Magnetohydrodynamic Turbulence

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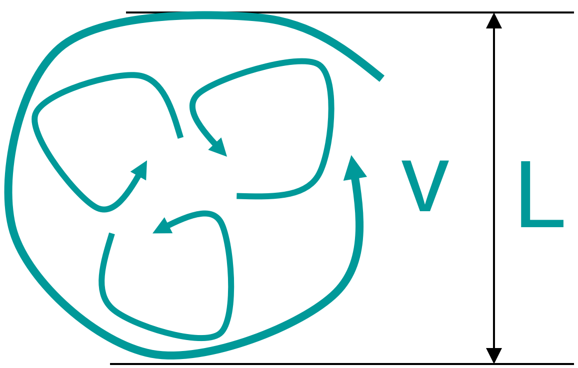
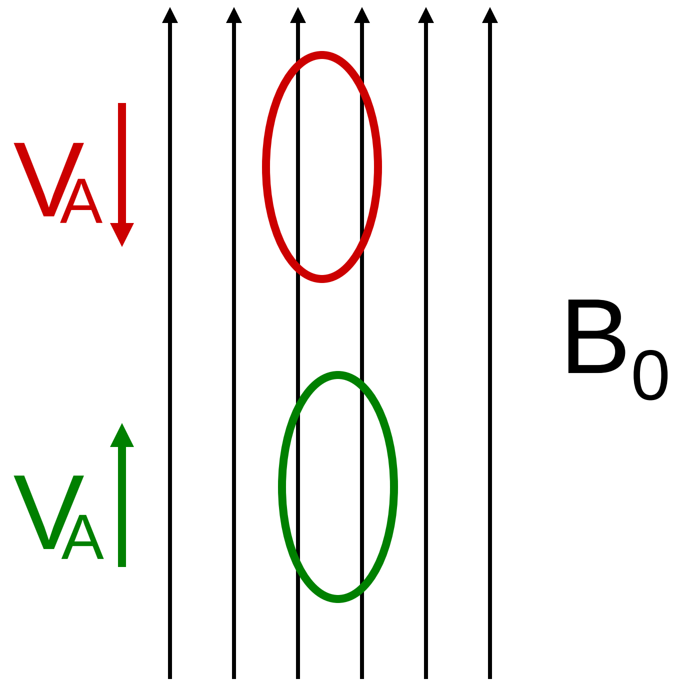
Turbulent cascades: MHD vs HD

HD turbulence:
interaction of eddies



MHD turbulence:
interaction of wave packets
moving with Alfven velocities



$$\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$$

Magnetohydrodynamic (MHD) equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{4\pi\rho_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{v}$$
$$\partial_t \mathbf{B} = \nabla\times[\mathbf{v}\times\mathbf{B}] + \eta\nabla^2\mathbf{B}$$

Separate the uniform magnetic field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

Introduce the Elsasser variables: $\mathbf{z}^{\pm} = \mathbf{v} \pm \frac{1}{\sqrt{4\pi\rho_0}}\mathbf{b}$

Then the equations take a symmetric form:

$$\partial_t \mathbf{z}^{+} - (\mathbf{v}_A\cdot\nabla)\mathbf{z}^{+} + (\mathbf{z}^{-}\cdot\nabla)\mathbf{z}^{+} = -\nabla P$$
$$\partial_t \mathbf{z}^{-} + (\mathbf{v}_A\cdot\nabla)\mathbf{z}^{-} + (\mathbf{z}^{+}\cdot\nabla)\mathbf{z}^{-} = -\nabla P$$

With the Alfven velocity $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$

The uniform magnetic field mediates small-scale turbulence

MHD turbulence: Alfvenic cascade

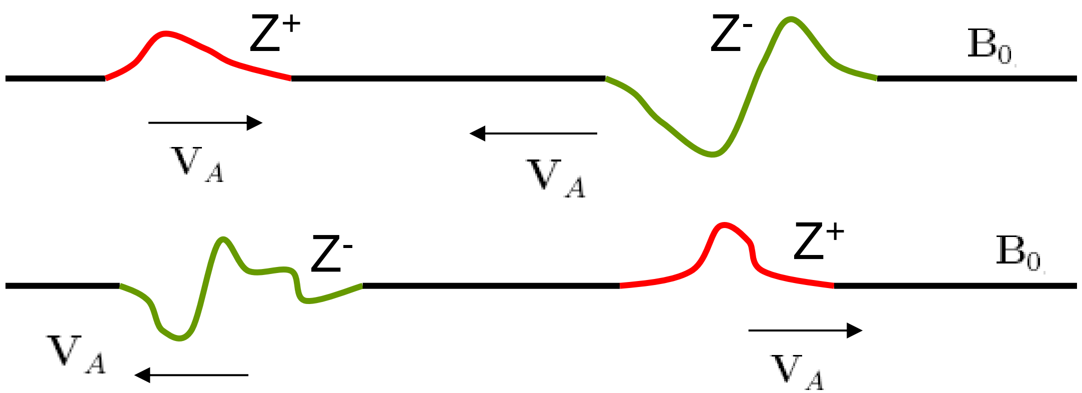
$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

Ideal system conserves the Elsasser energies

$$E^{+} = \int (\mathbf{z}^{+})^2 \, d^3x \qquad E = \frac{1}{2}\int (v^2 + b^2) d^3x$$

$$\equiv$$

$$E^{-} = \int (\mathbf{z}^{-})^2 \, d^3x \qquad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$



$E^{+} \sim E^{-}$: balanced case.

$E^{+} \gg E^{-}$: imbalanced case

$$H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x = \frac{1}{4}(E^{+} - E^{-}) \neq 0$$

Strength of interaction in MHD turbulence

$$\partial \mathbf{z}^{\pm} \mp \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{||} v_A) z^{\pm}} + \underbrace{(\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{\perp} z^{\mp}) z^{\pm}} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

When $k_{||} v_A \gg k_{\perp} z^{\mp}$ turbulence is weak

When $k_{||} v_A \sim k_{\perp} z^{\mp}$ turbulence is strong

Weak MHD turbulence: Phenomenology

Three-wave interaction of shear-Alfven waves

$$\omega(k) = |k_z| v_A$$

$$\begin{cases} \omega(k) = \omega(k_1) + \omega(k_2) \\ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \end{cases}$$

Only counter-propagating waves interact, therefore, k_{1z} and k_{2z} should have opposite signs.

⇓

Either $k_{1z} = 0$ or $k_{2z} = 0$

Wave interactions change $k_\perp$ but not k_z

At large $k_\perp$: $E(k_z, k_\perp) \propto g(k_z) k_\perp^{-\beta}$

Montgomery & Turner 1981, Shebalin et al 1983

Weak turbulence: Analytic framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

In the zeroth approximation, waves are not interacting. and z^+ and z^- are independent:

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}')\rangle = e^+(k_z, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}')\rangle = e^-(k_z, k_\perp)\delta(\mathbf{k}+\mathbf{k}')$$

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}')\rangle = 0$$

When the interaction is switched on, the energies slowly change with time: $e^\pm(k_z, k_\perp, t)$

$$\partial_t \mathbf{z}^\pm - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^\pm + (\mathbf{z}^\mp \cdot \nabla)\mathbf{z}^\pm = -\nabla P$$

$$\partial_t \langle z^+ z^+\rangle = ...\langle z^- z^+ z^+\rangle + \langle z^+ z^- z^+\rangle ...$$

$$\partial_t \langle z^- z^+ z^+\rangle = ...\underbrace{\langle z^+ z^- z^+ z^+\rangle} + \underbrace{\langle z^- z^- z^+ z^+\rangle} + \underbrace{\langle z^- z^+ z^- z^+\rangle}...$$

split into pair-wise correlators using Gaussian rule

Weak turbulence: Analytic framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t \langle z^+ z^+ \rangle = \ldots \langle z^- z^+ z^+ \rangle + \langle z^+ z^- z^+ \rangle \ldots$$

$$\partial_t \langle z^- z^+ z^+ \rangle = \ldots \langle \cancel{z^+ z^- z^+ z^+} \rangle + \langle z^- z^- z^+ z^+ \rangle + \langle z^- z^+ z^- z^+ \rangle \ldots$$

split into pair-wise correlators using Gaussian rule

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp) \right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

$$M_{k,pq} = \frac{\pi}{v_A} \frac{(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2 (\mathbf{k}_\perp \cdot \mathbf{p}_\perp)^2}{k_\perp^2 p_\perp^2 q_\perp^2}$$

This kinetic equation has all the properties discussed in the phenomenology: it is scale invariant, z^+ (z^-) interact only with z^- (z^+), k_z does not change during interactions.

