissipation range
- Floating point precision
- Parallelization
- What can be done with a supercomputer?

Introduction

Discretization: $v(x,t) \rightarrow v^n_{ijk} \equiv v(x_i, x_j, x_k, t_n)$

Here only **uniform** grids:

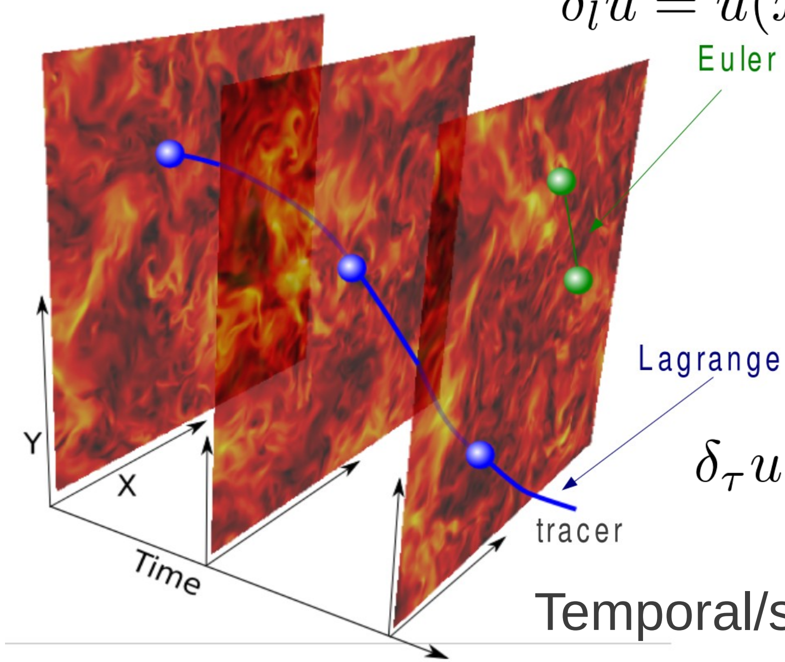


Time integration: $v^{n+1}_{ijk} = F(v^n_{ijk})$

- Runge-Kutta
- Adams-Bashforth
- Leap-Frog
- ...

Solve accurately the MHD equations!

$$\delta_l u = u(x+l,t) - u(x,t),$$



Tracer (fluid element) trajectories

$$\frac{d\boldsymbol{X}(\boldsymbol{x},t)}{dt} = \boldsymbol{v}(\boldsymbol{X}(\boldsymbol{x},t))$$

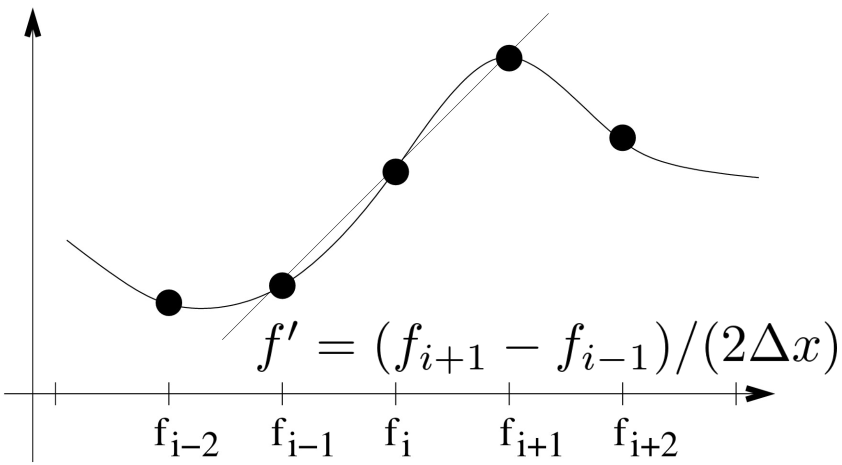
$\boldsymbol{v}$ solution of the MHD equations

$$\delta_\tau u = u(X(x,t+\tau),t+\tau)) - u(x,t),$$

Temporal/spatial averages → statistical data: $S_p(l) = \langle(\delta_l u)^p\rangle$

Numerical derivatives

How to approximate a derivative: $f' \equiv \partial_x v$



Finite differences: $f'_i = \sum_{j=-N}^{N} g_j f_j$

advantages: explicit (high speed)
flexibility: (boundary conditions)
drawbacks: order p → stencil has p+m(-1) points
high order → high communication

Compact schemes: $\sum_{j=-m}^{m} h_j f'_j = \sum_{j=-n}^{n} g_j f_j$

advantages: balance of left and right hand side terms → smaller stencil
drawback: implicit → more complicated to solve : m=1 → tridiagonal matrix

Spectral scheme: $f_K(x) = \sum_{k=-K}^{K} \widehat{f}_k e^{ikx} \quad f' = \widehat{i\,k\,\widehat{f_K}}$

advantages: exponential accuracy: for a periodic $f(x) \in C^m$

$$\|f - f_K\|_{L^p_{(0,2\pi)}} \le C K^{-m} \|f^{(m)}\|_{L^p_{(0,2\pi)}}.$$

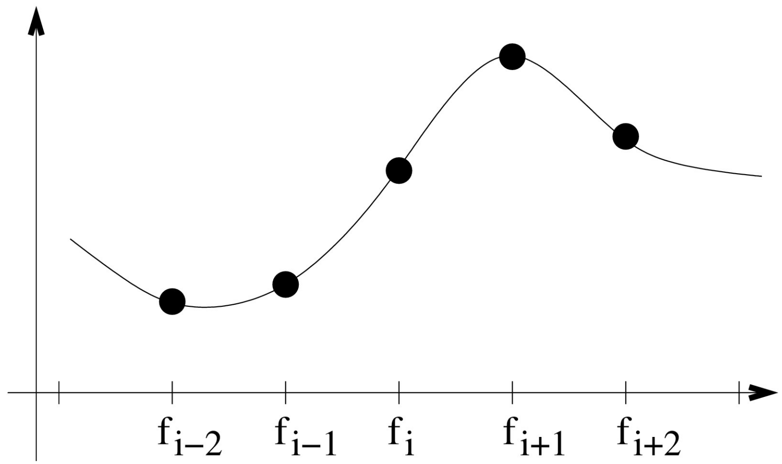
drawbacks: convolutions (N^2 operations) for non-linearities → speudo-spectral with FFTs
problems for discontinues functions
computation of modes $\widehat{f_K}$ = global operation

Fast Fourier Transforms

Quadratic term $f(x) = a(x) \cdot b(x) \widehat{=} (v \cdot \nabla v)$

$$(\widehat{f(x)})_k = \int a(x) b(x) e^{ikx} dx$$
$$= \int \hat{a}(k - k_1) \hat{b}(k_1) dk_1$$

→ N² operations → numerically expensive



Speudo-spectral: Derivatives in Fourier-Space, Products in real space
Transformations with Fast Fourier Transformations (FFT)

Idea: Discrete Fourier Transform = sum of 2 transforms of half length

$$\hat{f}_k = \sum_{j=0}^{N/2-1} e^{2\pi i k(2j)/N} f_{2j} + \sum_{j=0}^{N/2-1} e^{2\pi i k(2j+1)/N} f_{2j+1}$$
$$= \sum_{j=0}^{N/2-1} e^{2\pi i k j/(N/2)} f_{2j} + W^k \sum_{j=0}^{N/2-1} e^{2\pi i k j/(N/2)} f_{2j+1} = \hat{f}_k^{even} + W^k \hat{f}_k^{odd}$$

Fast Fourier Transforms (2)

Iterate this decomposition $\log_2 N$ times in even and odd transforms …

Second level: $\hat{f}_k = \hat{f}_k^{ee} + W^k \hat{f}_k^{eo} + W^k \hat{f}_k^{oe} + (W^k)^2 \hat{f}_k^{oo}$

Number of operations of a FFT: $N \log_2 N$

Why?