Weak turbulence: Analytic framework

[Galtier, Nazarenko, Newell, Pouquet, 2000]

$$\partial_t e^{\pm}(k_z, k_\perp) = \int M_{k,pq} e^{\mp}(0, q_\perp) \left[e^{\pm}(k_z, k_\perp) - e^{\pm}(k_z, p_\perp) \right] \delta(\mathbf{k}_\perp - \mathbf{p}_\perp - \mathbf{q}_\perp) d^2p d^2q$$

Consider statistically balanced case: $e^+ = e^-$

The general balanced solution of the Galtier et al Eqs is:

$$e^+(k_z, k_\perp) = e^-(k_z, k_\perp) = g(k_z) k_\perp^{-3}$$

where $g(k_z)$ is an arbitrary function smooth at k_z=0.

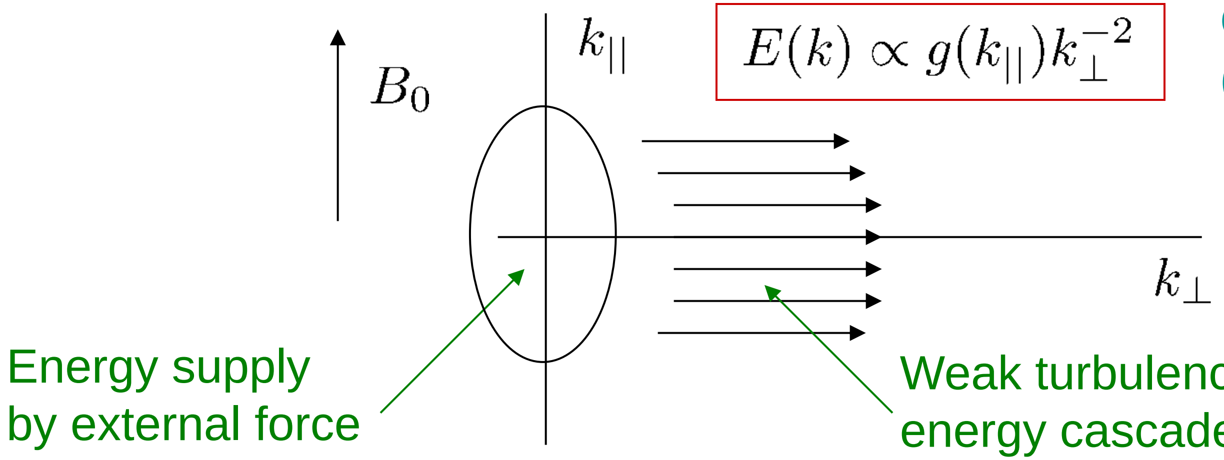
The spectrum of weak balanced MHD turbulence is therefore:

$$E^{\pm}(k_z, k_\perp) = e^{\pm}(k_z, k_\perp) 2\pi k_\perp \propto k_\perp^{-2}$$

Ng & Bhattacharjee 1996, Goldreich & Sridhar 1997

Weak turbulence spectrum

In weak MHD turbulence, energy is transferred to small scales in the field-perpendicular direction:



Galtier et al (2002)

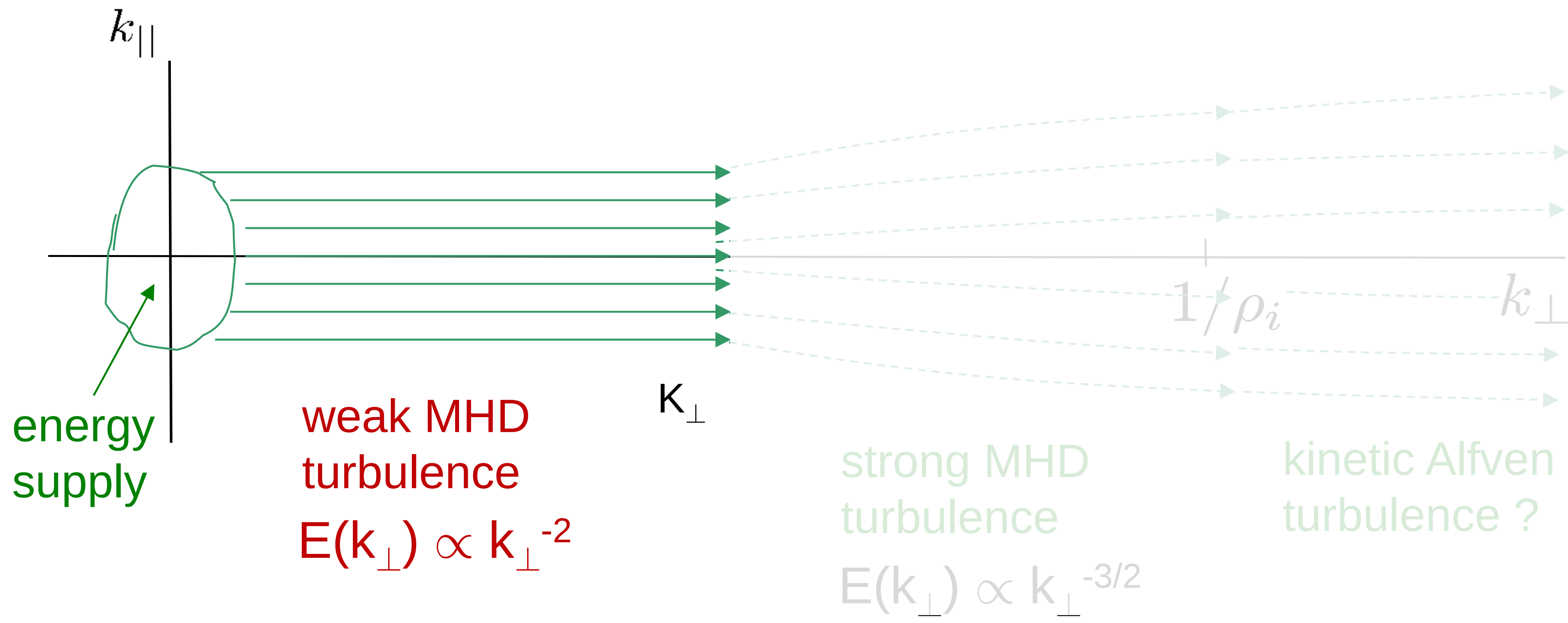
$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

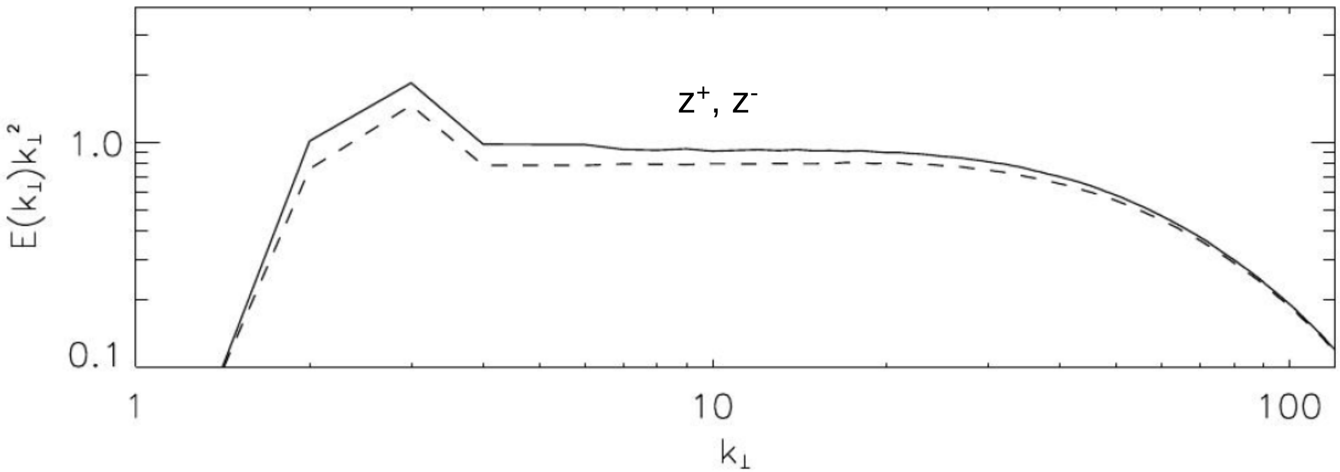
Linear terms do not change

Nonlinear terms increase

Eventually, turbulence becomes strong!