- $\log_2 N$ levels
- N operations per level s: $0 \leq k \leq N/2^s - 1$ (period halving)
  for the computation of 2^s functions

Resolution issues: inertial range

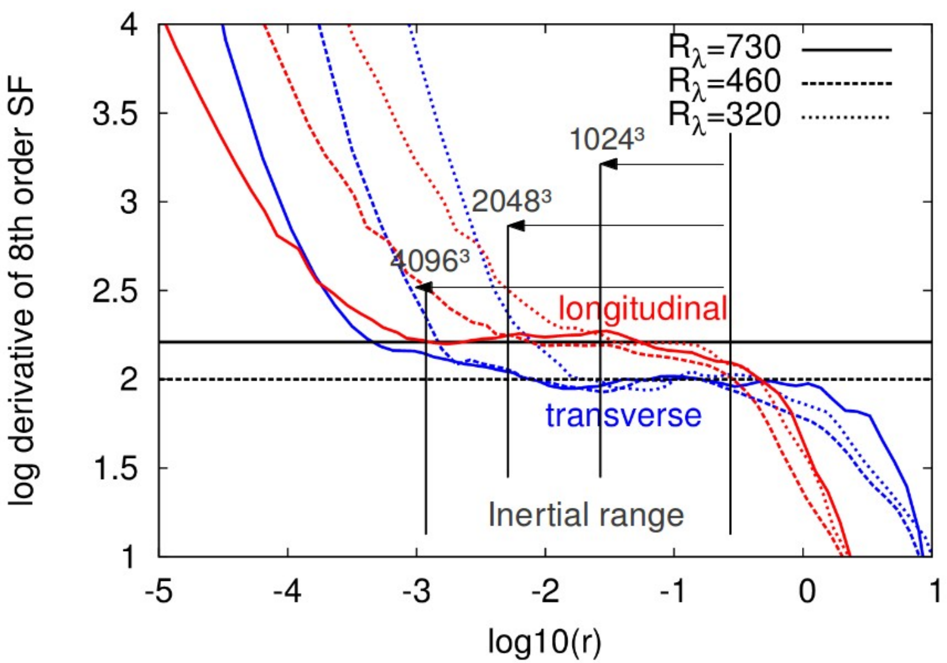
Which resolution of the turbulent flow? A) **inertial range statistics**

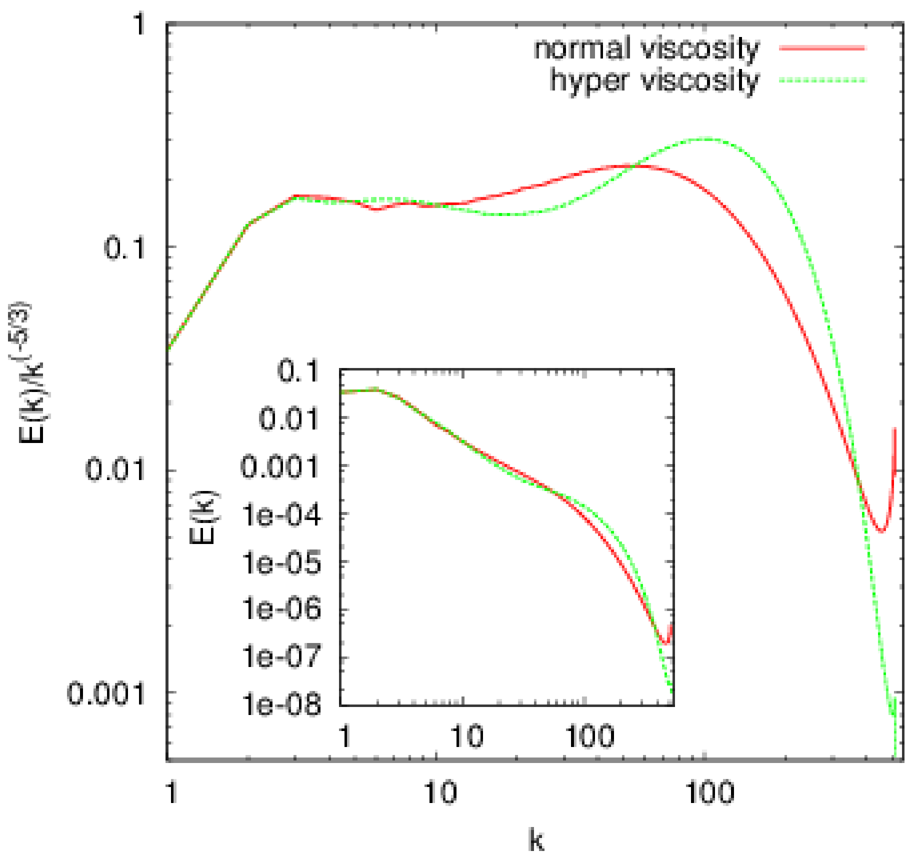
Forcing at large scales fixes L. Variation of ν determines $Re = \frac{L\, u_{rms}}{\nu}$

Respect $k_{max}\eta \geq 1.5$

Small differences in the scaling behavior of two different types of structure functions

large Re → inertial range broadens



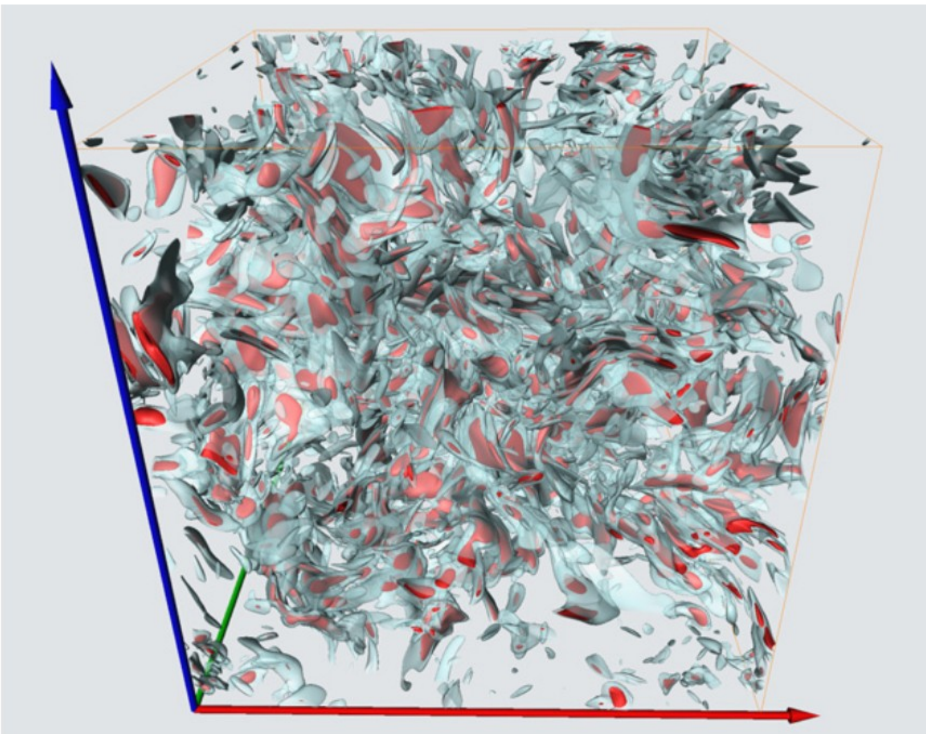


Hyper viscosity: $\Delta u \rightarrow \Delta^2 u$

Grauer, Homann et al. (2012) NJP

Resolution issues: dissipation range

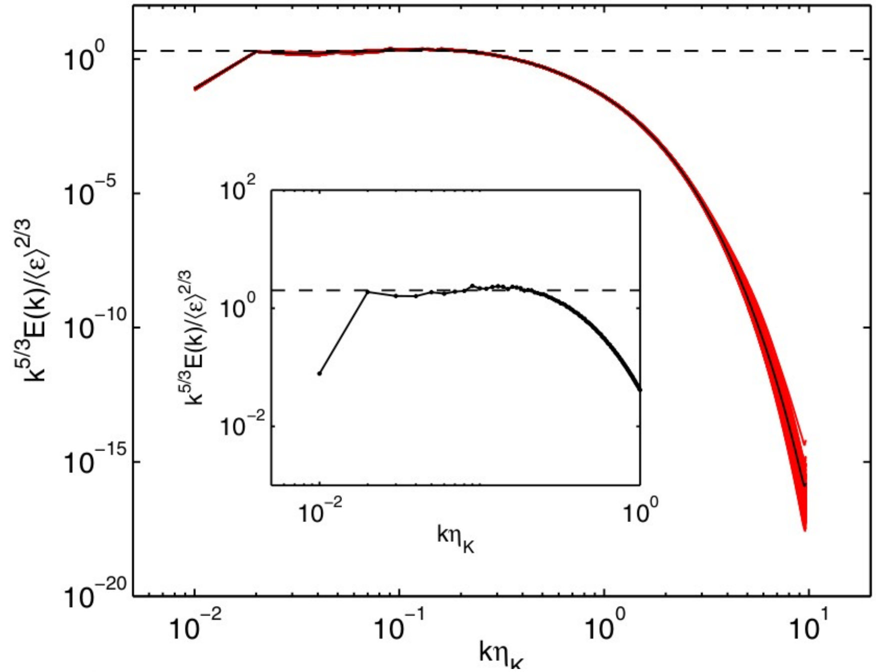
Which resolution of the turbulent flow?