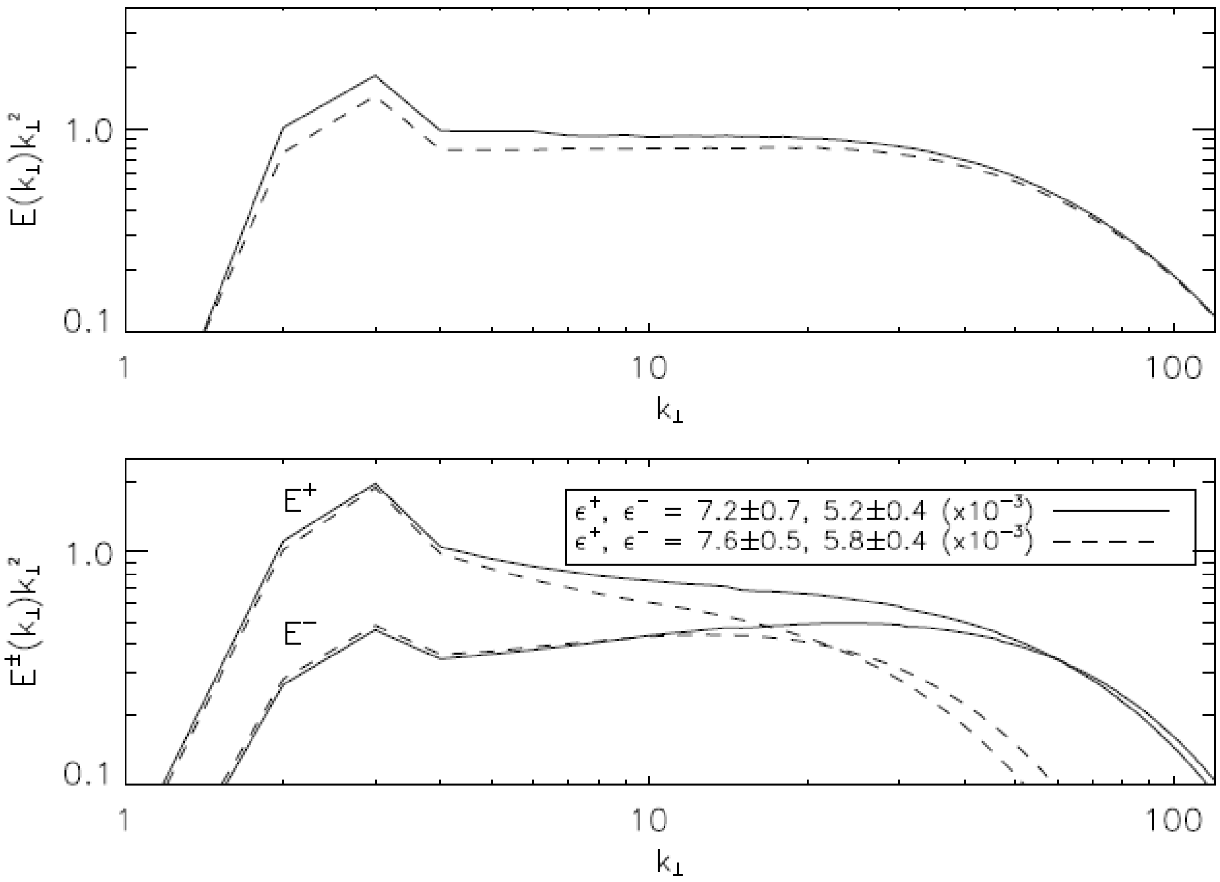
Weak MHD turbulence





SB & J. C. Perez (2009)

Imbalanced weak MHD turbulence: Numerical results



Balanced

Imbalanced—
Inconsistent with the theory!

Residual energy in weak MHD turbulence

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^+(\mathbf{k}') \rangle = e^+(k_{||}, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$ √

$$\langle \mathbf{z}^-(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = e^-(k_{||}, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$ √

$$\langle \mathbf{z}^+(\mathbf{k}) \cdot \mathbf{z}^-(\mathbf{k}') \rangle = q^r(k_{||}, k_\perp)\delta(\mathbf{k} + \mathbf{k}')$$ $\neq 0$

since the waves are not independent!

$$\langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 - b^2 \rangle$$

What is the equation for the residual energy?

Residual energy in weak MHD turbulence

- Waves are almost independent –
  one would not expect any residual energy!
- Analytically tractable:

$$\partial_t q^r = 2ik_{||} v_A q^r - \gamma_k q^r +$$
$$+\int R_{k,pq}\{e^+(\mathbf{q})\,[e^-(\mathbf{p}) - e^-(\mathbf{k})] + e^-(\mathbf{q})\,[e^+(\mathbf{p}) - e^+(\mathbf{k})]\}\delta(q_{||})\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d^3p\,d^3q$$

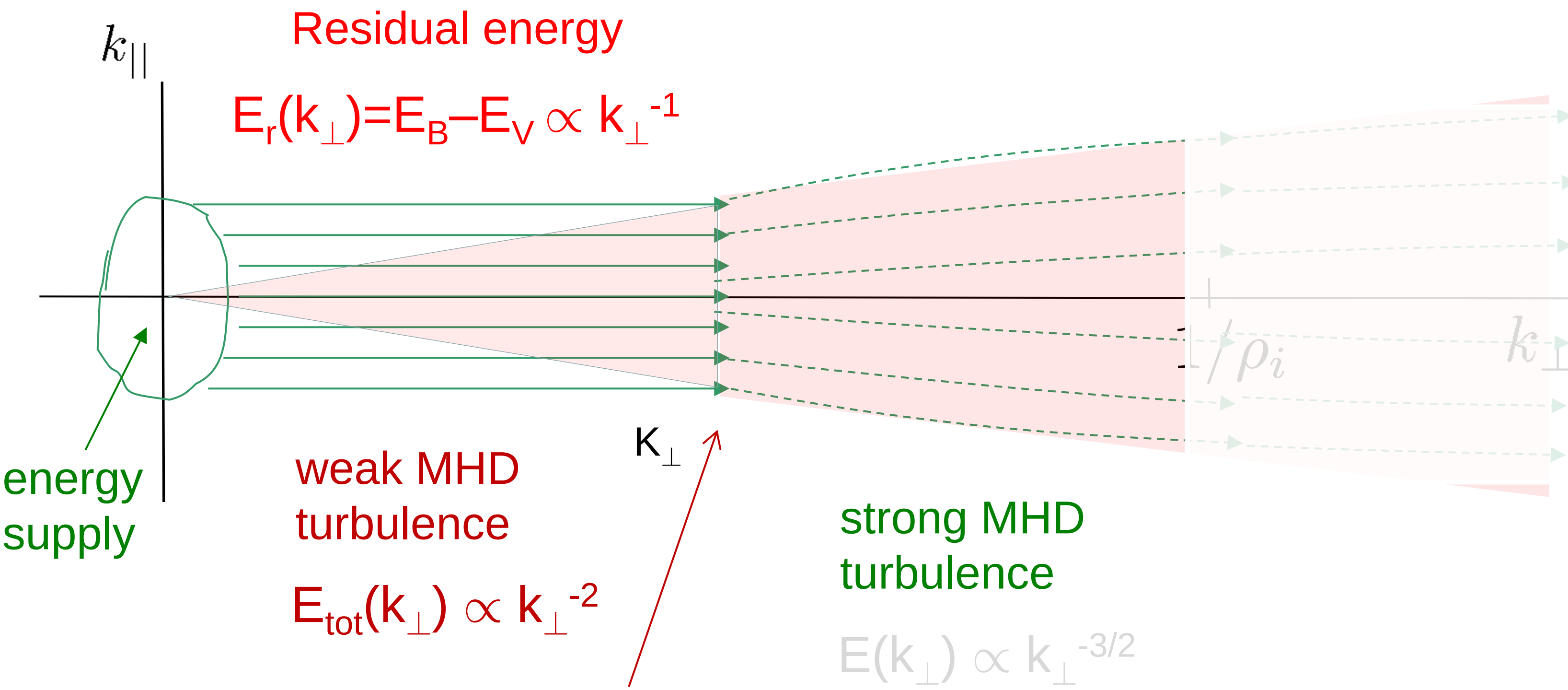
where: $R_{k,pq} = (\pi v_A/2)(\mathbf{k}_\perp \times \mathbf{q}_\perp)^2(\mathbf{k}_\perp \cdot \mathbf{p}_\perp)(\mathbf{k}_\perp \cdot \mathbf{q}_\perp)/(k_\perp^2 p_\perp^2 q_\perp^2)$