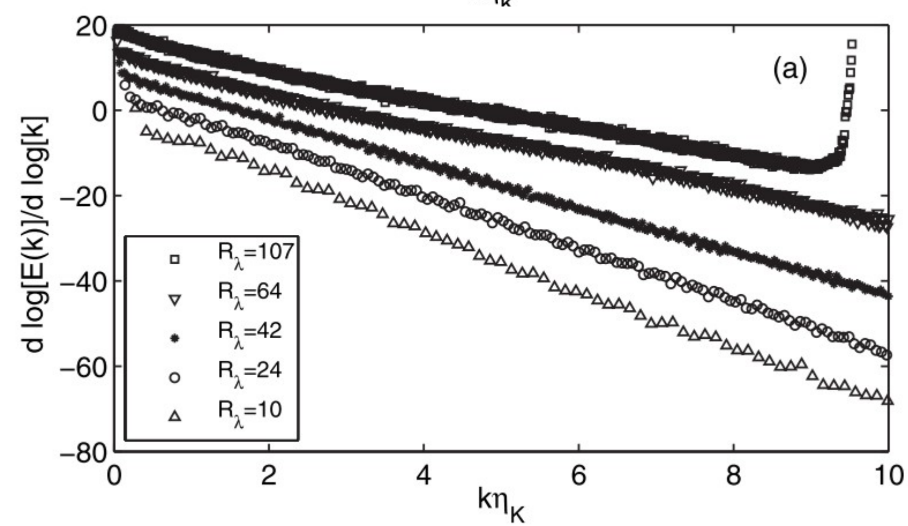


$$k_{max}\eta = 10 - 34$$

Schumacher (2007) EPL

B) **dissipation range statistics**





Floating point precision

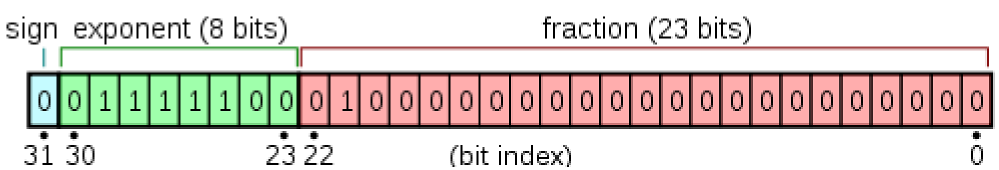
Which floating point precision is needed?

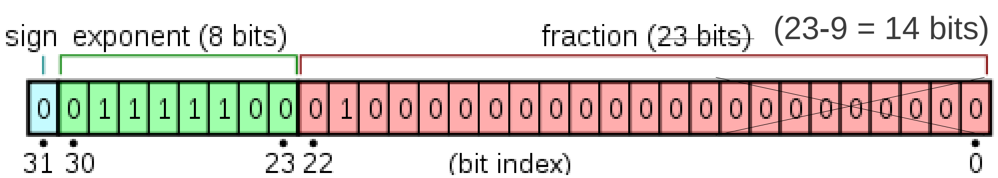
Navier-Stokes simulation with 256^3 grid-points

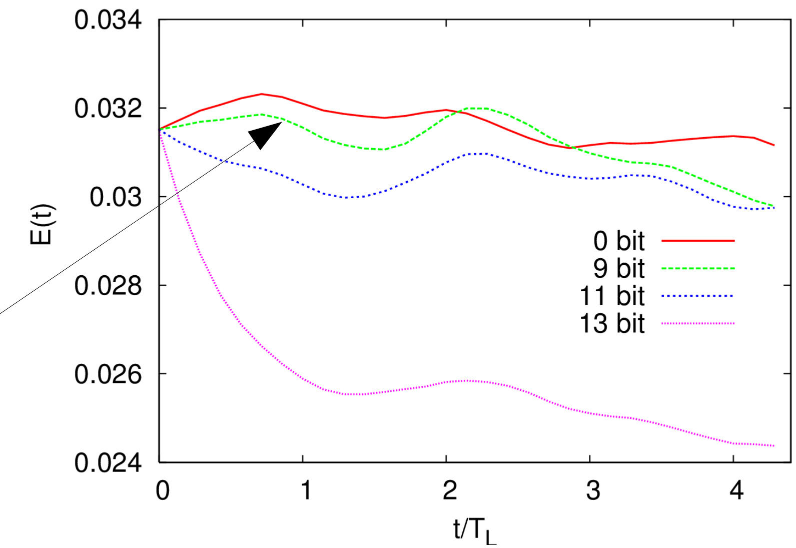
single-precision

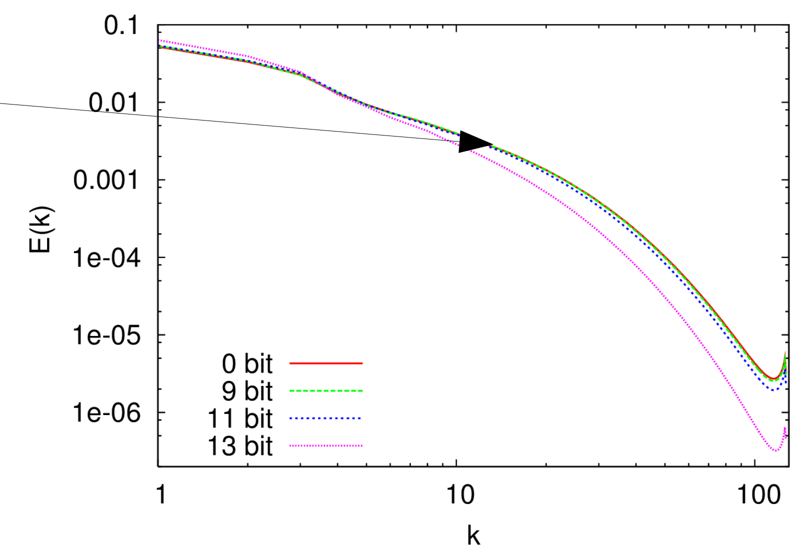
$$k_{max}\eta = 1.5$$

artificial noise on n bits









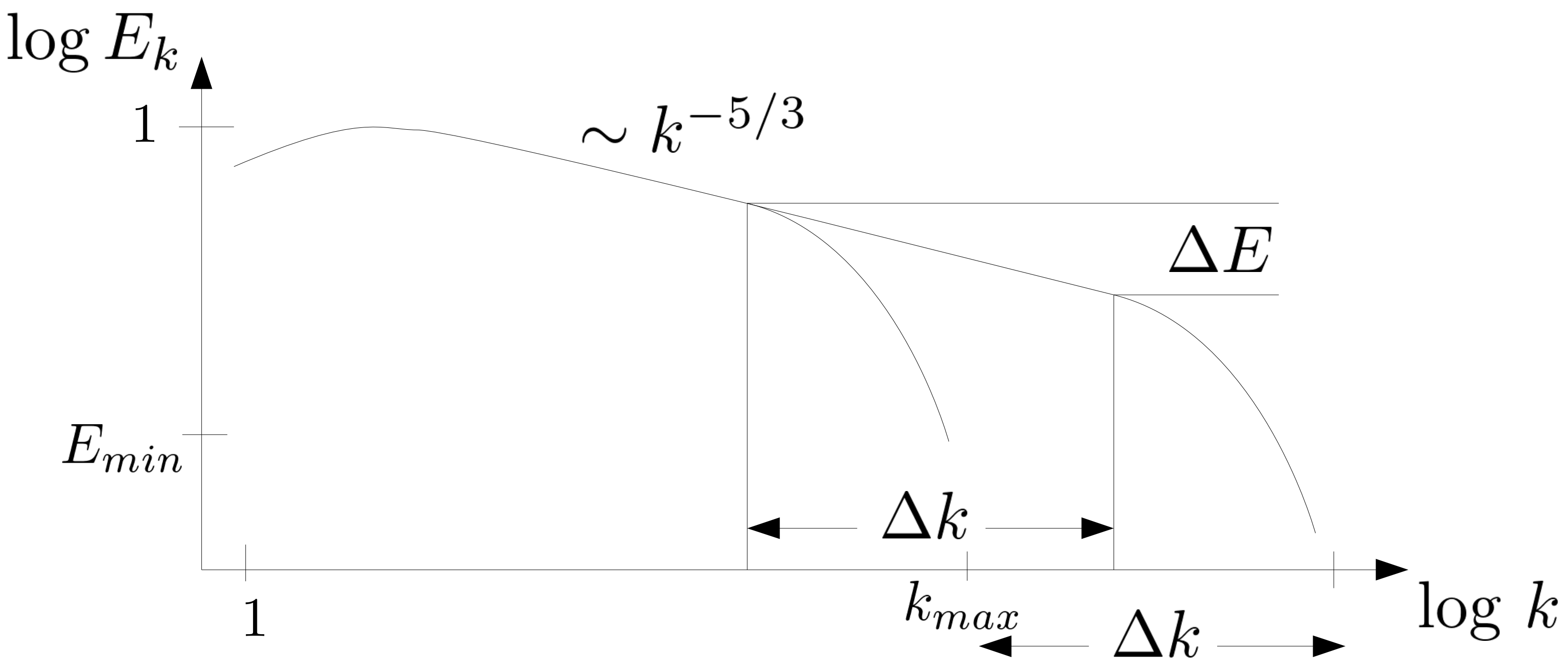
14 bits in fraction are sufficient for 256^3 simulation

Homann, Dreher et al. (2007) Comp. Phys. Comm

Floating point precision (2)

What maximal resolution can be run in single precision?

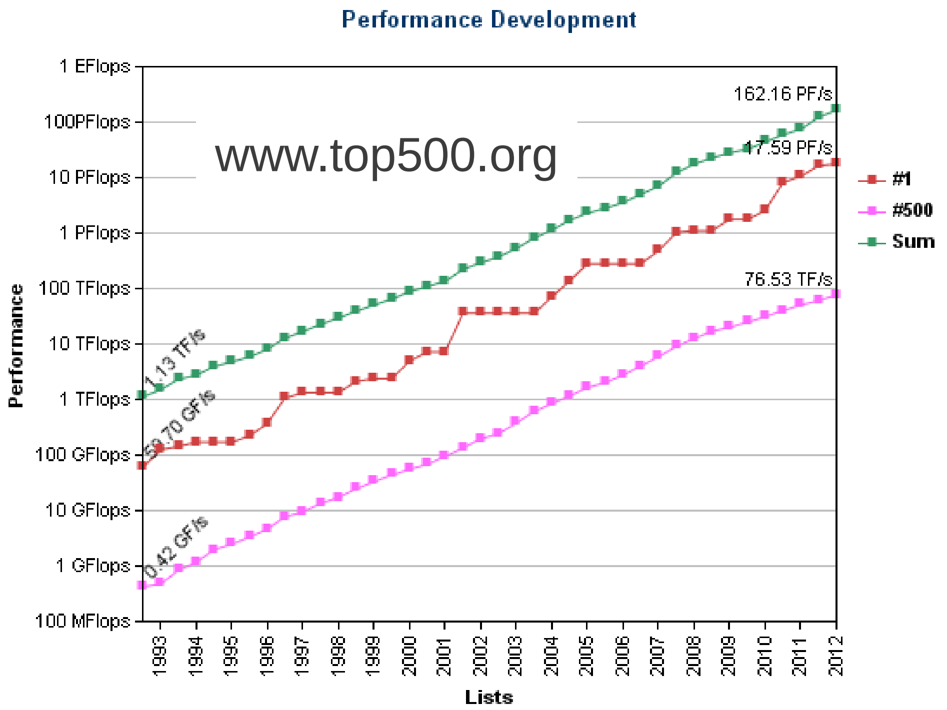
- 9 bits more → 512 times larger number
- Assume Kolmogorov spectrum $E_k \sim k^{-5/3}$

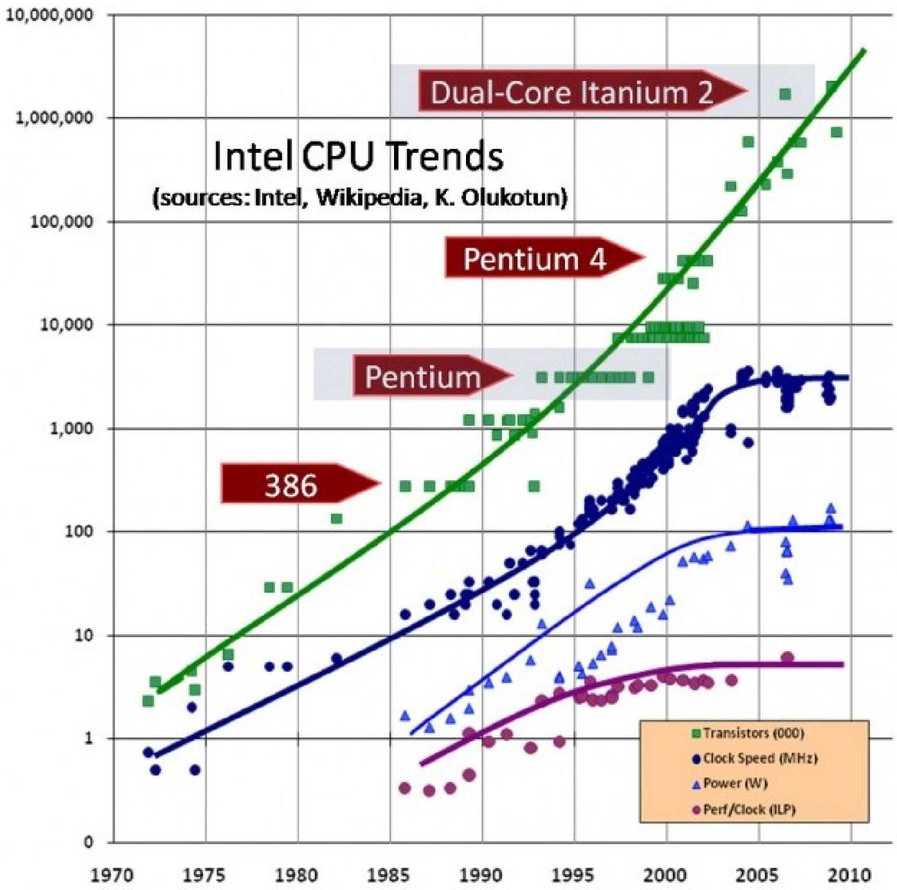


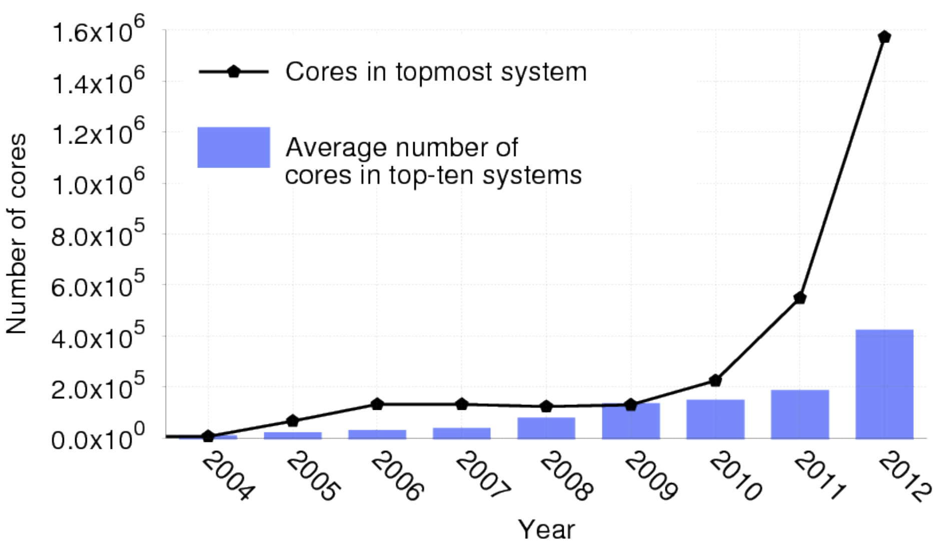
$$\Delta E = 512 = (\Delta k)^{5/3} \qquad k_{max} = 256\,\Delta k = 256 \cdot 512^{3/5} = 10752$$

8192^3 simulations are possible in single precision

Parallelization







At the end … huge number of standard computing cores connected with a very fast network with an essentially distributed memory

Parallelization (2)

Message Passing Interface (MPI): Data transfer between cores with different ram

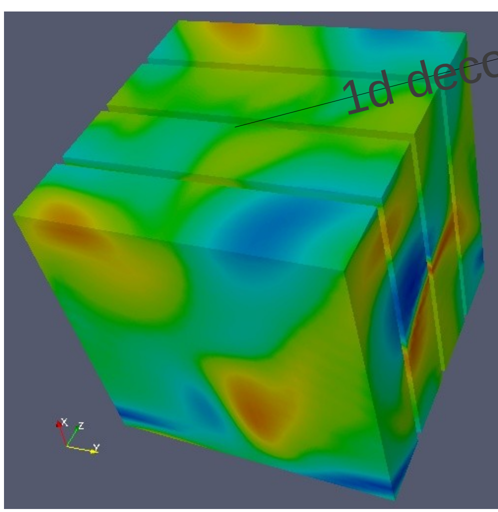
Library with functions:

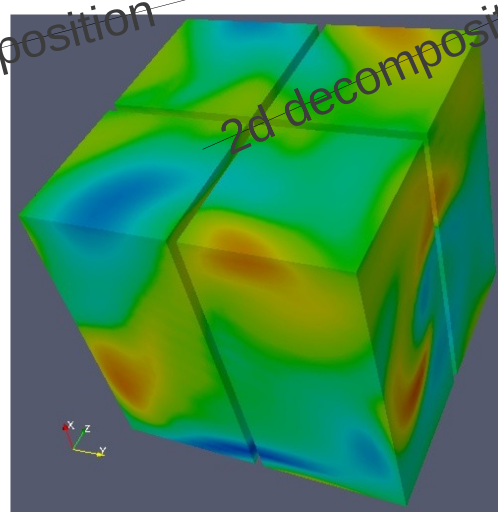
```
int MPI_Send(void *buf, int count, MPI_Datatype datatype, int dest, int tag, MPI_Comm comm)
int MPI_Recv(void *buf, int count, MPI_Datatype datatype, int source, int tag, MPI_Comm comm, MPI_Status *status)
```