Conclusions:

- *Residual energy is always generated by interacting waves!*
- $\int \ldots < 0$, *so the residual energy is negative:*
  magnetic energy dominates! $e^r = \langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 - b^2 \rangle < 0$

Y. Wang, S. B. & J. C. Perez (2011)
S.B, J. C. Perez & V. Zhdankin (2011)

Residual energy in MHD turbulence



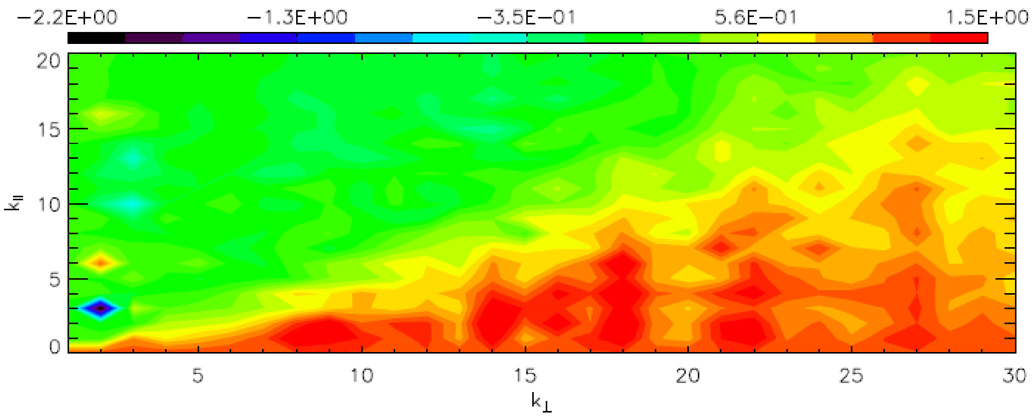
Here $E_r \sim E_{tot}$!

Also, turbulence becomes strong!

Wang et al 2011

Residual energy in weak MHD turbulence

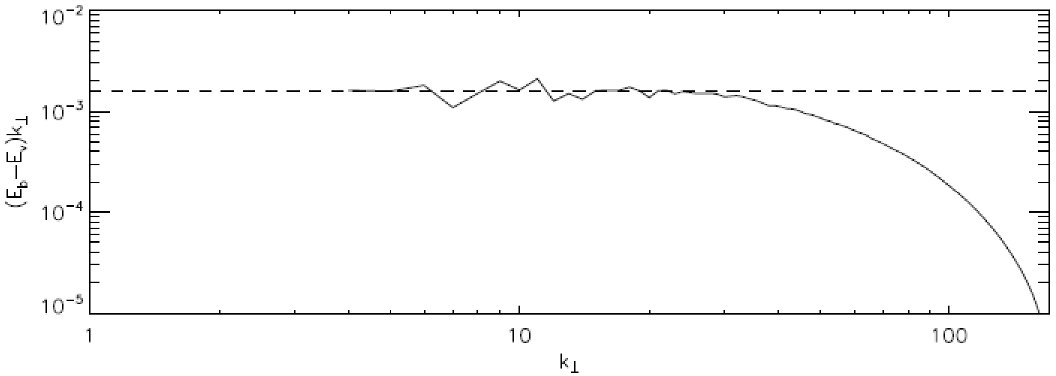
$$e^r = \langle \mathbf{z}^+ \cdot \mathbf{z}^- \rangle = \langle v^2 - b^2 \rangle \propto -\epsilon^2 k_\perp^{-2} \Delta(k_\parallel)$$



$\Delta(k_\parallel)$ is concentrated at

$$k_\parallel < C\epsilon^2 k_\perp$$

"condensate"

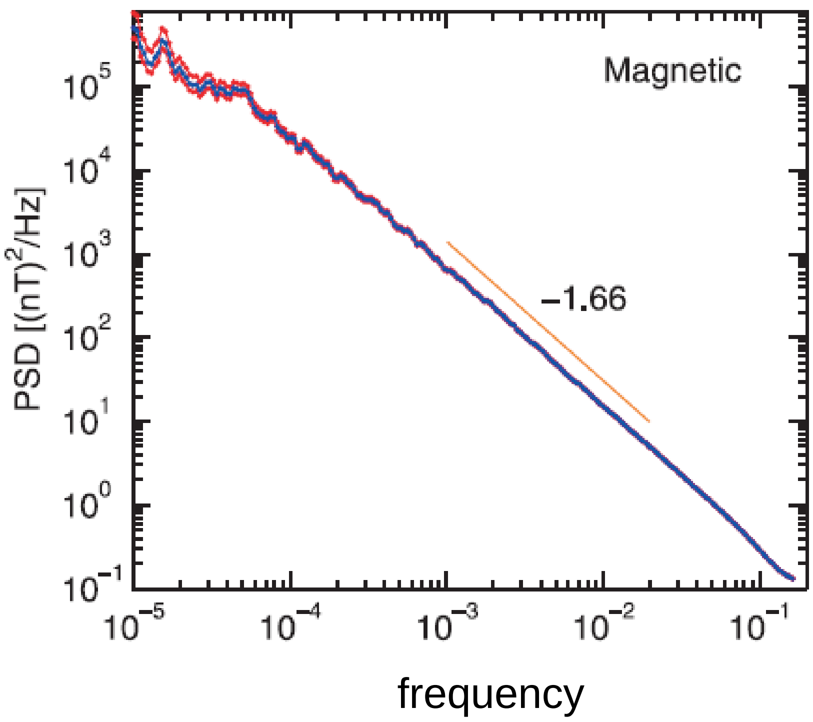


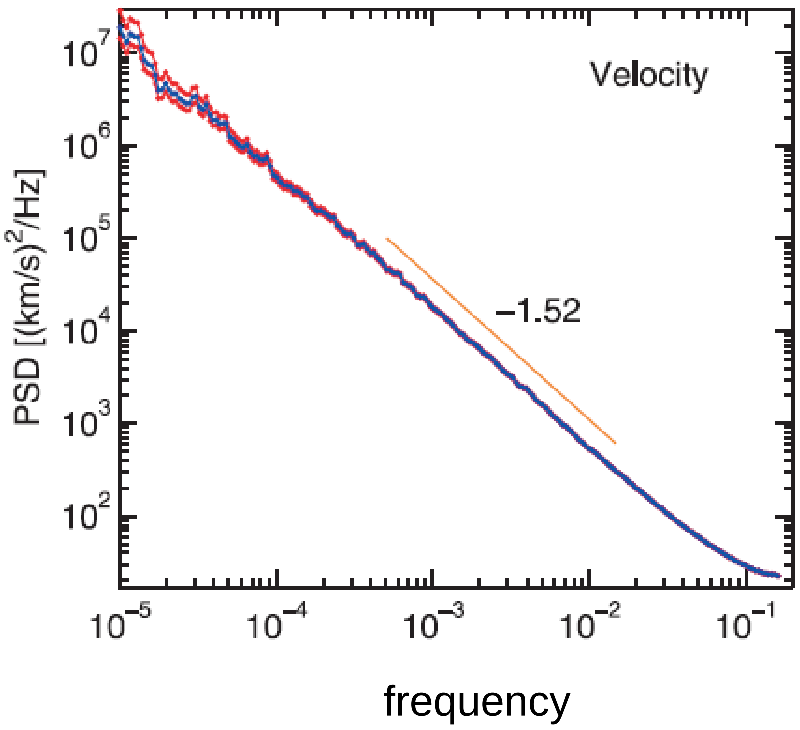
$$E^R(k_\perp) = \int e^r(k_\parallel, k_\perp)\, dk_\parallel$$