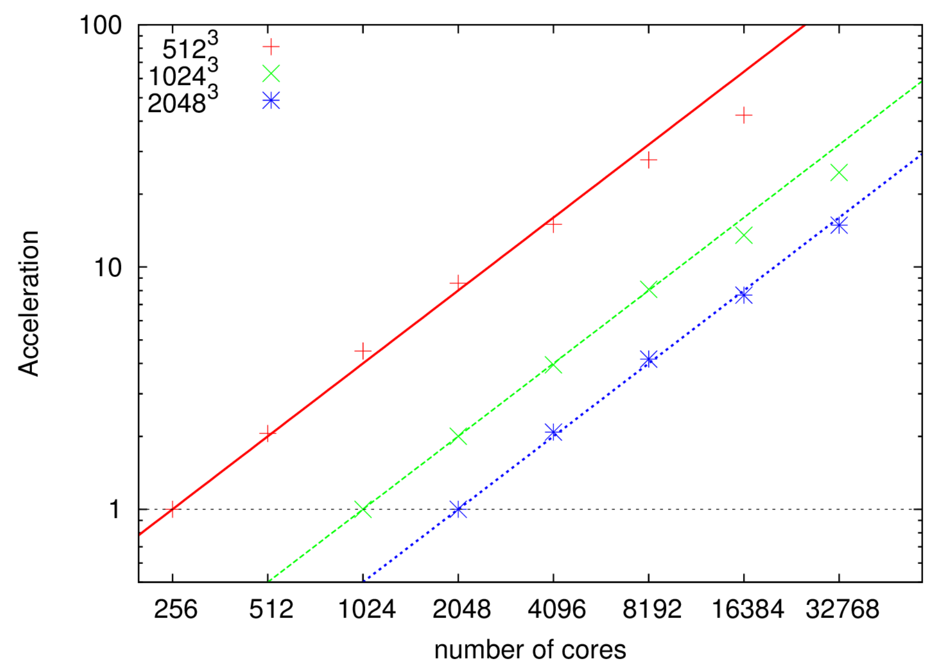
+ the same in non-blocking + global communications such as

```
int MPI_Alltoall(void *sendbuf, int sendcount, MPI_Datatype sendtype, void *recvbuf, int recvcount, MPI_Datatype recvtype,
                 MPI_Comm comm)
```

Free parallel FFT libraries such as FFTW and P3DFFT







MPI + OpenMP/Threads (on shear memory)

What is possible

TURING (IDRIS, 31st of TOP500):

BlueGene/Q: 0.84 Pflop, 65 Tbyte RAM, 400 kW energy consumption

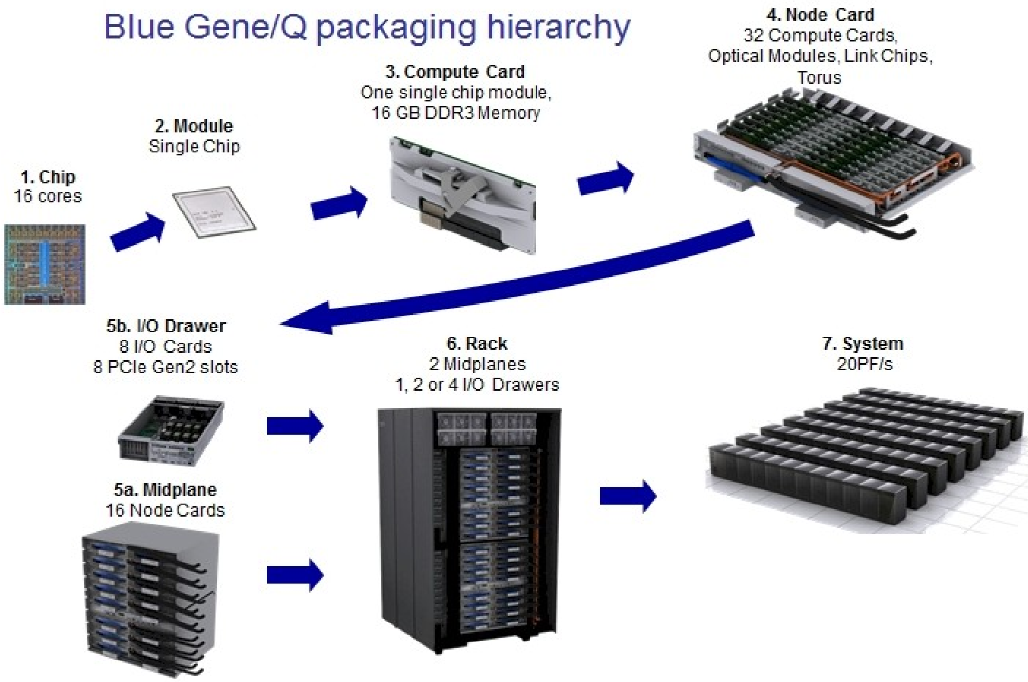
TURING (IDRIS) 65536 cores

Simulation:

4096^3 grid-points ($Re \simeq 40000$)

Approx. 8TByte RAM

10 LET = 40 Days = 60 Million CPU h



PRACE (Partnership for advanced computing in Europe)

Electricity: 400000 kWh = 45 k euros (if 1kWh = 0.11 cents as in France, else ...)

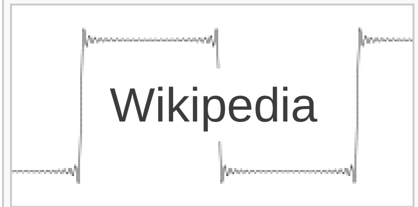
BlueGene/Q is the most efficient Supercomputer: 2GFlop/W (Green500)

Discontinuities

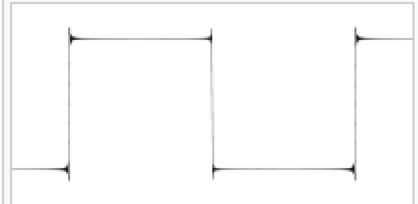
Incompressible Navier-Stokes: Δv → smooth solution

Highly compressible Flows → shocks = discontinuities → Gibbs oscillations

Functional approximation of square wave using 5 harmonics



Functional approximation of square wave using 25 harmonics

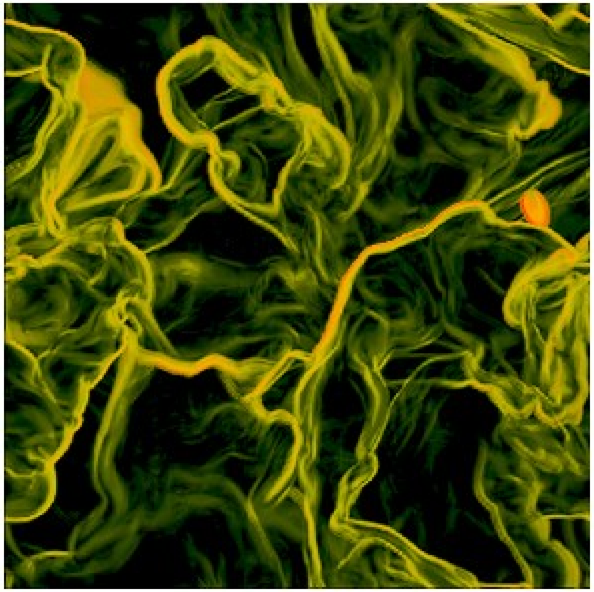


Functional approximation of square wave using 125 harmonics

Finite volume scheme

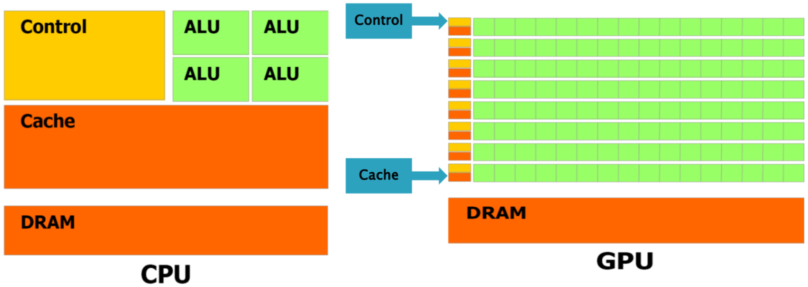
Central Weighted Essentially Non-Oscillatory scheme

Beetz, Schwarz et al. (2008) Phys. Lett. A



New trends: Graphic cards (CUDA, OpenCL)
- 2688 simple cores arranged in multiprocessors
- small cache per multiprocessor
- big (6TByte) global memory

very fast multiprocessors slow memory bandwidth



APPLICATIONS

- Universality
- Planet formation
- Turbulent dynamo
- Transport of tracers and Impurities

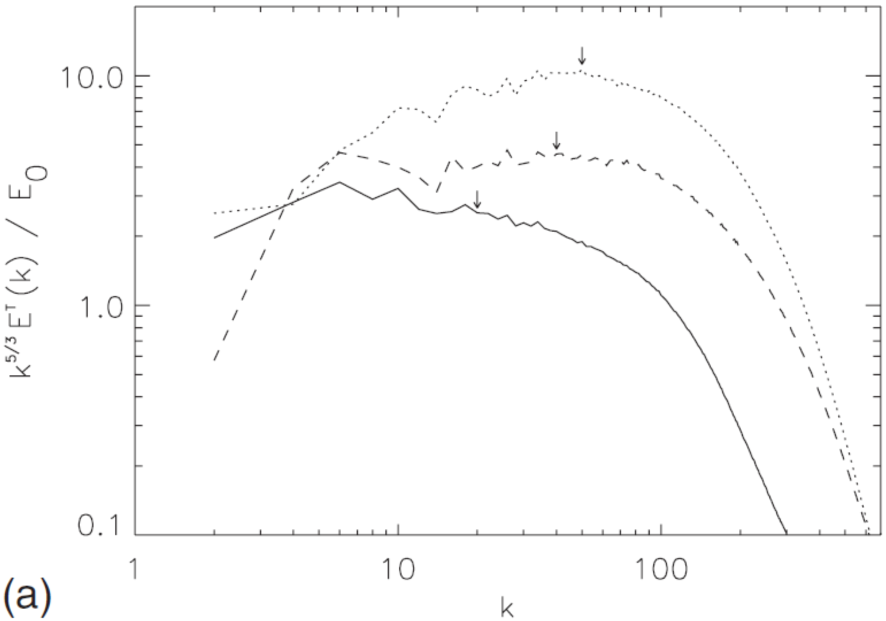
Universality

Problem: Are scaling laws universal ?

Speudo-spectral simulations of decaying MHD turbulence

Different initial conditions

Energy spectra:



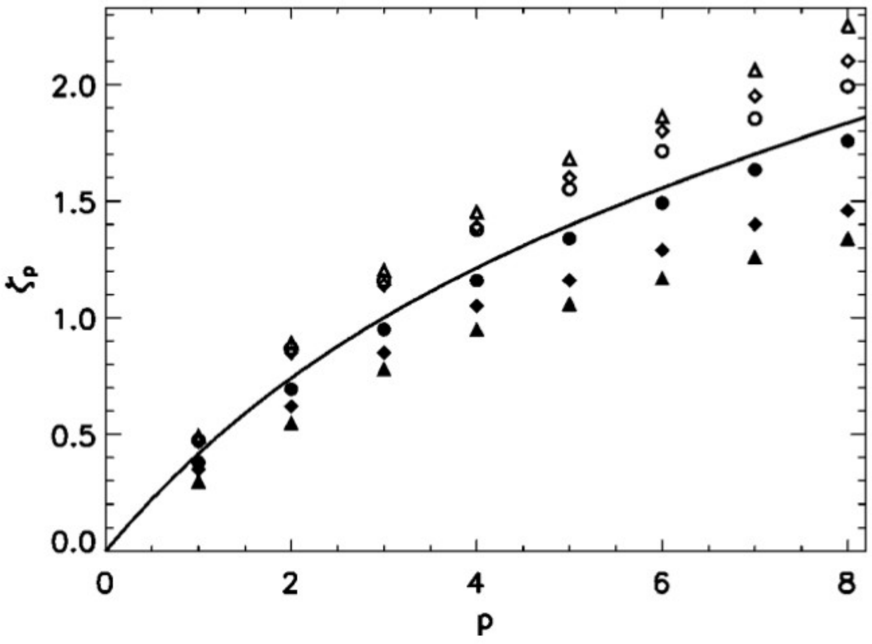
(a)

Lee, Brachet et al. (2012) PRE

Different mean magnetic fields

$$S_p(l) = \langle (\delta z_l)^p \rangle \sim l^{\zeta_p}$$

$$\boldsymbol{z}^{\pm} = \boldsymbol{v} \pm \boldsymbol{B}$$

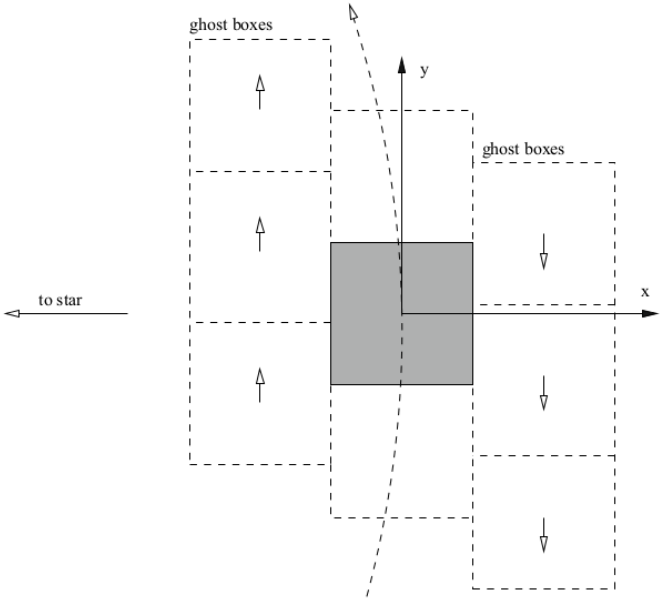


Müller, Biskamp et al. (2003) Phy. Rev. E

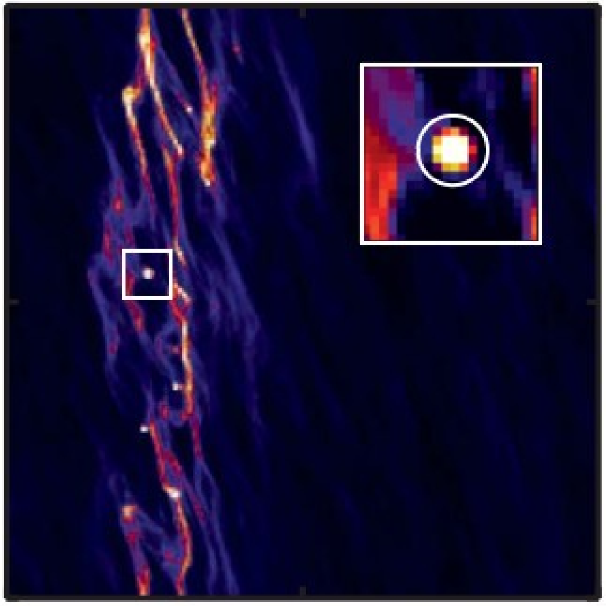
Planet formation

Problem: How to grow m-size boulders to km-size planetisimals?

- DNS of MHD shearing box
- Heavy particles
- 2-way coupling
- High order finite differences
  (pencil code, http://www.nordita.org/pencil-code/))



- Particles concentrate in high-pressure regions
- Acceleration of gravitational collapse
- Accelerated boulder growth



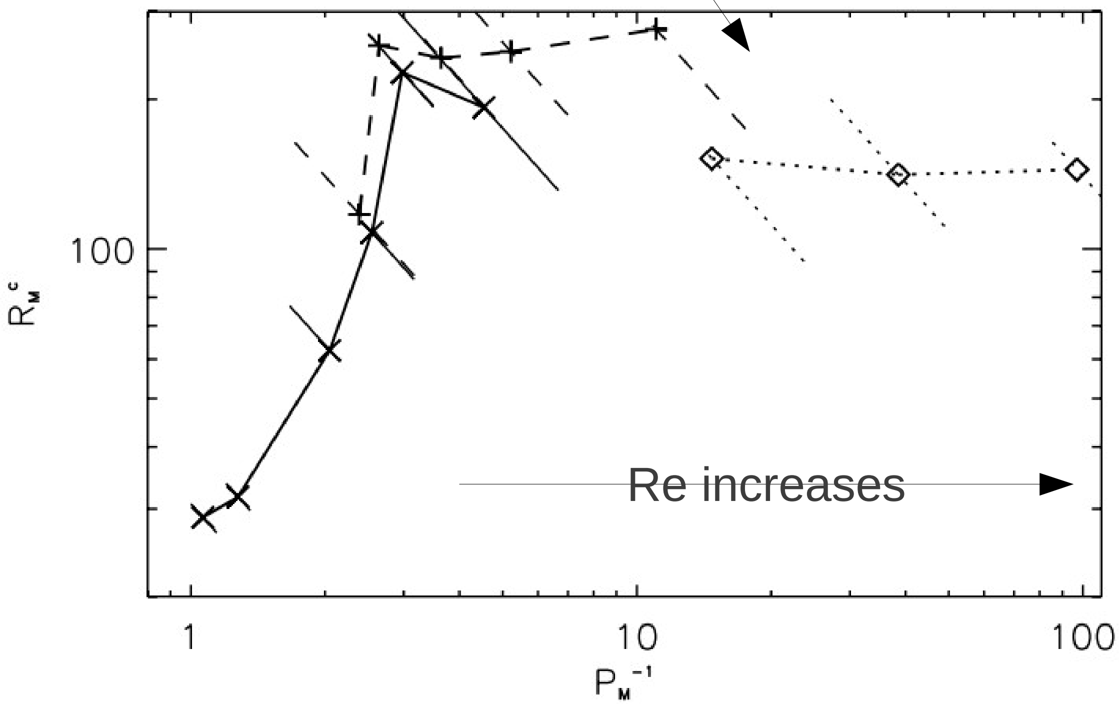
Johansen, Oishi et al. (2007) Nature
Johansen, Klahr et al. (2011) A&A

Turbulent dynamo

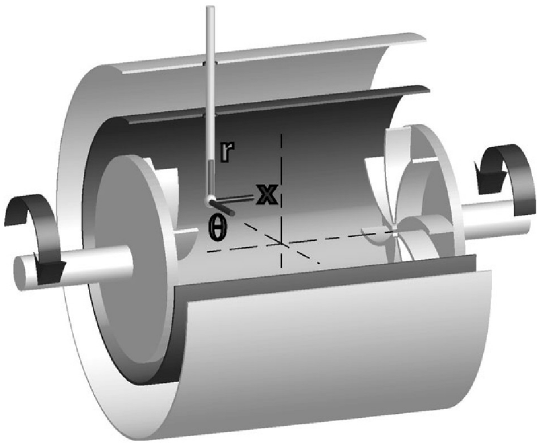
Problem: How does the dynamo onset scale with the Reynolds number?