$$\propto k_\perp^{-1}$$

Y. Wang, S. B. & J. C. Perez (2011)
S.B, J. C. Perez & V. Zhdankin (2011)

MHD turbulence in the solar wind

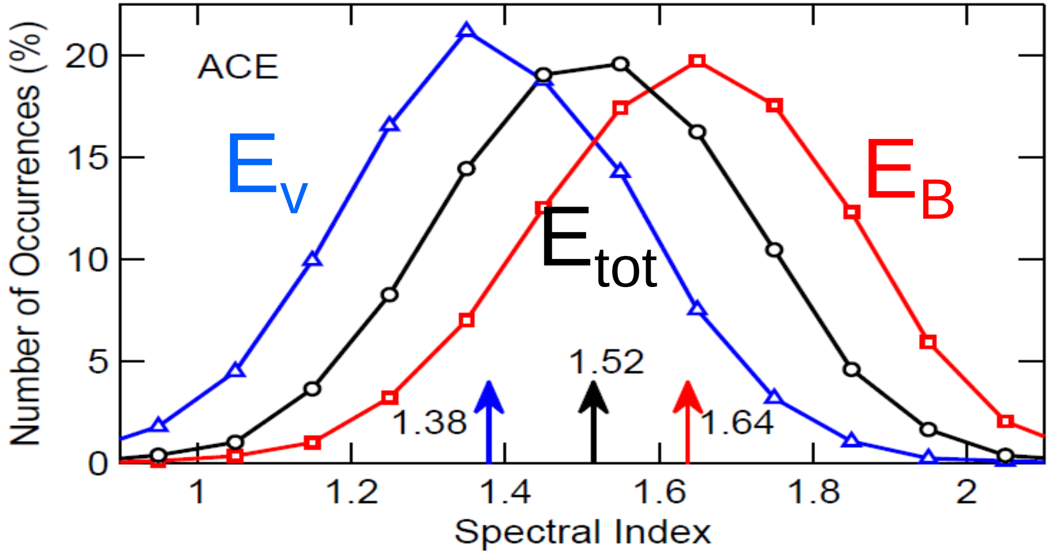




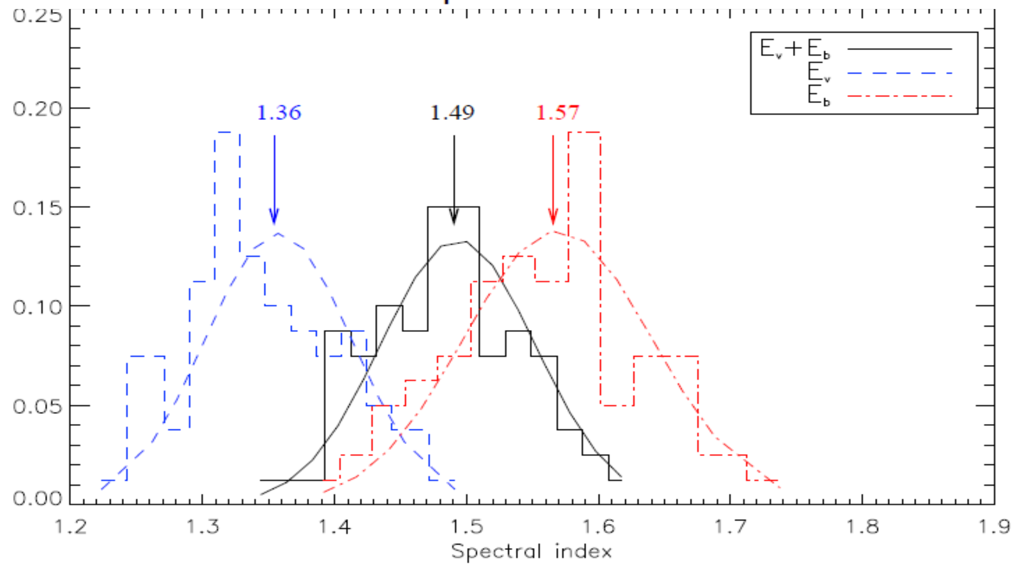
Podesta et al (2007)

Energy spectra in the solar wind and in numerical simulations

Solar wind observations: spectral indices in 15,472 independent measurements. (From 1998 to 2008, fit from 1.8 $\times$ 10^{-4} to 3.9 $\times$ 10^{-3} Hz)

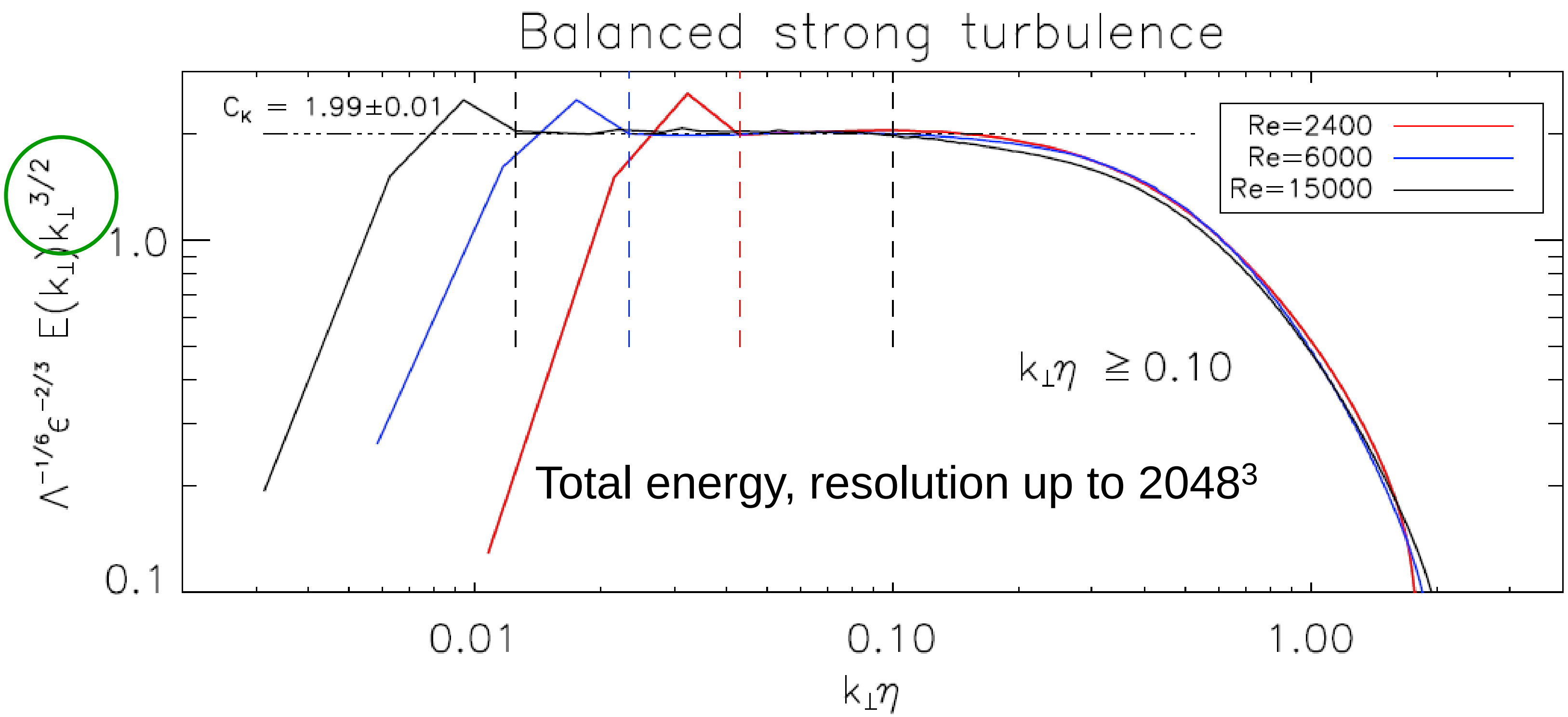


Numerical simulations: spectral indices in 80 independent snapshots, separated by a turnover time.