- DNS of MHD flow in periodic box with Taylor Green forcing
- First, onset increases with Re
- Later, onset saturates

$$\boldsymbol{F} = F_0 \begin{pmatrix} \sin(k_0 x)\cos(k_0 y)\cos(k_0 z) \\ -\cos(k_0 x)\sin(k_0 y)\cos(k_0 z) \\ 0 \end{pmatrix}$$



VKS experiment



Ponty, Minninni et al. (2005) Phys. Rev. Lett.

Monchaux et al., Phys. Fluids 21, 035108 (2009)

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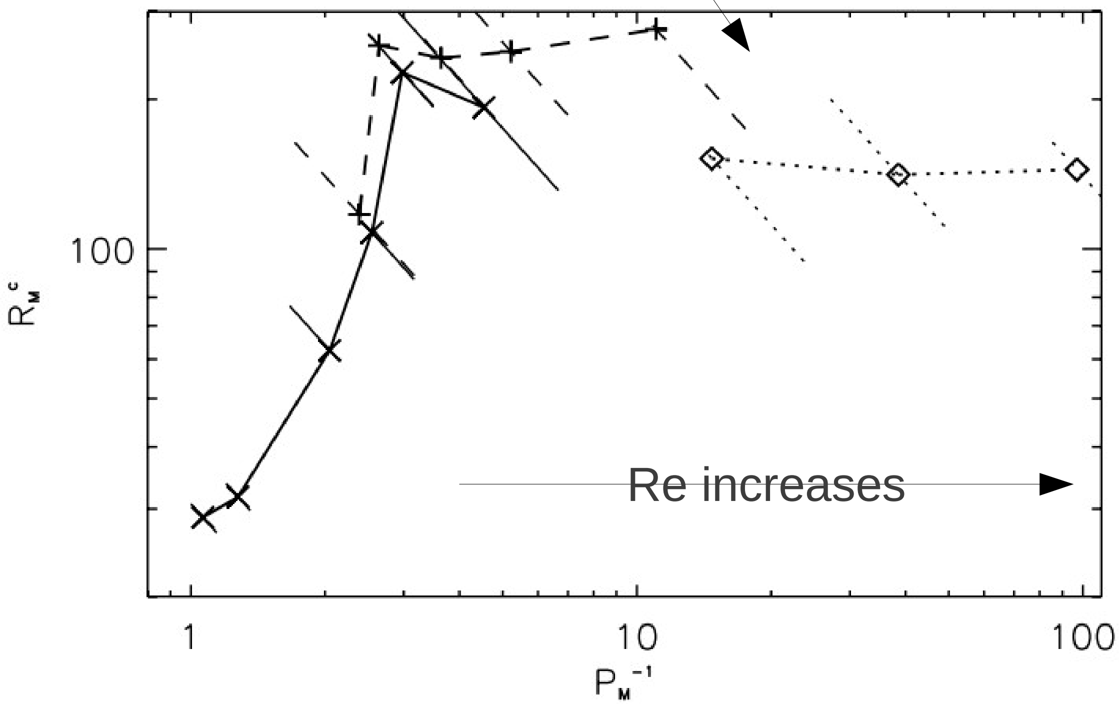
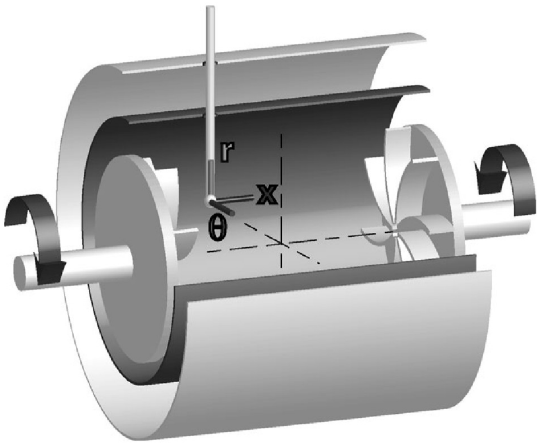


VKS experiment



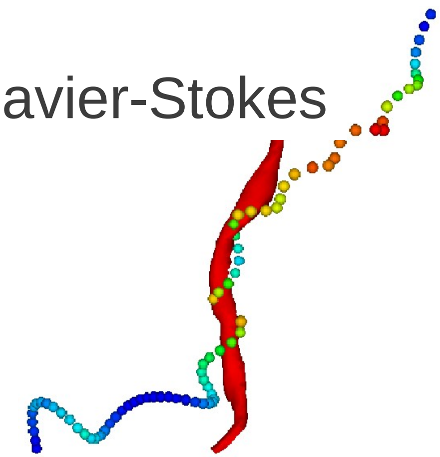
Ponty, Minninni et al. (2005) Phys. Rev. Lett.

Monchaux et al., Phys. Fluids 21, 035108 (2009)

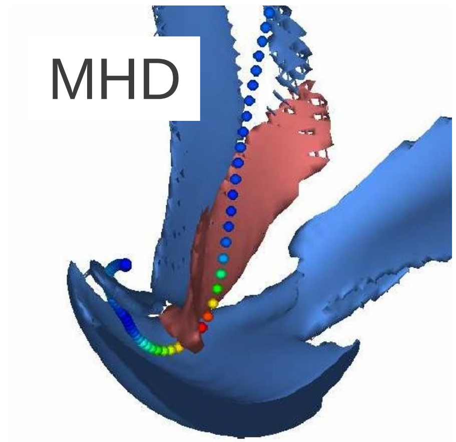
Transport of tracers

Problem: Statistical properties of fluid element trajectories

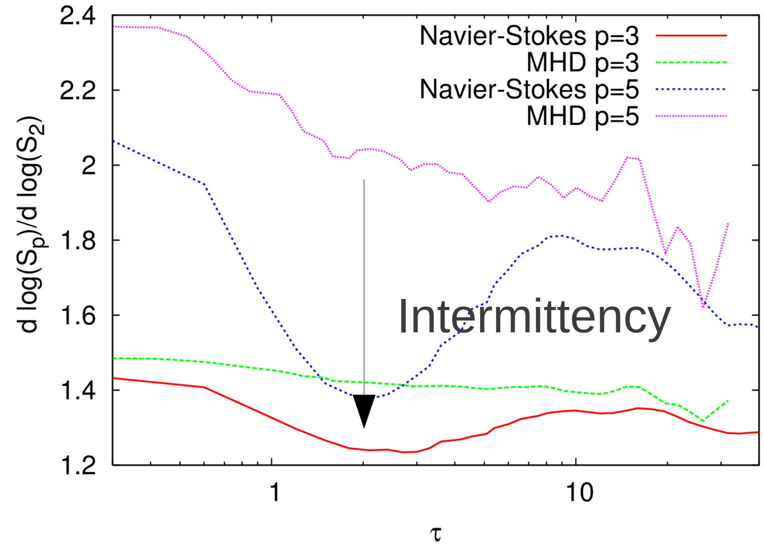
Navier-Stokes



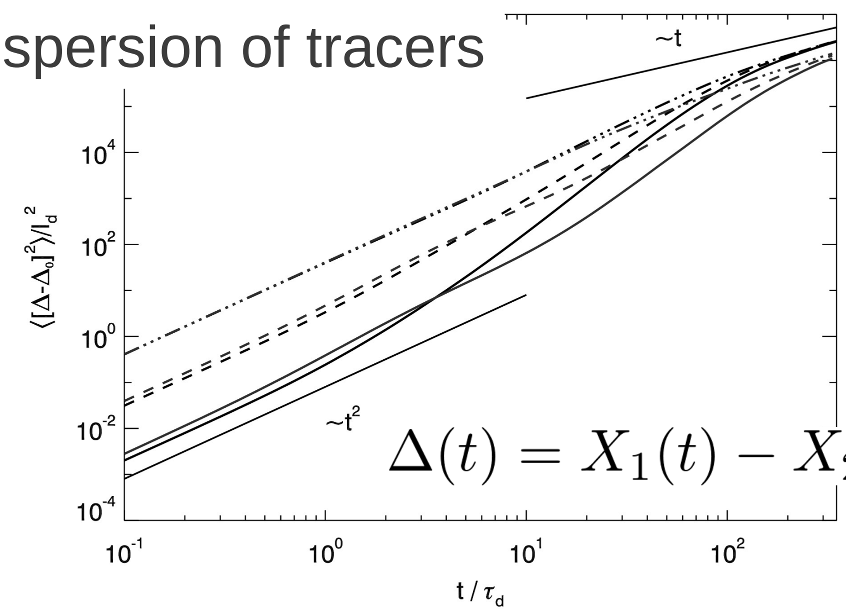
MHD



$$S_p(\tau) = \langle (\delta_\tau u)^p \rangle$$



Dispersion of tracers



$$\Delta(t) = X_1(t) - X_2(t)$$

- MHD velocity increments are less intermittent than Navier-Stokes ones
- Accelerated (Richardson) MHD dispersion is delayed due to alignment to mean magnetic field
- B-transverse dispersion is reduced