S.B., J. Perez, J Borovsky & J. Podesta (2011)

Spectrum of strong MHD turbulence: balanced case

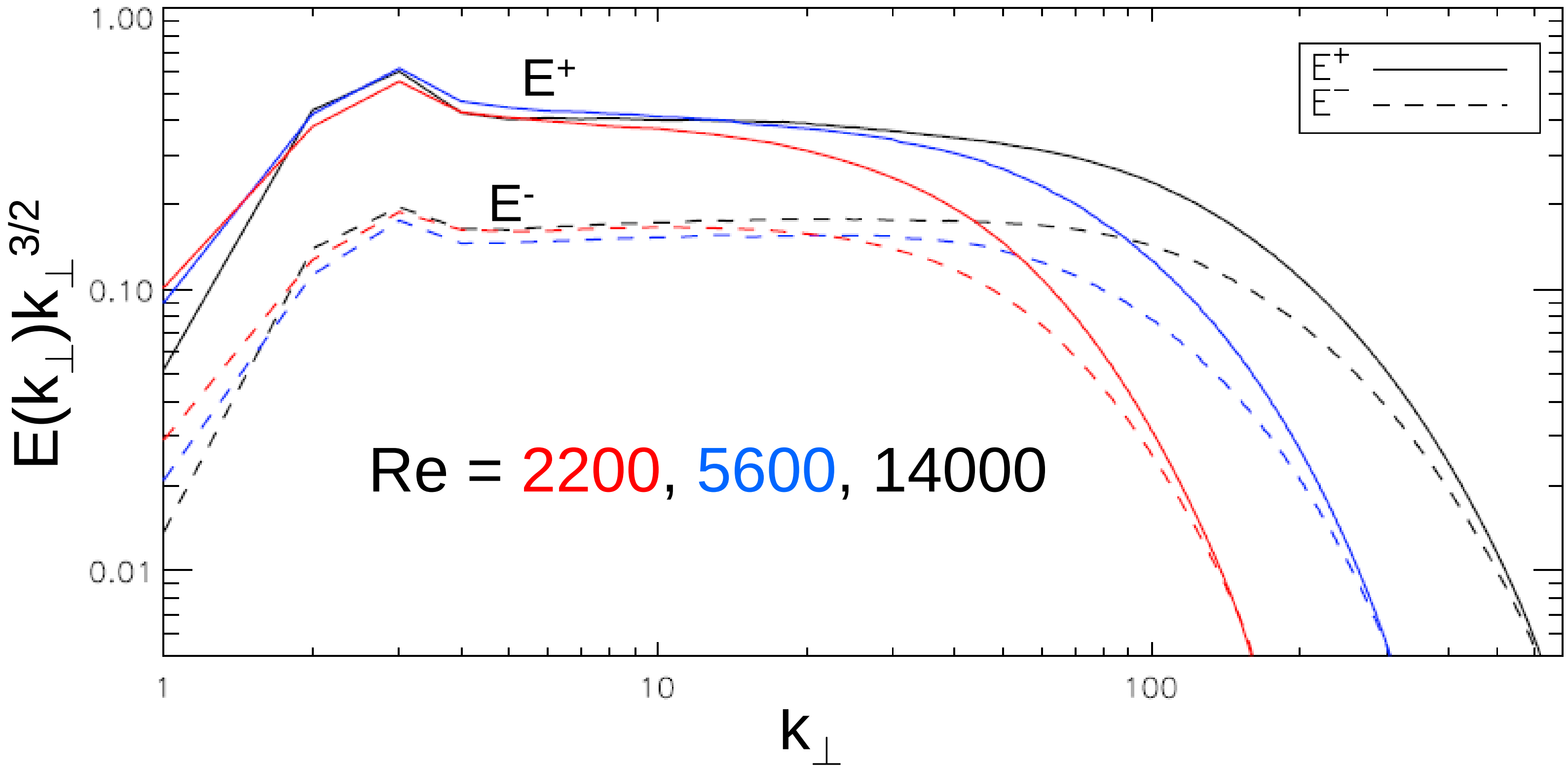


Computational resources: DoE 2010 INCITE,
Machine: Intrepid, IBM BG/P at Argonne Leadership Computing Facility

Perez et al, Phys Rev X (2012)

Recall the phenomenological prediction for the spectrum : -5/3

Spectrum of strong MHD turbulence: imbalanced case



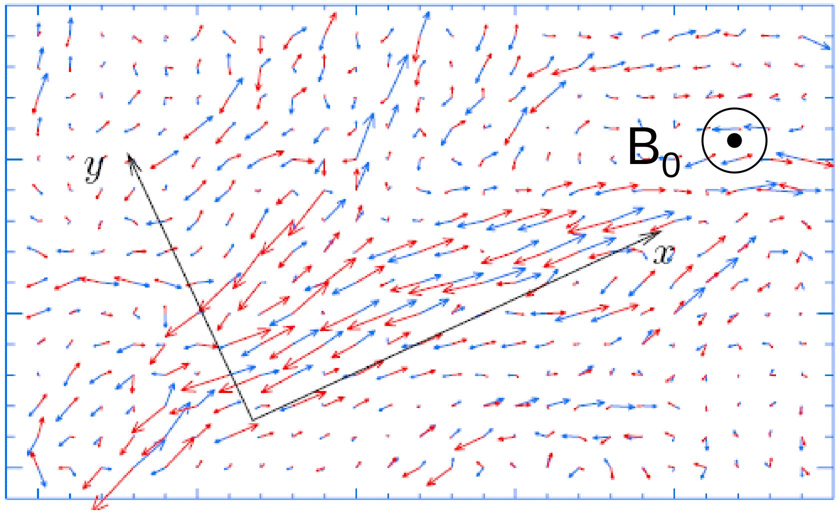
Computational resources: DoE 2010 INCITE,
Machine: Intrepid, IBM BG/P at Argonne Leadership Computing Facility

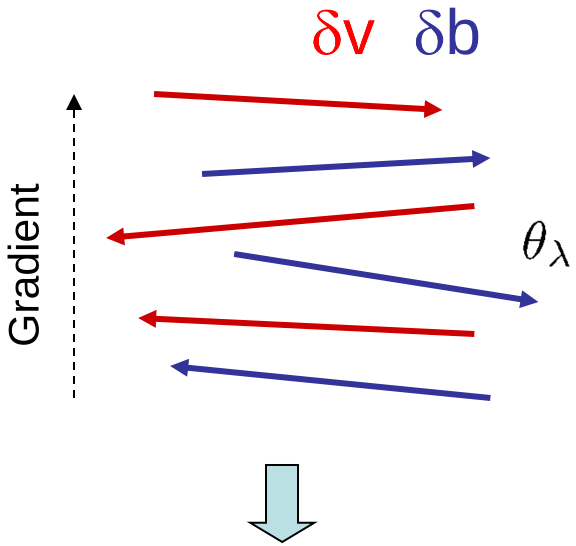
Perez et al, Phys Rev X (2012)

Possible explanation of the -3/2 spectrum

Dynamic Alignment theory

Fluctuations $\delta \mathbf{v}_\lambda$ and $\delta \mathbf{b}_\lambda$ become spontaneously aligned in the field-perpendicular plane within angle θ_λ





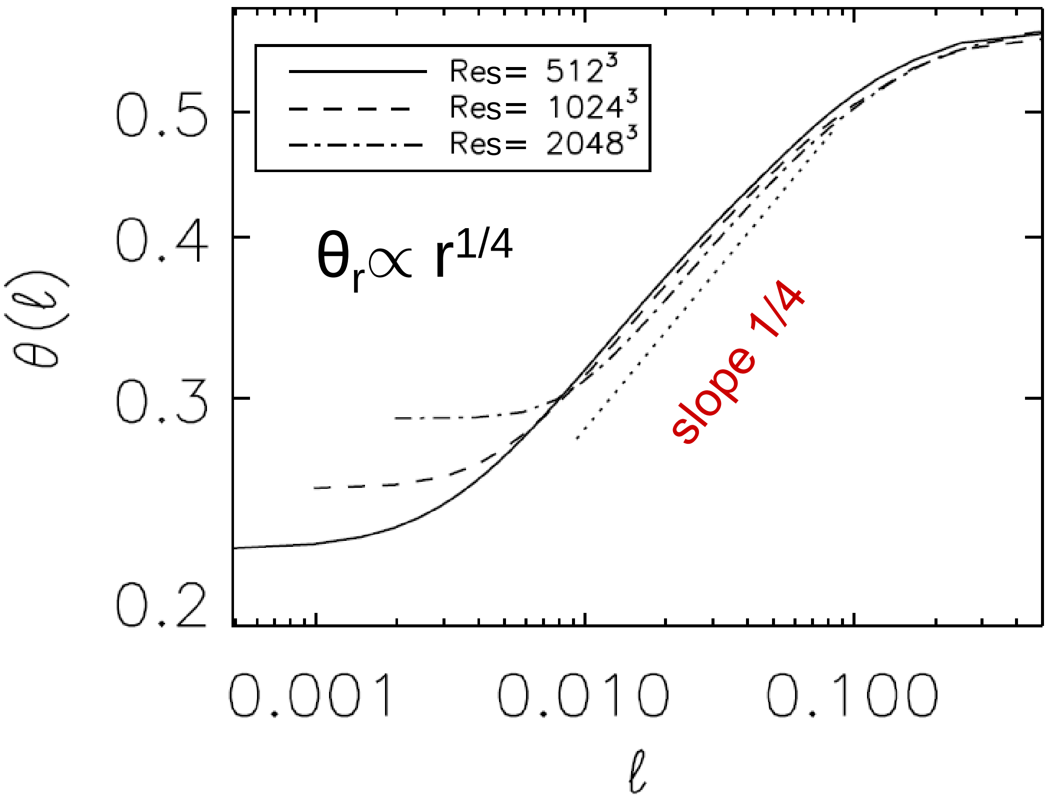
$$(\mathbf{z}^{\pm} \cdot \nabla)\mathbf{z}^{\mp} \sim \theta_\lambda \delta v_\lambda^2 / \lambda$$

Nonlinear interaction is depleted

Numerical verification of dynamic alignment

$$S_{cross}(r) = \langle|\delta\tilde{\mathbf{v}}_r \times \delta\tilde{\mathbf{b}}_r|\rangle \qquad S_2(r) = \langle|\delta\tilde{\mathbf{v}}_r||\delta\tilde{\mathbf{b}}_r|\rangle$$

Alignment angle: $\theta_r \approx \sin(\theta_r) \equiv S_{cross}(r)/S_2(r)$



Magnetic and velocity fluctuations build progressively stronger correlation at smaller scales.

Form sheet-like structures

Mason et al 2011,
Perez et al 2012

Physics of the dynamic alignment

Hydrodynamics: $$\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p + \nu\nabla^2\mathbf{v}$$

$$E = \frac{1}{2}\int \mathbf{v}^2(\mathbf{x})\, d^3x$$

MHD: $$\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{4\pi\rho_0}(\mathbf{B}\cdot\nabla)\mathbf{B} + \nu\nabla^2\mathbf{v}$$

$$\partial_t\mathbf{B} = \nabla\times[\mathbf{v}\times\mathbf{B}] + \eta\nabla^2\mathbf{B}$$

$$E = \frac{1}{2}\int (v^2 + b^2) d^3x \qquad H^C = \int (\mathbf{v}\cdot\mathbf{b}) d^3x$$

Energy E is dissipated faster than cross-helicity H^C

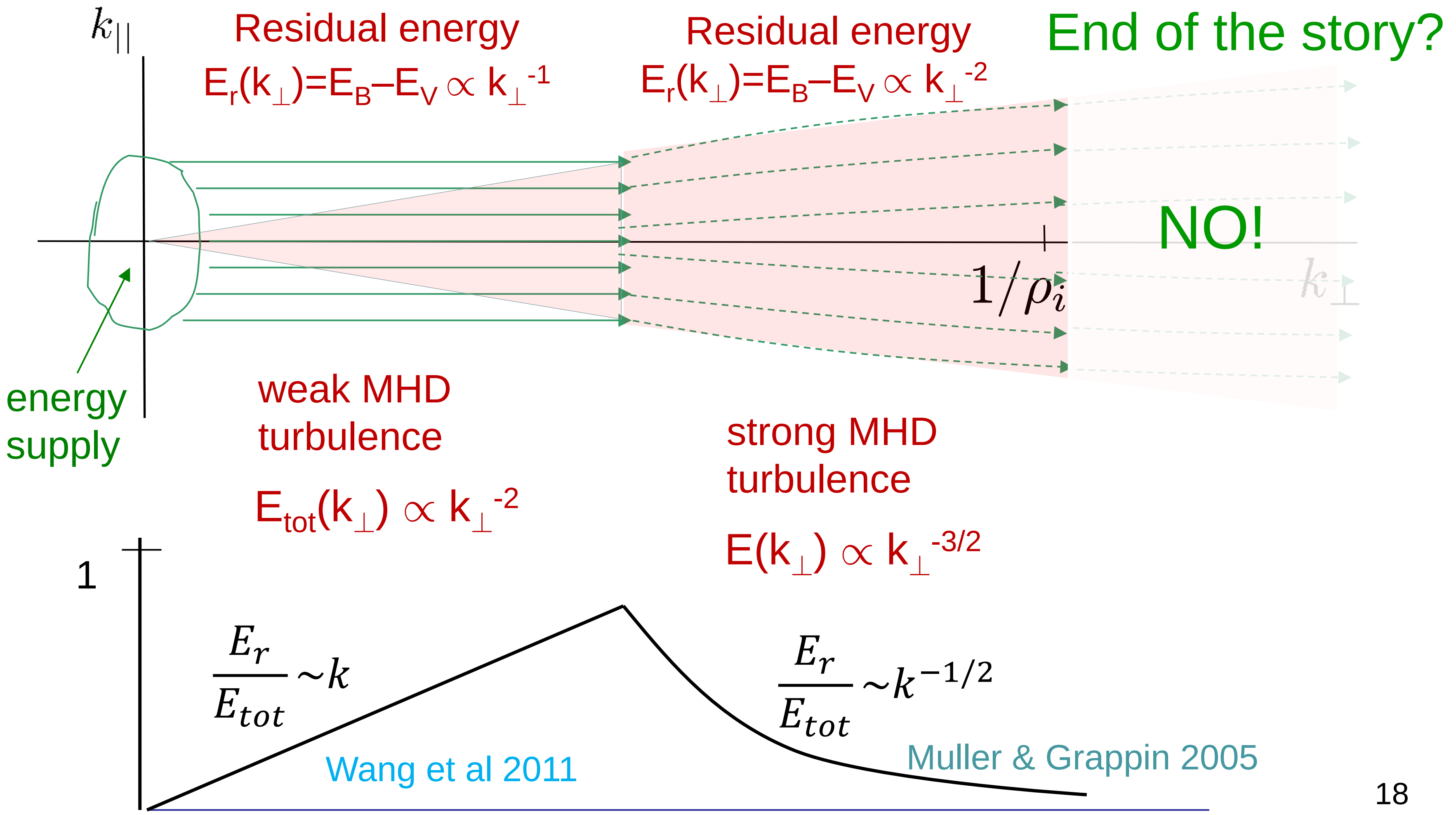
$$\frac{\delta}{\delta\mathbf{v}}\left[\int (v^2+b^2)d^3x - \lambda\int(\mathbf{v}\cdot\mathbf{b})\,d^3x\right] = 0$$

$$\frac{\delta}{\delta\mathbf{b}}\left[\int (v^2+b^2)d^3x - \lambda\int(\mathbf{v}\cdot\mathbf{b})\,d^3x\right] = 0$$