Busse, Müller et al. (2007) Phys. Plasmas

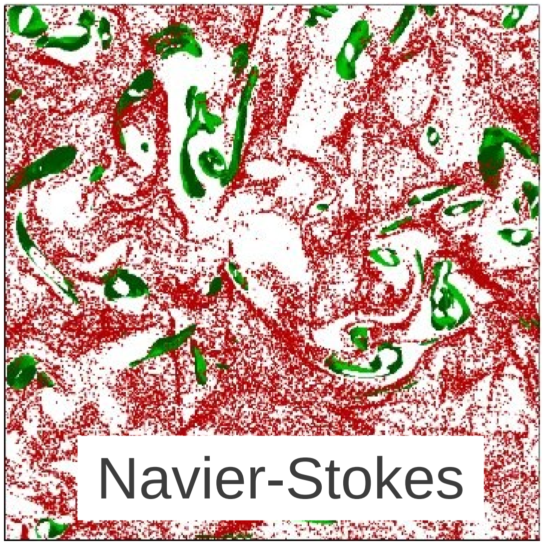
Transport of impurities

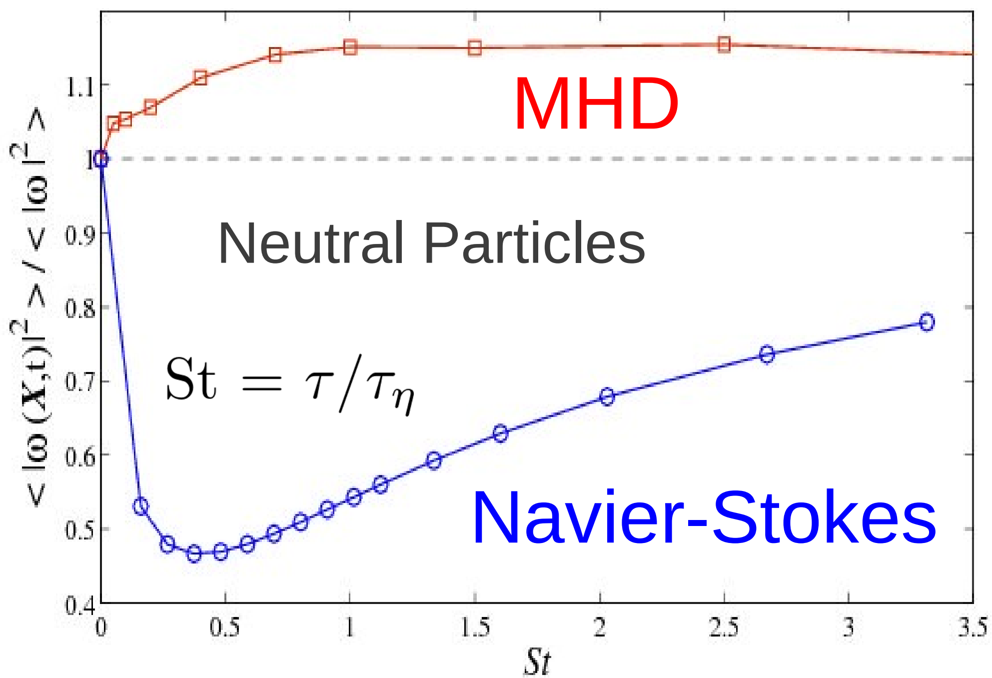
Problem: Where do inertial particles go in MHD turbulence?

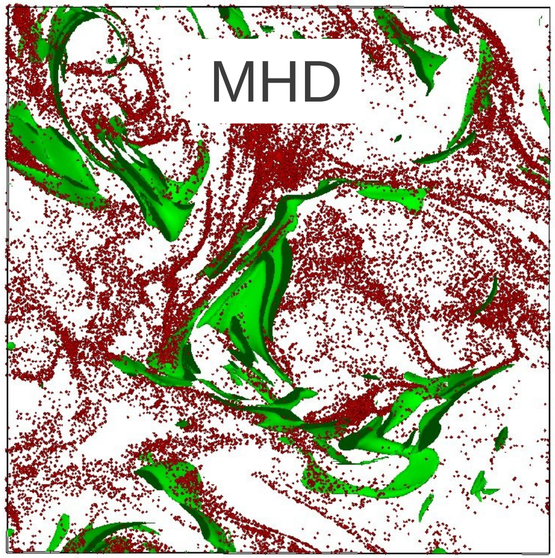
$$\ddot{X} = \underbrace{\frac{1}{\tau}\left[u(X,t) - \dot{X}\right]}_{\text{viscous drag}} + \underbrace{\frac{1}{\ell}\left[E(X,t) + \dot{X} \times B(X,t)\right]}_{\text{Lorentz force}},$$

$\tau = 2\rho_\mathrm{p} a^2/(9\rho_\mathrm{f}\nu)$: particle response time a: radius

$\ell = \rho_\mathrm{p}\, c/(\rho_\mathrm{c}\sqrt{4\pi\rho_\mathrm{f}})$: length-scale

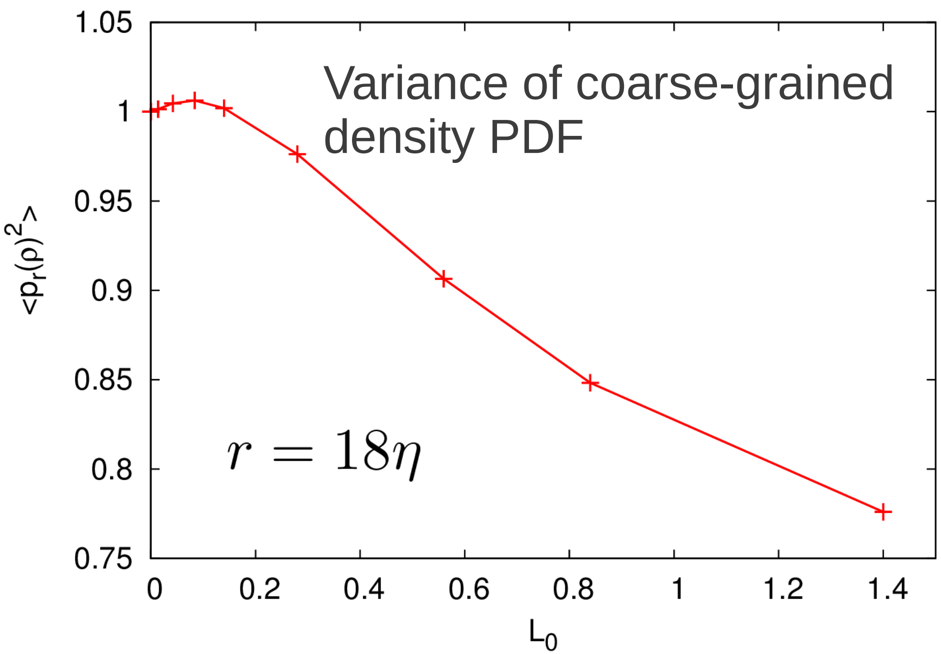






Transport of charged impurities

Clustering of **charged** particles $\ell > 0$

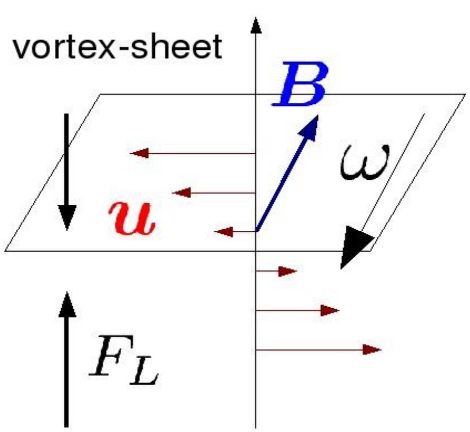


Lorentz-number

$$\mathrm{Lo} = \tau\, B_{\mathrm{rms}}/\ell$$

Effect of Lorentz-force with respect to inertial drag

Vortex-sheet selection



$\omega \cdot \boldsymbol{B} < 0$: attractive sheet

$\omega \cdot \boldsymbol{B} > 0$: repulsive sheet