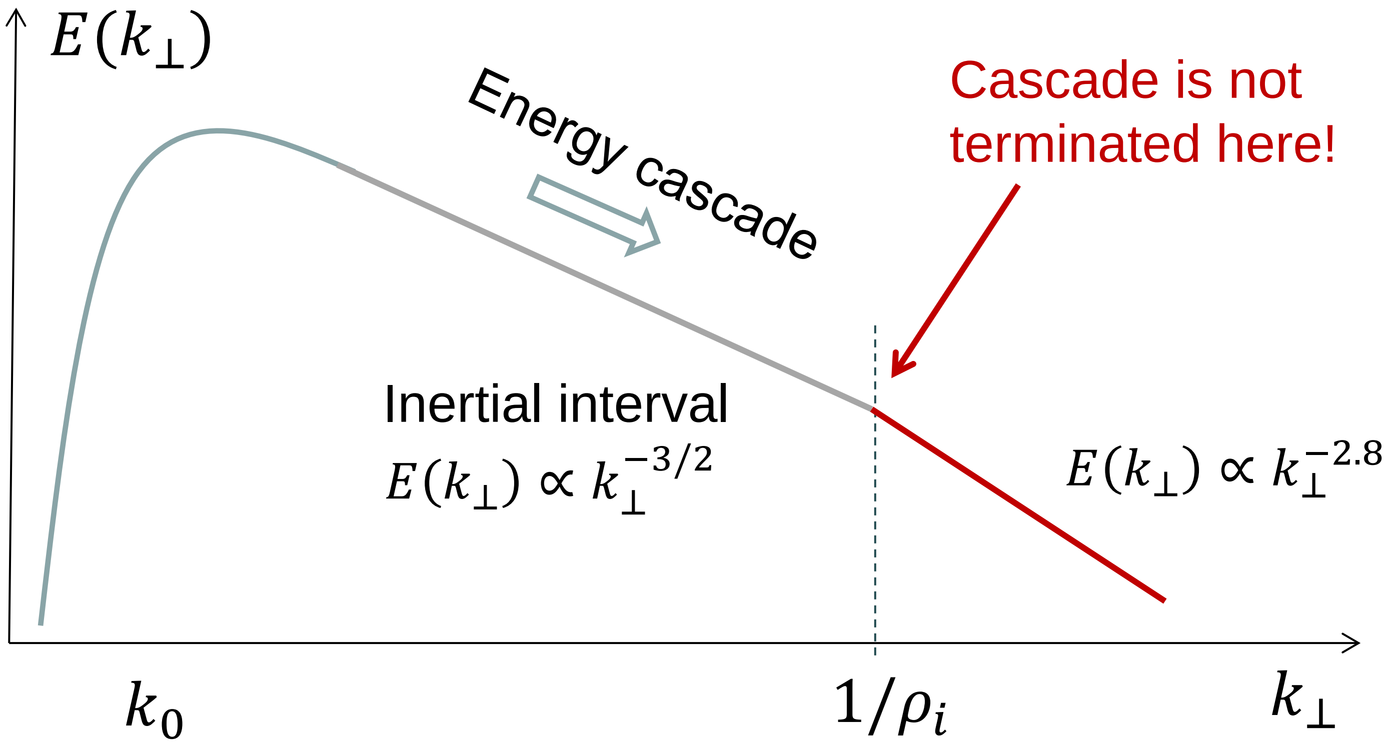
⇓

$$\mathbf{v}(\mathbf{x}) = \pm\mathbf{b}(\mathbf{x})$$

Residual energy in weak MHD turbulence



Spectrum of MHD turbulence



$\omega \approx k_{||} V_A \ll \Omega_i$ when $k_\perp \sim 1/\rho_i$

At subproton scales turbulence is highly anisotropic: $k_\perp \gg k_{||}$

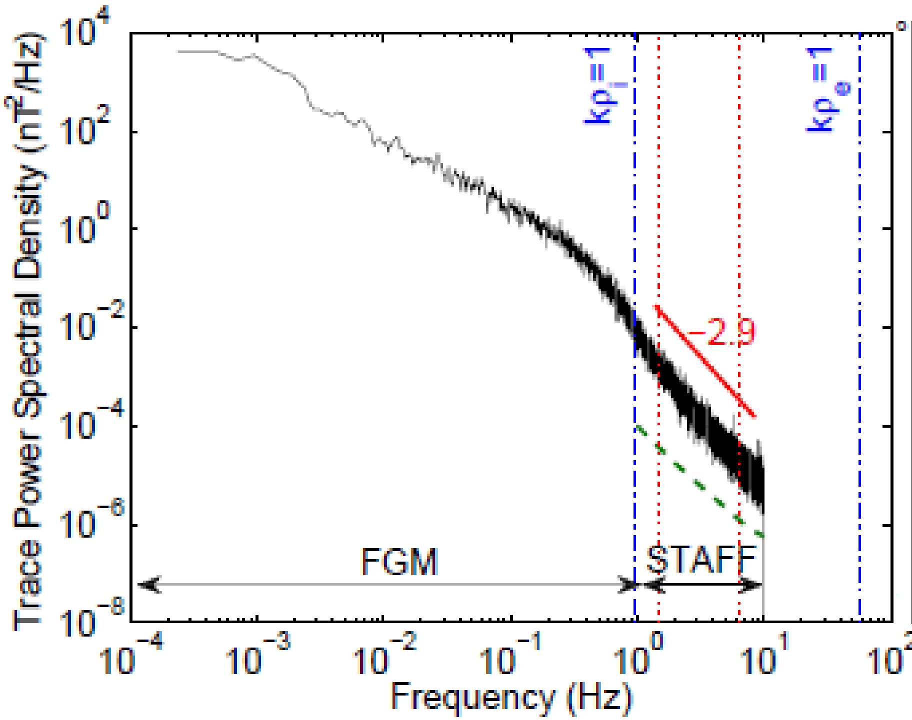
cascade continues as Kinetic-Alfven turbulence!

This may explain how plasma is heated.

[Howes et al 2006]

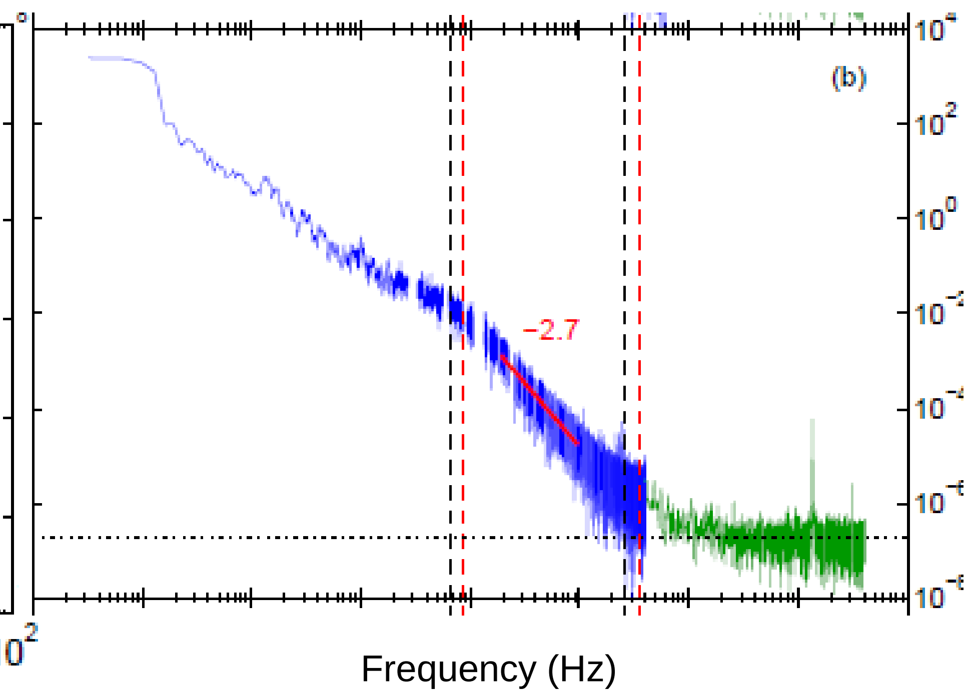
Sub-proton fluctuations in the solar wind

Magnetic field



Chen, et al (2010)

Density



Chen et al (2012);
Also: Safrankova et al (2013)

Conclusions

- Magnetic plasma turbulence goes through the three universal regimes:
  - weak MHD turbulence
  - strong MHD
  - strong kinetic Alfven

  distinguished by their energy spectra and v-b correlation.

- Effects of self-organization are crucial for understanding driven, steady-state plasma turbulence, in particular the effects of: magnetic energy condensation and residual energy generation; small-scale intermittency

- High-resolution magnetohydrodynamic numerical simulations become very efficient and they allow for direct comparison with observational data.